Topics: sections I.5, II.1 of Freitag & Busam has much wisdom to offer, however I am happy to assign some problems from Churchill which should round things out nicely. This assignment covers complex-differentiability where we can calculate either by a limiting process, laws of calculus like product, quotient chain or the uniquely complex construction of the CR-equations. We also study some basic complex integrals. Notice I have added considerable depth to the notes which I sketched in lecture, I hope you read them. On the other hand, I hope Churchill helps when I don't make sense and/or when Freitag is a bit much... of course, ask me if you are lost, I am here to help.

Problem 23 **Prove:** If $f'(z)$ and $g'(z)$ exist then $(f+g)'(z) = f'(z) + g'(z)$.

Problem 24 **Prove:** If f is complex-differentiable at z_o then f is continuous at z_o.

Problem 25 Suppose $f(z) = (z+3i)^2$ calculate $f'(z)$ by explicitly calculating and resolving the limit of the difference quotient.

Problem 26 Show that $\cos^2(z) + \sin^2(z) = 1$.

Problem 27 Show that $\sin(z+w) = \cos(z)\sin(w) + \sin(z)\cos(w)$. Then, differentiate with respect to z and to derive the adding-angle formula for $\cos(z+w)$.

Problem 28 Show $f(z) = \cosh(z)$ is entire via the Cauchy-Riemann equations (you should comment that u, v are clearly continuously differentiable once you compute their formulas).

Problem 29 problem 10c of section 22 of Churchill (page 63).

Problem 30 problem 13 of section 22 of Churchill (page 63). (note: I derive the polar CR-equations on pages 77-78 of my notes)

Problem 31 problem 17 of section 24 of Churchill (page 72).

Problem 32 problem 2b of section 25 of Churchill (page 74).

Problem 33 problem 14b of section 25 of Churchill (page 75).

Problem 34 problem 10a of section 29 of Churchill (page 85).

Problem 35 problem 14 of section 29 of Churchill (page 85).

Problem 36 Freitag I.5 problem 17

Problem 37 problem 1a of section 33 of Churchill (page 102).

Problem 38 problem 6 of section 33 of Churchill (page 102).

Problem 39 problem 10 of section 33 of Churchill (page 103).

Problem 40 problem 14 of section 33 of Churchill (page 103).

PROBLEM SET 3 SOLUTION

PROBLEM 23 Suppose $f'(z), g'(z) \in \mathbb{C}$. Consider,

$$\lim_{h\to 0}\left[\frac{(f+g)(z+h)-(f+g)(z)}{h}\right] = \lim_{h\to 0}\left[\frac{f(z+h)-f(z)+g(z+h)-g(z)}{h}\right]$$

$$\overset{*}{=} \lim_{h\to 0}\left[\frac{f(z+h)-f(z)}{h}\right] + \lim_{h\to 0}\left[\frac{g(z+h)-g(z)}{h}\right]$$

$$= f'(z)+g'(z).$$

$*$: we note this step is valid because the resulting limits exist.

PROBLEM 24 See the notes. I presented this on 40. Suppose $f'(z_0)$ exists. Consider

$$\lim_{h\to 0}\left(f(z_0+h)-f(z_0)\right) = \lim_{h\to 0}\left(\frac{f(z_0+h)-f(z_0)}{h}\cdot h\right)$$

$$= \lim_{h\to 0}\left(\frac{f(z_0+h)-f(z_0)}{h}\right)\lim_{h\to 0}(h)$$

$$= f'(z_0)\cdot 0$$

$$= 0. \Rightarrow \lim_{h\to 0} f(z_0+h) = f(z_0).$$

Thus complex-diff. at $z_0 \Rightarrow$ continuity at z_0.

PROBLEM 25 Suppose $f(z) = (z+3i)^2$.

$$f'(z) = \lim_{h\to 0}\left(\frac{(z+h+3i)^2-(z+3i)^2}{h}\right)$$

$$= \lim_{h\to 0}\left[\frac{\cancel{(z+3i)^2}+2(z+3i)h+h^2-\cancel{(z+3i)^2}}{h}\right]$$

$$= \lim_{h\to 0}\left[2(z+3i)+h\right]$$

$$= \underline{2(z+3i)}.$$

<u>Remark</u>: we can replicate many arguments from calculus I in this context, however, we also have CR-eq^{n}'s when not directed otherwise.

