Show your work for computations and write complete sentences for proofs. Partial credit is available, but I would like for these homeworks to help you improve your proof-writing skill. More importantly, I hope these problems make the lecture clear. Thanks and enjoy.

Topics: much of this corresponds to section I.1 of Freitag & Busam. Note the problems in the text have hints in the back of the text which are at times very insightful.

Problem 1 Write each of the complex numbers below in their cartesian form:

- **a.** $(8+i)-(5+i)$
- **b.** $2/i$
- **c.** $3/i+i/3$
- **d.** $\frac{2+3i}{1+2i}-\frac{8+i}{6-i}$
- **e.** $i^2(1+i)^2$

Problem 2 Let $z=\frac{3+4i}{3-4i}$ and $w=1+3i$.

- **a.** find the cartesian form of z
- **b.** find the polar form of z and w, in particular find $Arg(z)$ and $Arg(w)$.
- **c.** verify $|zw|\leq|z||w|$.
- **d.** verify $|z+w|\leq|z|+|w|$.

Problem 3 Desribe the solution sets of the following complex equations:

- **a.** $Im(z)\leq-2$
- **b.** $|2z-i|=4$
- **c.** $|z|=Re(z)+2$
- **d.** $|z-i|<2$
- **e.** $|z-1|+|z+1|=7$

Problem 4 Find all solutions of $z^3=27$. Factor $P(z)=z^3-27$.

Problem 5 Complete the proof of associativity of multiplication (item 2. of the Theorem on page 3 of my notes). I worked out $z_1(z_2z_3)$, you need to work out $(z_1z_2)z_3$ in the same fashion.

Problem 6 Prove property 3. on page 3 of my notes. Your proof should utilize the $*$ notation.

Problem 7 Prove properties 1 and 4 from page 4 of my notes.

Problem 8 If $X\in M$ then $X=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Show that M forms a complex number system with respect to the usual matrix operations.

a. show that there is a subset $M_{\mathbb{R}}$ of M which corresponds to real numbers. In particular, explicitly construct $\Phi : \mathbb{R} \to M_{\mathbb{R}}$ and show Φ is a bijection such that $\Phi(xy) = \Phi(x)\Phi(y)$ and $\Phi(x + cy) = \Phi(x) + c\Phi(y)$ for all $x, y, c \in \mathbb{R}$.

b. Solve $X^2 + I = 0$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

c. Remark on the number of solutions you found in the previous part.

Problem 9 Show $e^{z+i\pi} = -e^z$.

Problem 10 Show that $|z + w| \leq |z| + |w|$ and $|z| - |w| \leq |z - w|$ for all $z, w \in \mathbb{C}$. (see the hint to problem 3 of page 19 Freitag & Busam)

Problem 11 Work out problem 4 of page 19 Freitag & Busam.

Problem 12 Suppose $z \neq 0$ for the sake of $k < 0$. Prove by induction that $\cancel{(\bar{z})}^k = \overline{(z^k)}$ for all $k \in \mathbb{Z}$.

$(\bar{z})^k = \overline{z^k}$

PROBLEM 1

a.) $(8+i)-(5+i) = \boxed{13 + i(0)}$

b.) $\frac{2}{i} = \boxed{-2i}$ $\quad$ (note: $i^2 = -1 = ii \Rightarrow \frac{1}{i} = -i$)

c.) $\frac{3}{i} + \frac{i}{3} = -3i + \frac{i}{3} = \boxed{0 + i\left(-\frac{11}{3}\right)}$

d.)
$$\frac{2+3i}{1+2i} - \frac{8+i}{6-i} = \frac{(2+3i)(1-2i)}{(1+2i)(1-2i)} - \frac{(8+i)(6+i)}{(6-i)(6+i)}$$
$$= \frac{1}{5}(2+3i-4i+6) - \frac{1}{37}(48+8i+6i-1)$$
$$= \frac{1}{5}(8-i) - \frac{1}{37}(47+14i)$$
$$= \boxed{\left(\frac{8}{5} - \frac{47}{37}\right) + i\left(\frac{-1}{5} - \frac{14}{37}\right)}$$

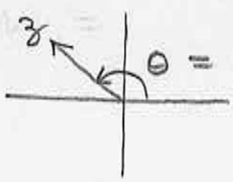
e.)
$$i^2(1+i)^2 = -1(1+2i+i^2)$$
$$= -2i$$
$$= \boxed{0 + i(-2)}$$

Remark: technically cartesian form is either $a+ib$ or perhaps $a+bi$. We gave you credit even if you omitted the 0 or moved the minus.

