Show your work for computations and write complete sentences for proofs. Partial credit is available, but I would like for these homeworks to help you improve your proof-writing skill. More importantly, I hope these problems make the lecture clear. Thanks and enjoy.

Topics: sections I.2 and I.3 of Freitag & Busam proved a bit too deep for our purposes. For this reason, I bring you a few easier exercises from other texts which mirror the topics discussed in week 2. (Incidentally, these probably should have been due 1-31, but I got a bit behind so you got a break... however, Problem Set 3 will be due 2-7 as a consequence)

Problem 13 Calculate $log(-2-3i)$ and find $Log(-2-3i)$.

Problem 14 Solve

a. $e^z = 2i$

b. $Log(z^2-1) = \frac{i\pi}{2}$

c. $e^{2z} + e^z + 1 = 0$

Problem 15 Find all complex solutions of $az^2 + bz + c = 0$ where $a, b, c \in \mathbb{R}$.

Problem 16 Let

$$x_1 = \frac{2Re(z)}{|z|^2+1}, \qquad x_2 = \frac{2Im(z)}{|z|^2+1}, \qquad x_3 = \frac{|z|^2-1}{|z|^2+1}.$$

Show that $(x_1, x_2, x_3) \in S_2 = \{v \in \mathbb{R}^3 \mid ||v|| = 1\}$ for each $z \in \mathbb{C}$. Furthermore, show the mapping $z \mapsto (x_1, x_2, x_3)$ sends the unit-circle in the complex plane to the equator of the sphere in three dimensions (it has equations $x_1^2 + x_2^2 = 1$ and $x_3 = 0$).

Note: these are the correct formulas to describe the stereographic projections from the sphere to the complex plane. My initial picture in class was set-up wrong. I'll provide a hand-out Tuesday from Nagle and Saff which shows you the meaning and geometry of the formulas I give in this problem. You don't need the handout to do this problem, it's *just* algebra.

Problem 17 Explain what it **wrong** with the following argument: since $z^2 = (-z)^2$ it follows that $Log(z^2) = Log(-z)^2$ whence $2Log(z) = 2Log(-z)$ thus $Log(z) = Log(-z)$. But, then $exp(Log(z)) = exp(Log(-z))$ which indicates $z = -z$. Which step here is bogus and why?

Problem 18 Show $Log(e^z) = z$ iff $-\pi < Im(z) \leq \pi$.

Problem 19 Show that $\lim_{z\to 0} \frac{\bar{z}^2}{z} = 0$.

Problem 20 Calculate $\lim_{z\to 0} \left(\frac{z}{\bar{z}}\right)^2$ or give an argument as to why it does not exist as a complex number.

Problem 21 Let U, V be open sets in $\mathbb{C}$. Prove that $U \cup V$ is open.

Problem 22 Let U, V be open sets in $\mathbb{C}$. Prove that $U \cap V$ is open.

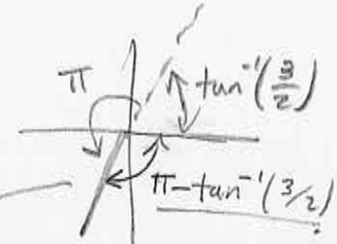
BONUS ! Your mission, should you choose to accept it, prove the equivalence of the following:

a. (epsilonic definition:) $f(z) \to f(z_o)$ as $z \to z_o$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $z \in \mathbb{C}$ with $0 < |z - z_o| < \delta$ implies $|f(z) - f(z_o)| < \epsilon$.

b. (topological definition:) $f(z) \to f(z_o)$ as $z \to z_o$ iff the inverse image of an open set containing $f(z_o)$ is an open set containing z_o.

c. (sequential definition:) $f(z) \to f(z_o)$ as $z \to z_o$ iff for each sequence $z_n \to z_o$ as $n \to \infty$ the corresponding sequence $f(z_n) \to f(z_o)$ as $n \to \infty$.