PROBLEM 26 We recall the det^{n} of $\sin z$, $\cos z$ over $\mathbb{C}$ in what follows:

$$\cos^2 z + \sin^2 z = \left[\tfrac{1}{2}(e^{iz}+e^{-iz})\right]^2 + \left[\tfrac{1}{2i}(e^{iz}-e^{-iz})\right]^2$$

$$= \tfrac{1}{4}\left[\cancel{(e^{iz})^2} + 2e^{iz}e^{-iz} + \cancel{(e^{-iz})^2}\right] - \tfrac{1}{4}\left[\cancel{(e^{iz})^2} - 2e^{iz}e^{-iz} + \cancel{(e^{-iz})^2}\right]$$

$$= \tfrac{1}{4}\, 4e^{iz}e^{-iz}$$

$$= 1.$$

PROBLEM 27

$$\cos z \sin w + \sin z \cos w = \tfrac{1}{2}(e^{iz}+e^{-iz})\tfrac{1}{2i}(e^{iw}-e^{-iw}) + \tfrac{1}{2i}(e^{iz}-e^{-iz})\tfrac{1}{2}(e^{iw}+e^{-iw})$$

$$= \tfrac{1}{4i}\left(e^{i(z+w)} - \cancel{e^{i(z-w)}} + \cancel{e^{-i(z-w)}} - e^{-i(z+w)}\right) + \tfrac{1}{4i}\left(e^{i(z+w)} + \cancel{e^{i(z-w)}} - \cancel{e^{-i(z-w)}} - e^{-i(z+w)}\right)$$

$$= \tfrac{1}{2i}\left(e^{i(z+w)} - e^{-i(z+w)}\right)$$

$$= \sin(z+w).$$

Therefore,

$$\frac{d}{dz}\left(\sin(z+w)\right) = \frac{d}{dz}\left(\cos z \sin w + \sin z \cos w\right)$$

$$\Rightarrow \boxed{\cos(z+w) = -\sin z \sin w + \cos z \cos w}$$

PROBLEM 28 $\sin(iz) = \frac{1}{2i}\left(e^{i(iz)} - e^{-i(iz)}\right) = i\left(\frac{1}{2}(e^{z}-e^{-z})\right) = i\sinh z \quad (\star)$

$$f(z) = \cosh(z) = \tfrac{1}{2}(e^{z}+e^{-z}) = \tfrac{1}{2}\left(e^{-i(iz)}+e^{i(iz)}\right) = \cos(iz)$$

Use the previous problem $iz = i(x+iy) = ix - y$.

$$f(z) = \cos(ix-y) = \cos(ix)\cos(-y) - \sin(ix)\sin(-y)$$

$$= \underbrace{\cosh(x)\cos(y)}_{u} + i\,\underbrace{\sinh(x)\sin(y)}_{v} \quad \text{by } \star$$

Observe $u_x = \sinh x \cos y = v_y$ and $u_y = -\cosh x \sin y = -v_x$ and u_x, u_y, v_x, v_y are continuous hence $f = u + iv$ is entire.

$\boxed{\text{PROBLEM 29}}$ (10c of §22 Churchill p. 63)

Find harmonic conj. of $u(x,y) = \sinh x \sin y$ after showing u harmonic

$$u_{xx} + u_{yy} = \sinh x \sin y - \sinh x \sin y \cancel{\neq} 0$$

We seek $v(x,y)$ such that $u_x = v_y$ and $u_y = -v_x$ or,

$$\frac{\partial v}{\partial y} = \cosh x \sin y \Rightarrow v = -\cosh x \cos y + c_1(x)$$

$$\frac{\partial v}{\partial x} = -\sinh x \cos y \Rightarrow v = -\cosh x \cos y + c_2(y)$$

Thus $\boxed{v(x,y) = -\cosh x \cos y}$ will do nicely.

$\boxed{\text{PROBLEM 30}}$ (#13 of §22) Show $u(r,\theta) = \ln r$ is harmonic in domain $r > 0$, $0 < \theta < 2\pi$ then derive $v(r,\theta)$ conj. to u

Need to check $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ ($u_{xx} + u_{yy} = 0$ changed to polars)

$$u_r = \frac{1}{r}, \quad u_{rr} = \frac{-1}{r^2}, \quad u_{\theta\theta} = 0$$

Thus $r^2 u_{rr} + r u_r + u_{\theta\theta} = \frac{-r^2}{r^2} + r\frac{1}{r} \cancel{\neq} 0.$

Polar CR-eq$^{\underline{n}}$'s are

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad \frac{1}{r} u_\theta = -v_r$$