PROBLEM 2 Let $z = \frac{3+4i}{3-4i}$ and $w = 1+3i$

a.) $z = \frac{1}{25}(3+4i)(3+4i) = \underline{\frac{-7}{25} + i\left(\frac{24}{25}\right)}$. cartesian form of z

b.) $|z| = \frac{1}{25}\sqrt{7^2+(24)^2} = \frac{25}{25} = 1$ $\quad$ (well, duh $|3+4i| = |3-4i|$ and $|z| = \frac{|3+4i|}{|3-4i|}$.)



$\theta = \tan^{-1}\left(\frac{24}{-7}\right) + \pi \cong 1.855$

$\boxed{z \cong e^{i(1.855)}}$ where $|z| = 1$, $Arg(z) = 1.855$

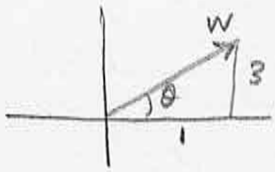
continued

Remark: $(-\pi, \pi] \cong (-3.14, 3.14]$
note $1.855 \in (-\pi, \pi]$ this is how I knew $Arg(z)$ was θ.

PROBLEM 2

b.) continued,

$$|w| = \sqrt{1+9} = \sqrt{10}$$



$$\theta = \tan^{-1}\left(\frac{3}{1}\right) \cong \boxed{1.249 = \operatorname{Arg}(w)}$$

$$\boxed{w = \sqrt{10}\, e^{i(1.249)}}$$ ← polar form of w

c.) $$zw = \frac{1}{25}(-7+24i)(1+3i)$$
$$= \frac{1}{25}(-7-72+24i-21i)$$
$$= \frac{1}{25}(-79+3i)$$

Hence $|zw| = \frac{1}{25}\sqrt{(79)^2+9} = \frac{1}{25}25\sqrt{10} = \sqrt{10}$

However, $|z||w| = 1 \cdot \sqrt{10}$ thus $|zw| = |z||w|$.

Remark: we can replace $|zw| \leq |z||w|$ with $|z||w|$!
More generally a set S which is an algebra with a norm $\|\cdot\| : S \to \mathbb{R}$ is a Banach Algebra if $\|zw\| \leq \|z\|\cdot\|w\|$.
This special case $S = \mathbb{C}$ has equality rather than $\leq$.
Sorry, I had Banach Algebras on brain when wrote problem (ii)

d.) $$z+w = \frac{-7}{25} + i\left(\frac{24}{25}\right) + 1 + 3i = \frac{18}{25} + i\left(\frac{99}{25}\right)$$

$$|z+w| = \sqrt{\left(\frac{18}{25}\right)^2 + \left(\frac{99}{25}\right)^2} = \frac{1}{25}\sqrt{(18)^2+(99)^2} \cong 4.025$$

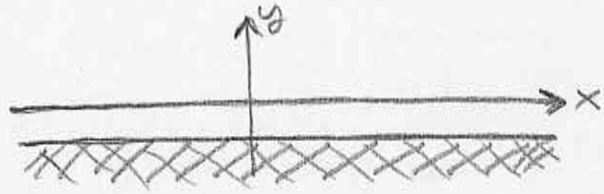
$|z| = 1$ and $|w| = \sqrt{1^2+3^2} = \sqrt{10} \cong 3.162$

$|z+w| \cong 4.025 < 4.162 \cong |z|+|w|$. (it works!)

PROBLEM 3 Describe the sol^{n}'s to the following complex eq^{n}'s.

a.) $\operatorname{Im}(z) \leq -2$

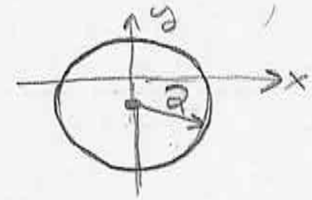
$z = x + iy$ with $\operatorname{Im}(z) = y \leq -2$



b.) $|2z - i| = 4 \Rightarrow |2(z - i/2)| = 4$

$\Rightarrow \underbrace{|z - i/2| = 2}$

$d(z, i/2) = 2$, a circle at $z_0 = i/2$ with radius 2.