PROBLEM 13



$$\begin{aligned}\operatorname{Log}(-2-3i) &= \ln|-2-3i| + i\operatorname{Arg}(-2-3i)\\ &= \ln\sqrt{13} + i\left(\tan^{-1}\left(\tfrac{3}{2}\right) - \pi\right)\\ &\cong \boxed{\tfrac{1}{2}\ln(13) - 2.159\, i}\end{aligned}$$

Whereas,

$$\begin{aligned}\log(-2-3i) &= \ln|-2-3i| + i\arg(-2-3i)\\ &= \boxed{\left\{ \tfrac{1}{2}\ln(13) - i(2.159 + 2\pi k) \;\middle|\; k \in \mathbb{Z}\right\}}\end{aligned}$$

PROBLEM 14 Solve:

(a.) $e^z = 2i \Rightarrow \log(e^z) = \log(2i)$

$\Rightarrow z \in \log(2i)$

$\Rightarrow \boxed{z \in \left\{ \ln 2 + i\left(\frac{\pi}{2} + 2\pi k\right) \;\middle|\; k \in \mathbb{Z}\right\}}$

Remark: $\operatorname{Log}(e^z) = \operatorname{Log}(2i)$ finds only one of the sol^{n}'s we exhibit above.

(b.) $\operatorname{Log}(z^2 - 1) = \frac{i\pi}{2} \Rightarrow z^2 - 1 = e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$

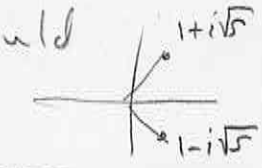
Thus $z^2 = 1 + i$. Now we need to calculate the square roots of $1 + i = \sqrt{2}\, e^{i\pi/4}$

$(1+i)^{1/2} = \left\{ \sqrt{\sqrt{2}}\, e^{i\pi/8},\ \sqrt{\sqrt{2}}\, e^{i(\pi/8 + \pi)} \right\}$

$\Rightarrow \boxed{z \in \left\{ 2^{1/4} e^{i\pi/8},\ 2^{1/4} e^{9i\pi/8} \right\}}$

(c.) $e^{2z} + e^z + 1 = 0 \Rightarrow \left(e^z - \frac{1}{2}\right)^2 = \frac{-5}{4}$

Thus $e^z - \frac{1}{2} \in (-5/4)^{1/2} = \left\{ \pm \frac{i\sqrt{5}}{2} \right\}$ we should



solve $e^z = \frac{1 \pm i\sqrt{5}}{2} \Rightarrow z \in \log\left(\frac{1 \pm i\sqrt{5}}{2}\right)$

Hence,

$$\boxed{z \in \left\{ \tfrac{1}{2}\ln\left(\tfrac{3}{2}\right) + i\left(\tan^{-1}\left(\tfrac{\sqrt{5}}{2}\right) + 2\pi k\right) \;\middle|\; k \in \mathbb{Z}\right\} \cup \left\{ \tfrac{1}{2}\ln\left(\tfrac{3}{2}\right) + i\left(\tan^{-1}\left(\tfrac{-\sqrt{5}}{2}\right) + 2\pi k\right) \;\middle|\; k \in \mathbb{Z}\right\}}$$

B.t.w., $\tan^{-1}\left(\frac{\sqrt{5}}{2}\right) \cong 0.841$

$\boxed{\text{PROBLEM 15}}$ Find all complex sol$^{\underline{n}}$'s: $(a, b, c \in \mathbb{R})$

$$az^2 + bz + c = 0$$

If $a = 0$ then $bz + c = 0$.
If $b = 0$ then $c = 0$ hence the sol$^{\underline{n}}$ set is either $\mathbb{C}$ or ϕ depending on whether $c = 0$ or $c \neq 0$.
On the other hand, if $b \neq 0$ then $bz + c = 0 \Rightarrow \underline{z = -c/b}$.
This completes the $a = 0$ case.