We need to solve:

$$\left.\begin{aligned} \frac{\partial v}{\partial r} &= 0 &&\Rightarrow v(r,\theta) = c_1(\theta) \\ \frac{\partial v}{\partial \theta} &= r\left(\frac{1}{r}\right) = 1 &&\Rightarrow v(r,\theta) = \theta + c_2(r) \end{aligned}\right\} \boxed{v = \theta}$$

Remark: this exercise reveals that $f(z) = u + iv$ with $u = \ln|z| \Rightarrow v = \arg(z)$. In other words, $f(z) = \log(z) = \ln|z| + i\arg(z)$ (we have to choose a branch to be careful, until then understand the result holds in some slit-plane like $\mathbb{C}_-$)

$\boxed{\text{PROBLEM 31}}$ #17 of §24. Solve $\cos z = 2$

$$\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \underbrace{\cos x \cosh y}_{2} - i\underbrace{\sin x \sinh y}_{0} = 2$$

Note: either $y = 0$ or $x = n\pi$ for $n \in \mathbb{Z}$ for $\text{Im}(\cos z) = 0$.
Also $\cos x \cosh y = 2$. If $y = 0$ then $\cos x = 2 \Rightarrow \nexists x \in \mathbb{R}$.
Thus $y \neq 0$ hence $x = n\pi$ for $n \in \mathbb{Z}$,

$$\cos(n\pi)\cosh y = 2$$
$$\cosh y = 2(-1)^n$$
$$y = \cosh^{-1}(2(-1)^n), \quad x = n\pi, \ n \in \mathbb{Z}.$$
$$\Rightarrow \boxed{x = n\pi,\ n \in \mathbb{Z},\ y = \cosh^{-1}(2)}$$
aka $\boxed{n\pi + i\cosh^{-1}(2),\ n \in \mathbb{Z}}$ (better)

$\boxed{\text{PROBLEM 32}}$ #2b of §25

Show $\sin(2z) = 2\sin z \cos z \Rightarrow \sinh(2z) = 2\sinh z \cosh z$

Note: $\cos(iz) = \cosh z$ and $\sin(iz) = i\sinh z$ hence,

$$\sin(2z) = 2\sin z \cos z \Rightarrow \sin(2iz) = 2\sin(iz)\cos(iz)$$
$$\Rightarrow i\sinh(2z) = 2i\sinh z \cosh(z)$$
$$\Rightarrow \boxed{\sinh(2z) = 2\sinh z \cosh z}$$

$\boxed{\text{PROBLEM 33}}$ #14b of §25 Solve $\sinh(z) = i$

$$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$$
$$\sinh(x+iy) = \sinh x \cosh(iy) + \cosh x \sinh(iy)$$
$$= \sinh(x)\cos(y) + i\cosh(x)\sin(y)$$

$\cosh(iz) = \cos z$
$\sinh(iz) = i\sin z$

However, $\sinh(x+iy) = i$ by assumption hence

(I.) $\sinh(x)\cos(y) = 0 \Rightarrow \sinh(x) = 0$ or $\cos(y) = 0$ (☆)

(II.) $\cosh(x)\sin(y) = 1$

$x = 0$ only sol^{n} ; $y = \frac{\pi}{2}(2n-1)$, $n \in \mathbb{Z}$.

If $x = 0$ then (II) $\sin(y) = 1 \Rightarrow y = \frac{\pi}{2}, \frac{5\pi}{2}, \dots = \frac{\pi}{2}(4k+1), k \in \mathbb{Z}$.
thus only some of ☆ gives $\sin(y) = 1$. (Half of ☆ makes $\sin y = -1$)
therefore, $\boxed{z = i(2k + \frac{1}{2})\pi}$ for $k \in \mathbb{Z}$

If $x \neq 0$ then $\cos(y) = 0$ from (I) hence $y = \frac{\pi}{2}(2n-1)$ for $n \in \mathbb{Z}$.
Note $\cosh(x) \neq 1 \Rightarrow \sin(y) \neq 1$ and $\cosh(x) \geq 1$ so $\sinh(y) < 0$ not a sol^{n} hence this case impossible since $\sin(y) = \pm 1$ for $y = \frac{\pi}{2}(2n-1)$.