c.) $|z| = \operatorname{Re}(z) + 2$

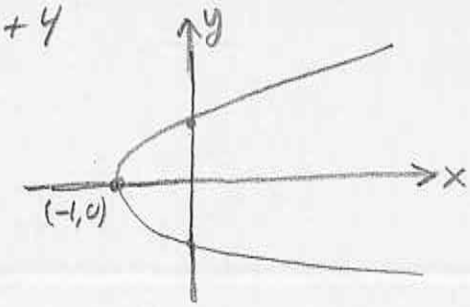
☆ $\sqrt{x^2+y^2} = x + 2 \Rightarrow x+2 \geq 0$ btw, keep this in mind for later.

$\cancel{x^2}+y^2 = \cancel{x^2} + 4x + 4$

$y^2 = 4x+4$

$y^2 = 4(x+1)$

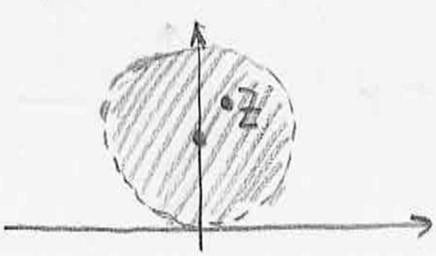
$x = \frac{1}{4}y^2 - 1$



But ! $x+2 \geq 0 \Rightarrow x \geq -2$ ok, no modification since $x = \frac{1}{4}y^2 - 1 \geq -2$ $\forall$ pts.

(whenever I square (☆) an eq^{n} it may add extraneous data)

d.) $|z - i| < 2$

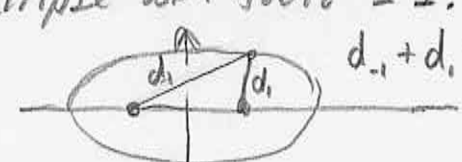


e.) $\underbrace{|z-1| + |z+1|} = 7$

sum of $d(z,1) + d(z,-1) = 7$

ellipse with foci ± 1.

$d_{-1} + d_1 = 7$



$\boxed{\text{PROBLEM 4}}$ Find all sol^{n}'s of $z^3 = 27$ and factor $P(z) = z^3 - 27$

$z^3 = 27$ solved nicely by polar rep. of $z = re^{i\theta}$

$$(re^{i\theta})^3 = 27 \Rightarrow r^3 e^{3i\theta} = 27e^0$$

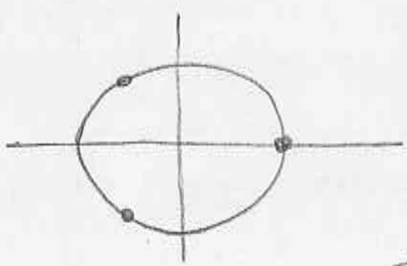
$$\Rightarrow r^3 = 27 \text{ and } 3\theta = 2\pi k \text{ for } k \in \mathbb{Z}$$

$$\Rightarrow r = 3 \text{ and } \theta = 0, \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \ldots$$

$$\Rightarrow z = 3e^{i\theta} \text{ for } \theta = 0, \tfrac{2\pi}{3}, \tfrac{4\pi}{3}.$$

Note: $\exp\left(\frac{2\pi i}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

$$e^{\frac{4\pi i}{3}} = \frac{-1}{2} - \frac{i\sqrt{3}}{2}.$$



The sol^{n}'s are $\boxed{z = 3, \tfrac{3}{2}(-1+i\sqrt{3}), \tfrac{3}{2}(-1-i\sqrt{3})}$

By factor Th^{m}. $\Rightarrow \boxed{P(z) = (z-3)\left(z - \tfrac{3}{2}(1-i\sqrt{3})\right)\left(z - \tfrac{3}{2}(1+i\sqrt{3})\right)}$

$\boxed{\text{PROBLEM 5}}$

$$(z_1 z_2) z_3 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) * (x_3, y_3)$$

$$= \left((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + x_2y_1)y_3, (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + x_2y_1)x_3\right.$$

$$= \left(x_1x_2x_3 - y_1y_2x_3 - x_1y_2y_3 - x_2y_1y_3, x_1x_2y_3 - y_1y_2y_3 + x_1y_2x_3 + x_2y_1x_3\right.$$

$= z_1(z_2z_3)$ by 3 of my posted notes from this semester.