If $a \neq 0$ then consider,

$$z^2 + \frac{b}{a}z + \frac{c}{a} = \frac{0}{a} = 0$$

$$\Rightarrow \left(z + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{(2a)^2}$$

$$\Rightarrow \left(z + b/2a\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow z + \frac{b}{2a} \in \left(\frac{b^2 - 4ac}{4a^2}\right)^{1/2}$$

$$\Rightarrow \boxed{z \in \frac{-b + (b^2 - 4ac)^{1/2}}{2a}}$$

$$\Rightarrow \boxed{z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

where $\sqrt{b^2 - 4ac}$ denotes the Principle $\sqrt{}$ fnct.

Remark: the main point is that the set of square roots includes two roots. Since $e^{\frac{2\pi i}{2}} = e^{\pi i} = -1 = \omega_o$ for $n = 2$,

$$(b^2 - 4ac)^{1/2} = \left\{ \sqrt{b^2 - 4ac}\, \exp(i \operatorname{Arg}(b^2 - 4ac)),\ -\sqrt{b^2 - 4ac}\, \exp(i \operatorname{Arg}(b^2 - 4ac)) \right\}.$$

Since $a, b, c \in \mathbb{R}$ it follows $\operatorname{Arg}(b^2 - 4ac) = \begin{cases} 0 & b^2 - 4ac > 0 \\ \pi & b^2 - 4ac < 0 \end{cases}$

Hence $\exp(i \operatorname{Arg}(b^2 - 4ac)) = \begin{cases} 1 & \text{if } b^2 - 4ac > 0 \\ i & \text{if } b^2 - 4ac < 0 \end{cases}$

The standard quadratic formula follows.

PROBLEM 16

Let $x_1 = \frac{2Re(z)}{|z|^2+1}$, $x_2 = \frac{2\,Im(z)}{|z|^2+1}$, $x_3 = \frac{|z|^2-1}{|z|^2+1}$

Claim: $x_1^2 + x_2^2 + x_3^2 = 1$ for all $z \in \mathbb{C}$ with x_1, x_2, x_3 defined as above $(z = z)$.

Proof: $x_1^2 + x_2^2 + x_3^2 = \left(\frac{2\,Re(z)}{|z|^2+1}\right)^2 + \left(\frac{2\,Im(z)}{|z|^2+1}\right)^2 + \left(\frac{|z|^2-1}{|z|^2+1}\right)^2$

$$= \frac{1}{(|z|^2+1)^2}\left[4\left(\frac{z+\bar{z}}{2}\right)^2 + 4\left(\frac{z-\bar{z}}{2i}\right)^2 + (z\bar{z}-1)^2\right]$$

$$= \frac{1}{(|z|^2+1)^2}\left[\cancel{z^2} + 2z\bar{z} + \cancel{\bar{z}^2} - (\cancel{z^2} - 2z\bar{z} + \cancel{\bar{z}^2}) - (z\bar{z})^2 - \cancel{2z\bar{z}} + 1\right]$$

$$= \frac{1}{(|z|^2+1)^2}\left[|z|^4 + 2|z|^2 + 1\right]$$

$$= \frac{(|z|^2+1)^2}{(|z|^2+1)^2}$$

$$= 1 \quad \left(\text{since } |z| \geq 0 \Rightarrow (|z|^2+1)^2 > 0.\right)$$

Claim: If $|z| = 1$ then $x_1^2 + x_2^2 = 1$ and $x_3 = 0$.

Proof: $x_3 = \frac{|z|^2-1}{|z|^2+1} = \frac{1-1}{2} = 0.$

$$x_1^2 + x_2^2 = \left(\frac{2\,Re(z)}{|z|^2+1}\right)^2 + \left(\frac{2\,Im(z)}{|z|^2+1}\right)^2$$

$$= \frac{4}{4}\left(Re(z)^2 + Im(z)^2\right)$$

$$= |z|^2$$

$$= 1.$$

PROBLEM 17 Note: $Log(1^2) = 2Log(1) = 0$ But, $2Log(-1) = 2\pi i$.