PROBLEM 34 | #10a of § 29 | Find values of $\tan^{-1}(2i)$

$$\tan^{-1}(2i) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right) \quad \text{for} \quad z = 2i$$

$$= \frac{i}{2}\log\left(\frac{3i}{-i}\right)$$

$$= \frac{i}{2}\log(-3)$$

$$= \frac{i}{2}\left[\ln(3) + i\arg(-3)\right]$$

$$= \left\{\frac{i}{2}\left[\ln(3) + i(\pi + 2\pi k)\right] \;\middle|\; k \in \mathbb{Z}\right\}$$

$$= \boxed{\left\{ -\frac{\pi}{2}(1+2k) + \frac{i}{2}\ln(3) \;\middle|\; k \in \mathbb{Z}. \right\}} \quad \text{(same as Churchill's)}$$

PROBLEM 35 | #14 of § 29 | derive formula for $\tan^{-1}(z)$ used in 10a

Show: $\tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$

Proof: Let $z = \tan(w)$ then $\tan^{-1}(z) = w$.

$$\tan(w) = \frac{\sin w}{\cos w} = \frac{\frac{1}{2i}(e^{iw} - e^{-iw})}{\frac{1}{2}(e^{iw} + e^{-iw})} = i\left(\frac{e^{-iw} - e^{iw}}{e^{iw} + e^{-iw}}\right) = z$$

Hence, $i(e^{-iw} - e^{iw}) = z(e^{iw} + e^{-iw})$

$$\Rightarrow i\left(1 - (e^{iw})^2\right) = z\left((e^{iw})^2 + 1\right)$$

$$\Rightarrow (i+z)(e^{iw})^2 = i - z$$

$$\Rightarrow \log (i+z)(e^{iw})^2 = \log(i-z)$$

$$\log(i+z) + 2\log(e^{iw}) = \log(i-z)$$

$$2iw = \log(i-z) - \log(i+z)$$

$$w = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right) = \underline{\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)}.$$

Remark: giving up on single-valued seems like a good idea for these type of algebraic questions! I imagine sorting through all this with Log...

PROBLEM 36 (Freitag I.5 problem 17)

Let $\underbrace{\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}}_{\text{upper half-plane}}$ and $\underbrace{\mathbb{E} = \{q \in \mathbb{C} \mid |q| < 1\}}_{\text{unit-disk}}$

<u>Show</u>: $f(z) = \dfrac{z-i}{z+i}$ provides globally conformal map of $\mathbb{H}$ <u>onto</u> $\mathbb{E}$. What is its inverse map? (f is the Cayley Map) 1846

Clearly $f'(z) = \dfrac{z+i-(z-i)}{(z+i)^2} = \dfrac{2i}{(z+i)^2}$ for $z \neq -i$

hence f is analytic thus conformal on $\mathbb{H}$ $(-i \notin \mathbb{H})$.

Let $q \in \mathbb{E}$ we seek $z \in \mathbb{H}$ such that $f(z) = q$.

Scratch work:

$$q = \frac{z-i}{z+i} \quad \text{vs.} \quad \underbrace{\tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)}$$

$$\exp\left(\frac{2\tan^{-1}(z)}{i}\right) = \frac{i+z}{i-z} = \left(\frac{i-z}{i+z}\right)^{-1}$$

$$\Rightarrow -\exp\left(2i\tan^{-1}(z)\right) = \frac{z-i}{z+i} = q$$

$$\Rightarrow 2i\tan^{-1}(z) = \log(-q)$$

$$\Rightarrow z = \tan\left(\frac{1}{2i}\log(-q)\right)$$

$$\therefore f\left(\tan\left(\tfrac{1}{2i}\log(-q)\right)\right) = q.$$

Clearly we should conjecture, for $q \neq 0$,

$$f\left(\tan\left(\tfrac{1}{2i}\operatorname{Log}(-q)\right)\right) = q.$$

Suppose $q \in \mathbb{E}$ then $|q| < 1$ hence $\ln|q| < 0$.

$$\operatorname{Log}(-q) = \ln|q| + i\operatorname{Arg}(-q)$$

$$\frac{1}{2i}\operatorname{Log}(-q) = \underbrace{-\frac{i}{2}\ln|q|}_{\alpha} + \underbrace{\frac{1}{2}\operatorname{Arg}(-q)}_{\beta}$$

PROBLEM 36 continued

$$\sin(i\theta) = \frac{1}{2i}\left(e^{-\theta} - e^{+\theta}\right) = i\left(\tfrac{1}{2}\left(e^{\theta}-e^{-\theta}\right)\right) = i\sinh\theta$$
$$\cos(i\theta) = \tfrac{1}{2}\left(e^{-\theta}+e^{\theta}\right) = \cosh\theta. \qquad \bigstar$$

$$\tan\left(\frac{1}{2i}\operatorname{Log}(-q)\right) = \tan(\alpha+\beta)$$
$$= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$\alpha = i\theta$, $\theta = \dfrac{-\ln|q|}{2}$