PROBLEM 6 Show $z_1 + z_2 = z_2 + z_1$

(sorry I meant Property 4, this one is just addition: $z_1+z_2 = (x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$

$$= (x_2+x_1, y_2+y_1) \quad \text{prop. of } \mathbb{R}.$$
$$= (x_2,y_2)+(x_1,y_1)$$
$$= z_2+z_1$$

(the proof $z_1 z_2 = z_2 z_1$ involves $*$ notation

$$z_1 z_2 = (x_1,y_1)*(x_2,y_2) = (x_1x_2 - y_1y_2, x_1y_2+y_1x_2)$$
$$= (x_2x_1 - y_2y_1, x_2y_1+y_2x_1)$$
$$= (x_2,y_2)*(x_1,y_1)$$
$$= z_2z_1.$$

PROBLEM 7 Let $z = x+iy$ and $w = a+ib$,

1.) $\overline{z+w} = \overline{x+a+i(y+b)} = x+a-i(y+b)$
$$= (x-iy)+(a-ib)$$
$$= \bar{z}+\bar{w}.$$

4.) $z\bar{z} = (x+iy)(x-iy) = x^2+iyx-ixy-i^2y^2 = x^2+y^2.$

PROBLEM 8 If $M = \{ X \in \mathbb{R}^{2\times 2} \mid X = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ for } a,b \in \mathbb{R}\}.$

a.) Let $M_{\mathbb{R}} = \{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R}\}$

$\Phi(a) = aI$ clearly $\Phi^{-1}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a$ hence Φ is a bijection. Moreover,

$$\Phi(xy) = xyI = (xI)(yI) = \Phi(x)\Phi(y)$$
$$\Phi(x+cy) = (x+cy)I = xI+cyI = \Phi(x)+c\Phi(y).$$

alternatively

[you can show bijective a few different ways)
$\Phi(a) = \Phi(b) \Rightarrow aI = bI \Rightarrow a=b \therefore \Phi$ 1-1,

PROBLEM 8 continued

surjectivity of $\Phi : \mathbb{R} \to M_{\mathbb{R}}$ is easy.

If $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in M_{\mathbb{R}}$ then $\Phi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Thus Φ is 1-1 and onto $\therefore \Phi$ is bijective.

finishing-up bijective argument.

(explicit inverse also $\Rightarrow$ bijectivity that was my initial argument)

b.) $$X^2 + I = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left[\begin{array}{c|c} x^2 - y^2 + 1 & -2xy \\ \hline 2xy & -y^2 + x^2 + 1 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad x^2 - y^2 + 1 = 0 \ \text{(I)} \quad \text{and} \quad 2xy = 0 \ \text{(II)}$$

For (II) we need $x = 0$ or $y = 0$.

If $y = 0$ then (I) $\Rightarrow x^2 + 1 = 0 \Rightarrow x \notin \mathbb{R}$.

If $x = 0$ then (I) $\Rightarrow -y^2 + 1 = 0$

$\Rightarrow (1 - y)(1 + y) = 0$

$\Rightarrow y = 1$ or $y = -1$

$\Rightarrow X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

the only sol$^{\underline{n}}$'s of $X^2 + I = 0$.

c.) We found two sol$^{\underline{n}}$'s.

Moreover, while were at it, note:

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad -i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

in the 2×2 matrix formulation of complex #'s.

PROBLEM 9

$$e^{z+i\pi} = e^z e^{i\pi} = e^z(\underset{-1}{\cos\pi} + i\underset{0}{\sin\pi}) = -e^z.$$

PROBLEM 10 Show $|z+w| \leq |z| + |w|$ & $|z|-|w| \leq |z-w|$ for all $z, w \in \mathbb{C}$.