The wrong step is to assume both $z, -z \in dom(Log(z))$ only one of $Log(z^2) = 2Log(z)$ and $Log(-z)^2 = 2Log(-z)$ can be true for arbitrary z.

PROBLEM 18 Show $Log(e^z) = z$ iff $-\pi < Im(z) \leq \pi$.

Proof: $\Rightarrow$ Assume $Log(e^z) = z$. Let $z = x + iy$ thus

$$
\begin{aligned}
Log(e^{x+iy}) &= Log(e^x e^{iy}) \\
&= \ln|e^x e^{iy}| + i\,Arg(e^x e^{iy}) && \text{: def}^{\underline{n}} \text{ of Log.} \\
&= \ln|e^x||e^{iy}| + i\,Arg(e^{iy}) && \text{: prop. of modulus, geometry of Arg.} \\
&= \ln(e^x) + i\,Arg(e^{iy}) \\
&= x + i\,Arg(e^{iy})
\end{aligned}
$$

But $Log(e^z) = z$

$$
\begin{aligned}
&\Rightarrow x + i\,Arg(e^{iy}) = x + iy \\
&\Rightarrow Arg(e^{iy}) = y \\
&\Rightarrow y \in range(Arg) = (-\pi, \pi]. \\
&\Rightarrow -\pi < Im(z) \leq \pi.
\end{aligned}
$$

$\Leftarrow$ Suppose $-\pi < Im(z) \leq \pi$. Consider, $z = x + iy$

$$
\begin{aligned}
Log(e^z) &= \ln|e^z| + i\,Arg(e^z) && \text{: def}^{\underline{n}} \text{ of Log.} \\
&= \ln(e^x) + i\,Arg(e^x e^{iy}) && : |e^z| = |e^x e^{iy}| = |e^x| = e^x. \\
&= x + i\,Arg(e^{iy}) && \text{: Note } e^x \text{ just rescales. Geometrically } e^{iy} \text{ has the direction.} \\
&= x + iy && \text{: since } -\pi < y \leq \pi \text{ was given this follows.} \\
&= z.
\end{aligned}
$$

Remark: argument could reasonably be phrased as an $\Leftrightarrow$ calculation.

$\boxed{\text{PROBLEM 19}}$ Many ways to argue this.

① Note $0 \leq \left|\frac{\bar{z}^2}{z}\right| = |z|$ hence as $\lim_{z\to 0} 0 = 0$ and $\lim_{z\to 0} |z| = 0$ we find by the Squeeze Th$^{\underline{m}}$ for $\mathbb{R}$-valued limits of a $\mathbb{C}$-variable that $\lim_{z\to 0}\left|\frac{\bar{z}^2}{z}\right| = 0$. However, $\lim_{z\to z_0}|g| = 0 \Rightarrow \lim_{z\to z_0} g = 0$ hence we obtain $\lim_{z\to 0}\frac{\bar{z}^2}{z} = 0$. //

② Let $\epsilon > 0$ and choose $\delta = \epsilon$. Suppose $z \in \mathbb{C}$ and $0 < |z - 0| < \delta$. Consider,

$$\left|\frac{\bar{z}^2}{z} - 0\right| = \frac{|\bar{z}^2|}{|z|} = \frac{|\bar{z}|^2}{|z|} = \frac{|z|^2}{|z|} = |z| < \delta = \epsilon.$$

Therefore, $\lim_{z\to 0}\left(\frac{\bar{z}^2}{z}\right) = 0$.

Remark: the ϵ-δ argument is really better here since I have not proved the Squeeze Th$^{\underline{m}}$. For the curious, I believe the Squeeze Th$^{\underline{m}}$ can be proved just as it was in calculus I since the squeezing happens in the range of the real-valued functions.... same argument replacing $|x| = \sqrt{x^2}$ with $|x+iy| = \sqrt{x^2+y^2}$ should do...