$\theta > 0$ since $\ln|q| < 0$

$$= \frac{\sin(i\theta)\cos\beta + \cos(i\theta)\sin\beta}{\cos(i\theta)\cos\beta - \sin(i\theta)\sin\beta}$$
$$= \frac{i\sinh\theta\cos\beta + \cosh\theta\sin\beta}{\cosh\theta\cos\beta - i\sinh\theta\sin\beta} \qquad (\text{by } \bigstar)$$
$$= \frac{(\cosh\theta\sin\beta + i\sinh\theta\cos\beta)(\cosh\theta\cos\beta + i\sinh\theta\sin\beta)}{\cosh^2\theta\cos^2\beta + \sinh^2\theta\sin^2\beta}$$
$$= \left[\frac{\cosh^2\theta\sin\beta\cos\beta - \sinh^2\theta\cos\beta\sin\beta}{\cosh^2\theta\cos^2\beta + \sinh^2\theta\sin^2\beta}\right] + i\left[\frac{\sinh\theta\cosh\theta\cos^2\beta + \sin^2\beta\sinh\theta\cosh\theta}{\cosh^2\theta\cos^2\beta + \sinh^2\theta\sin^2\beta}\right]$$
$$= \frac{\sin\beta\cos\beta + i[\sinh\theta\cosh\theta]}{\cosh^2\theta\cos^2\beta + \sinh^2\theta\sin^2\beta}$$

$$\sinh\theta\cosh\theta = \tfrac{1}{2}\left(e^{\theta}-e^{-\theta}\right)\tfrac{1}{2}\left(e^{\theta}+e^{-\theta}\right)$$
$$= \tfrac{1}{4}\left(e^{2\theta}+1-1-e^{-2\theta}\right)$$
$$= \tfrac{1}{2}\cosh(2\theta)$$

Hence,

$$z = \tan\left(\frac{1}{2i}\operatorname{Log}(-q)\right) = \frac{\frac{1}{2}\sin(2\beta) + \frac{i}{2}\cosh(2\theta)}{\cosh^2\theta\cos^2\beta + \sinh^2\theta\sin^2\beta}$$

We observe $\operatorname{Im}\left(\tan\left(\frac{1}{2i}\operatorname{Log}(-q)\right)\right) > 0$ as $\cosh(2\theta) > 0$ and the denom. is clearly positive for $q \neq 0$. We find $z \in \mathbb{H}$ hence f onto $\mathbb{E}^{\bullet}$. Next $q=0$

PROBLEM 36 continued

We've shown $f : \mathbb{H} \longrightarrow \mathbb{E}$ is onto $\mathbb{E}^{\circ} = \mathbb{E} - \{0\}$. Consider $q = 0$. Observe $f(i) = \frac{i-i}{i+i} = 0$ hence f is onto $\mathbb{E}$. We have not shown f is <u>into</u> $\mathbb{E}$. We consider that question now.

$$\left| \frac{z-i}{z+i} \right|^2 = \left(\frac{z-i}{z+i}\right)\left(\overline{\frac{z-i}{z+i}}\right)$$

$$= \frac{(z-i)(\bar{z}+i)}{(z+i)(\bar{z}-i)}$$

$$= \frac{z\bar{z} + iz - i\bar{z} + 1}{z\bar{z} + i\bar{z} - iz + 1}$$

$\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$

$2\,\mathrm{Im}(z) = i(\bar{z} - z)$

$$= \frac{|z|^2 - 2\,\mathrm{Im}(z) + 1}{|z|^2 + 2\,\mathrm{Im}(z) + 1}$$

since $z \in \mathbb{H}$ we have $\mathrm{Im}(z) > 0$

thus,

$$< \frac{|z|^2+1}{|z|^2+1} = 1.$$

Thus $|f(z)| < 1 \quad \forall z \in \mathbb{H}$ hence $f : \mathbb{H} \rightarrow \mathbb{E}$ is well-defined, (and as we proved previously) and onto.

The <u>inverse map</u> is given by

$$\boxed{f^{-1}(q) = \tan\left(\frac{1}{2i}\,\mathrm{Log}(-q)\right)}$$

(I'll let Spencer point out gaps in this argument)

<u>Remark</u>: this problem illustrates why Churchill wins. The most innocuous looking Freitag problems do this to us.