Observe, $|Re(z)| \leq \sqrt{(Re(z))^2 + (Im(z))^2} = |z|$. Likewise it is clear that $|Im(z)| \leq |z|$. Observe,

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w}) \quad : \text{ since } \overline{z+w} = \bar{z}+\bar{w},$$
$$= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$$
$$= |z|^2 + z\bar{w} + w\bar{z} + |w|^2$$

Let $z = x+iy$ and $w = a+ib$ and consider,

$$z\bar{w} + w\bar{z} = (x+iy)(a-ib) + (a+ib)(x-iy)$$
$$= ax + yb + \cancel{i(ay-bx)} + ax + by + \cancel{i(-ay+bx)}$$
$$= 2(ax+yb)$$
$$= 2Re(z\bar{w})$$

$$\left(z\bar{w} = (x+iy)(a-ib) = \underbrace{xa+yb}_{Re(z\bar{w})} + i(ya-bx) \right)$$

Therefore,

$$|z+w|^2 = |z|^2 + 2Re(z\bar{w}) + |w|^2$$
$$\leq |z|^2 + 2|z\bar{w}| + |w|^2$$
$$= |z|^2 + 2|z||w| + |w|^2$$
$$= (|z|+|w|)^2$$

$|z\bar{w}| = |z||\bar{w}| = |z||w|$.

However, $|z+w| \geq 0$ hence $\underline{|z+w| \leq |z|+|w|}$.

Consider $|z| = |z-w+w| \leq |z-w| + |w|$ by our work above. Thus, $|z|-|w| \leq |z-w|$. //

$\boxed{\text{PROBLEM 11}}$ Work #4 of pg. 19 of Freitag & Busam

$\langle z, w \rangle = Re(z\bar{w}) = xu + yv$ for $z = x+iy$, $w = u+iv$

(A.) Claim: $\langle z, w\rangle^2 + \langle iz, w\rangle^2 = |z|^2|w|^2$.

Proof: $(Re(z\bar{w}))^2 + (Re(iz\bar{w}))^2 = (Re(z\bar{w}))^2 + (-Im(z\bar{w}))^2$

$$= |z\bar{w}|^2$$

$$= |z|^2|w|^2 //$$

I used a Lemma: $Re(iz) = -Im(z)$, this is easily verified $Re(i(x+iy)) = Re(ix - y) = -y = -Im(z)$.

(B.) Cauchy-Schwarz says $|\vec{A}\cdot\vec{B}| \le |\vec{A}||\vec{B}|$. Or, in our current notation $|\langle \vec{A}, \vec{B}\rangle| \le |\vec{A}||\vec{B}|$.

Hence,

Claim: $|\langle z, w\rangle|^2 = |xu + yv|^2 \le |z|^2|w|^2 = (x^2+y^2)(u^2+v^2)$

Proof: $\langle z, w\rangle^2 = \underbrace{|z|^2|w|^2 - \langle iz, w\rangle^2}_{\text{real \#'s}} \le \underbrace{|z|^2|w|^2}_{\text{clearly bigger.}}$

$$\Rightarrow \underline{|\langle z, w\rangle| \le |z|\cdot|w|}.$$

(Cauchy Schwarz followed from A.)

C.) Claim: $|z+w|^2 = |z|^2 + 2\langle z, w\rangle + |w|^2$.

Proof: $|z+w|^2 = \langle z+w, z+w\rangle$

$$= Re((z+w)(\bar{z}+\bar{w}))$$

$$= Re(z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w})$$

$$= Re(z\bar{z}) + Re(z\bar{w} + w\bar{z}) + Re(w\bar{w})$$

$$\star \quad = |z|^2 + 2Re(z\bar{w}) + |w|^2$$

$\star$ left to reader. $\quad = |z|^2 + 2\langle z, w\rangle + |w|^2$.

$\boxed{\text{Lemma: } \langle v, w\rangle = \langle w, v\rangle \text{ because } Re(z\bar{w}) = Re(\bar{z}w).}$

PROBLEM 11 Continued

D.) $|z-w|^2 = |z|^2 - 2\langle z,w\rangle + |w|^2$ (Claim)

Proof $|z-w|^2 = \langle z-w, z-w\rangle$

$= Re((z-w)(\bar{z}-\bar{w}))$

$= Re(z\bar{z} - w\bar{z} - \bar{w}z + w\bar{w})$

$= Re(z\bar{z}) - 2Re(w\bar{z}) + Re(w\bar{w})$

$= \langle z,z\rangle - 2\langle w,z\rangle + \langle w,w\rangle$

$= |z|^2 - 2\langle w,z\rangle + |w|^2$ //

E.) $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$ claim

Proof: add identities from parts C. & D.