$\boxed{\text{PROBLEM 20}}$ (I use two-path disagreement non-existence technique)

Let $z_1(t) = t + it$ and let $z_2(t) = t - it$ notice that

$$\left(\frac{z_1}{\bar{z}_1}\right)^2 = \left(\frac{t+it}{t-it}\right)^2 \longrightarrow \left(\frac{1+i}{1-i}\right)^2 \text{ as } t \to 0$$

however,

$$\left(\frac{z_2}{\bar{z}_2}\right)^2 = \left(\frac{t-it}{t+it}\right)^2 \longrightarrow \left(\frac{1-i}{1+i}\right)^2 \text{ as } t \to 0$$

Hence two paths differ. at zero $\Rightarrow \lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2$ d.n.e.

PROBLEM 21 Let U, V be open in $\mathbb{C}$. Prove $U \cup V$ open.

Let $z \in U \cup V$ hence $z \in U$ or $z \in V$. If $z \in U$ then z is an interior point of U as U is open hence $\exists \epsilon > 0$ such that $D(z, \epsilon) \subseteq U$. However, $\underline{U \subseteq U \cup V}$* hence $D(z, \epsilon) \subseteq U \cup V$ which shows z is interior to $U \cup V$. Now, if $z \in V$ a similar argument to the one just offered shows z interior to $U \cup V$. Consequently, each $z \in U \cup V$ is interior thus $U \cup V$ is open.//

*__Lemma__: $U \subseteq U \cup V$ (just in case you doubt this, I prove it)

__Pf__: Let $z \in U$ then $z \in U$ or $z \in V$ hence $z \in U \cup V$.//

PROBLEM 22 Let U, V be open. Prove $U \cap V$ open.

__Proof__: If $U \cap V = \emptyset$ then note that $U \cap V$ is open since each point in $U \cap V$ is interior (vacuously true!). Suppose $U \cap V \neq \emptyset$ then $\exists z \in U \cap V$. But $z \in U$ and $z \in V$ by def$^{\underline{n}}$ of intersection. Moreover, $\exists \epsilon_1, \epsilon_2$ such that $z \in D(z, \epsilon_1) \subseteq U$ and $z \in D(z, \epsilon_2) \subseteq V$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$ and note that (by Lemma**

$$D(z, \epsilon) \subseteq D(z, \epsilon_1) \subset U \text{ and } D(z, \epsilon) \subseteq D(z, \epsilon_2) \subseteq V$$

therefore, $D(z, \epsilon) \subseteq U$ and $D(z, \epsilon) \subseteq V$ which shows $D(z, \epsilon) \subseteq U \cap V$ hence z is interior to $U \cap V$. But, $z \in U \cap V$ was arbitrary hence __all__ points in $U \cap V$ are interior and we conclude $U \cap V$ is open.///

PROBLEM 22

<u>Lemma</u> ☆☆: If $a<b$ then $D(z,a) \subset D(z,b)$

<u>Proof</u>: Let $w \in D(z,a)$ then $|w-z| < a$ hence as $a<b$
$|w-z| < b \Rightarrow w \in D(z,b)$ so $D(z,a) \subset D(z,b)$. //

<u>Bonus</u>

a.) $\Rightarrow$ b.) Assume epsilonic def^n, and assume $f(z) \to f(z_o)$.
Let $V \subseteq \mathbb{C}$ be open and $f(z_o) \in V$. We must show
$f^{-1}(V)$ is open and $z_o \in f^{-1}(V)$. To show $z_o \in f^{-1}(V)$
simply note $f(z_o) \in V$. Consider $w \in f^{-1}(V)$ it follows
$\exists\, z \in V$ such that $f(z) = w$. But, V is open hence z
is interior and $\exists\, \epsilon > 0$ such that $D(z,\epsilon) \subseteq V$.

⋮

you can still turn
this in later

⋮

until
I finish it

⋮

fortunately, I'm
tired at the moment

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