PROBLEM 37 (#1a of §33)

$$\int_C \left(\frac{z+2}{z}\right) dz = \int_0^{\pi} \left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right) 2ie^{i\theta}\, d\theta \qquad : \text{def}^{n} \text{ of } \int_C f(z)dz.$$

$(C: z = 2e^{i\theta},\ 0 \le \theta \le \pi)$

$$= \int_0^{\pi} (4ie^{i\theta} + 2i)\, d\theta$$

$$= \left(4e^{i\theta} + 2i\theta\right)\Big|_0^{\pi}$$

$$= 4e^{i\pi} - 4e^{0} + 2i\pi \qquad e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$= \boxed{-8 + 2i\pi}$$

PROBLEM 38 #6 of §33

Let $f(z) = \exp((-1+i)\log(z))$ where $|z| > 0$, $\underbrace{0 < \arg < 2\pi}$ we pick this branch of log.

Integrate around ccw unit circle $|z| = 1$.

$$\int_C f(z)dz = \int_0^{2\pi} f(\gamma(\theta)) \frac{d\gamma}{d\theta}\, d\theta \quad : \gamma(\theta) = e^{i\theta} \leftarrow \text{gives } C.$$

$$= \int_0^{2\pi} \exp((-1+i)\log(e^{i\theta}))\, ie^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} \exp\left((-1+i)\left[\cancel{\ln|e^{i\theta}|}^{0} + i\underbrace{\arg(e^{i\theta})}_{\theta}\right]\right) ie^{i\theta}\, d\theta$$

because log was so-chosen for #6.

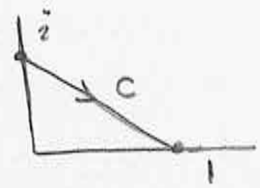
$$= \int_0^{2\pi} \exp((-1+i)\, i\theta)\, ie^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} ie^{-i\theta-\theta} e^{i\theta}\, d\theta$$

$$= i\int_0^{2\pi} e^{-\theta}\, d\theta$$

$$= \boxed{i(1 - e^{-2\pi})}$$

PROBLEM 39 #10 of §33 : Show $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$ for $C : i \to 1$



$m = \frac{1+i}{2}$ is closest to $z = 0$ of any pt. on C.

$$|m| = \sqrt{\tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{\tfrac{1}{2}} = \frac{1}{\sqrt{2}}$$

thus $z \in C \Rightarrow |z| < \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{|z|} < \sqrt{2}$.

Consider then, $z \in C \Rightarrow \left|\frac{1}{z^4}\right| = \frac{1}{|z|^4} < (\sqrt{2})^4 = 4$.

Applying the Th$^{\underline{m}}$ $\left|\int_C f(z)\,dz\right| \leq M\,l(C)$ where M bounds $|f(z)|$ on C,

$$\left|\int_C \frac{dz}{z^4}\right| \leq 4\,l(C) = 4\underbrace{|i-1|}_{\text{length of line-segment.}} = 4\sqrt{2}.$$

PROBLEM 40 #14 of §33 : $C_R : |z| = R$, $R > 2$, ccw upper-half-circle

Show $\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \leq \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$ and show the integral $\to 0$ as $R \to \infty$.

Just need to bound the integrand and apply the $M\,l(C)$ – Th$^{\underline{m}}$,

$$\left|\frac{2z^2-1}{z^4+5z^2+4}\right| = \frac{|2z^2-1|}{|z^2+1|\,|z^2+4|}$$

$$\leq \frac{2|z|^2+1}{\big||z^2|-|1|\big|\,\big||z^2|-|4|\big|}$$: apply Δ-inequality $|a+b| \leq |a|+|b|$ and $|a+b| \geq \big||a|-|b|\big|$.

$$= \frac{2R^2+1}{(R^2-1)(R^2-4)}$$: note $R > 2$ hence the quantities $R^2-1,\ R^2-4 > 0$.

Thus, as $l(C_R) = \pi R$ we find,

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \leq \frac{(2R^2+1)\pi R}{(R^2-1)(R^2-4)}.$$

Observe, (I don't think we need to show ↓ at this pt, it's obvious)

$$\lim_{R\to\infty}\left[\frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}\right] = \lim_{R\to\infty}\left[\frac{\pi(2R^3+R)}{R^4-5R^2+4}\right]$$

$$= \lim_{R\to\infty}\left[\frac{\pi(2+1/R^2)}{R-5+4/R^2}\right] = \boxed{0}$$