F.) Let $(z,w) \in \mathbb{C}^{\circ} \times \mathbb{C}^{\circ}$ meaning $z \neq 0$ and $w \neq 0$

Notice $|\langle z,w\rangle| \leq |z||w| \Rightarrow \left|\frac{\langle z,w\rangle}{|z||w|}\right| < 1$

since $|z|, |w| \neq 0$ $\quad -1 \leq \frac{\langle z,w\rangle}{|z||w|} \leq 1$

thus, $\exists! \omega \in (-\pi, \pi]$ such that $\cos\omega = \frac{\langle z,w\rangle}{|z||w|}$

(the uniqueness follows from the structure of cosine)

Likewise, $|\langle iz,w\rangle| \leq |iz||w| = |z||w|$ hence

$-1 \leq \frac{\langle iz,w\rangle}{|z||w|} \leq 1$. Moreover, $\cos\omega = \frac{\langle z,w\rangle}{|z||w|}$

$\Rightarrow \cos^2\omega = \frac{\langle z,w\rangle^2}{|z|^2|w|^2} = \frac{|z|^2|w|^2 - \langle iz,w\rangle^2}{|z|^2|w|^2} = 1 - \frac{\langle iz,w\rangle^2}{|z|^2|w|^2}$

$\Rightarrow \frac{\langle iz,w\rangle^2}{|z|^2|w|^2} = \sin^2\omega$

PROBLEM 11 continued

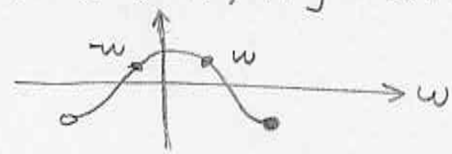
F.) We're attempting to find a <u>unique</u> real # $\omega = \omega(z, w) \in (-\pi, \pi]$ with

$$\cos \omega = \frac{\langle z, w \rangle}{|z||w|} \quad \text{and} \quad \sin \omega = \frac{\langle iz, w \rangle}{|z||w|}$$

We're given $w, z \neq 0$ hence $\left| \frac{\langle z, w \rangle}{|z||w|} \right| < 1$

So we may define, $\omega \in (-\pi, \pi]$ with

$$\cos(\omega) = \frac{\langle z, w \rangle}{|z||w|}$$



this does <u>not</u> uniquely fix ω, However, if we <u>also</u> insists

$$\sin(\omega) = \frac{\langle iz, w \rangle}{|z||w|}$$

$\boxed{\text{PROBLEM 12}}$ Suppose $z \neq 0$, prove $(\bar{z})^k = \overline{z^k}$ $\forall k \in \mathbb{Z}$ by induction.

To prove a statement for $\mathbb{Z}$ it's convenient to split into $\underbrace{-\mathbb{N}}_{\text{II}}$ and $\underbrace{\mathbb{N} \cup \{0\}}_{\text{I}}$.

(I.) If $k = 0$ then $z \neq 0$ has $(\bar{z})^0 = \overline{z^0} = 1$ by definition.
Suppose $(\bar{z})^k = \overline{z^k}$ for some $k > 0$ and consider,

$$\begin{aligned}
(\bar{z})^{k+1} &= (\bar{z})^k \bar{z} && \text{: def}^{\text{n}} \text{ of exponentiation.} \\
&= \overline{z^k}\, \bar{z} && \text{: apply induction hypothesis} \\
&= \overline{z^k z} && \text{: since } \overline{zw} = \bar{z}\bar{w}. \\
&= \overline{z^{k+1}} && \text{: def}^{\text{n}} \text{ of exponentiation.}
\end{aligned}$$

Hence we find $(\bar{z})^k = \overline{z^k}$ $\forall k \in \mathbb{N} \cup \{0\}$.

(II.) If $k \in -\mathbb{N}$ then $-k \in \mathbb{N}$ hence (I.) applies to $-k$.
Consider that

$$\begin{aligned}
(\bar{z})^k = \left(\frac{1}{\bar{z}}\right)^{-k} &= (\bar{w})^j && \text{: where } w = \frac{1}{z},\ j = -k \in \mathbb{N} \\
&= \overline{w^j} && \text{: by I.} \\
&= \overline{\left(\frac{1}{z}\right)^j} \\
&= \overline{(z^{-1})^{-k}} \\
&= \overline{(z^k)}.
\end{aligned}$$

(of course, you can also give an argument like that of I for II)

- Also, $z = re^{i\theta}$ gives nice argument