MATH 331: DUE THURSDAY BEFORE BREAK, BY 4:30PM. PROBLEM SET 5

Topics: mapping, a bit more on inverses, rotating coordinates. (this is not on the new material, that will be seen in Problem Set 6)

Problem 51 Derive the formula for $\tan^{-1}(z)$ in terms of the log. Calculate $\tan^{-1}(1+i)$. This will be a set of values since $\tan^{-1}(z)$ is multiply-valued unless we choose a branch.

Problem 52 On page 90d of my notes I relate $\int_C f(z)dz$ to the line and flux integrals of $< Im(f), Re(f) >$. Let C be the positively oriented unit-circle in each part of this problem. The purpose of this problem is to explore the geometric source of the formula $\int_C z^n\, dz = 2\pi i \delta_{n,-1}$. Why is it that only the $f(z) = 1/z$ gives a non-trivial integral?.

- **a.** find the flux and line integrals in $n = -1$ case.
- **b.** find the flux and line integrals in $n = -2$ case.
- **c.** find the flux and line integrals in $n = 0$ case.
- **d.** can you see why $n = -1$ is so special in view of the integrals above?

Problem 53 If $f(z)$ is complex-differentiable at z_o then $L(z) = f(z_o) + f'(z_o)(z - z_o)$ gives the best affine approximation to $f(z)$ near z_o. **Show that this linearization is conformal.** This requires you to consider two curves through z_o and show that the angle between the curves γ, α is the same as the angle between $f \circ \gamma$ and $f \circ \alpha$. You should assume $\gamma(0) = \alpha(0) = z_o$ hence $f \circ \gamma(0) = f \circ \alpha(0) = f(z_o)$. I guess the question you need to think about: how do we measure the angle between curves at a point?

Problem 54 Find a transformation which maps the unit disk onto the right half-plane such that it maps $z = -i$ to $w = 0$. Here we think of the map $z \rightarrow w = f(z)$. *Hint: read about Mobius transformations*

Problem 55 Find a linear transformation which takes the circle $|z| = 1$ and maps it onto the circle $|w - 5| = 3$ such that the point $z = i$ is mapped to $w = 2$.

Problem 56 If $\bar{z} = e^{i\theta} z$ then the barred-coordinates $\bar{x}, \bar{y}$ where $\bar{z} = \bar{x} + i\bar{y}$ are related to the coordinates x, y where $z = x + iy$ by a rotation. Discover the equation of an rotated equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$ by replacing x, y with $\bar{x}, \bar{y}$.

Problem 57 Is it easier to understand the rotated equations if we adopt a purely complex approach. Let the solution set of $|z - z_1| + |z - z_2| = k$ describe an ellipse with focii z_1, z_2. What angle and translation makes this ellipse look like $\bar{x}^2/a^2 + \bar{y}^2/b^2 = 1$? Here I believe we should study $\bar{z} = t + e^{i\theta} z$. The complex scalar t should allow us to recenter the ellipse at the origin of the barred-coordinates. I hope the spirit of the problem is clear, you don't have to follow my instructions precisely here. A solution of equivalent intent will be accepted if it has merit.

bonus provide proof that limit of the product of two convergent sequences does indeed converge to the product of their limits.

bonus Assuming the limit if the denominator sequence is nonzero, provide proof that limit of the quotient of two convergent sequences does indeed converge to the quotient of their limits.

bonus Prove that if $\lim_{n\to\infty} z_n = z_o$ then $\lim_{n\to\infty} |z_n| = |z_o|$. (Spencer forcasts this is really easy, give it a try.)

Problem Set 5

Problem 51 Let $w = \tan^{-1}(z)$

$$\tan(w) = z = \frac{\frac{1}{2i}(e^{iw} - e^{-iw})}{\frac{1}{2}(e^{iw} + e^{-iw})}$$

$$iz(e^{iw} + e^{-iw}) = e^{iw} - e^{-iw}$$

$$e^{iw}(iz - 1) = e^{-iw}(-1 - iz)$$

$$e^{2iw} = \frac{1+iz}{1-iz} = \frac{i-z}{i+z}$$

$$\log(e^{2iw}) = \log\left(\frac{i-z}{i+z}\right)$$

$$\Rightarrow \quad w = \boxed{\frac{1}{2i}\log\left(\frac{i-z}{i+z}\right) = \tan^{-1}(z)}$$

Problem 52 for each $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$ for $0 \le \theta \le 2\pi$.

a.) $$\oint_c z^{-1}\,dz = \int_0^{2\pi} \cancel{e^{-i\theta}}\, i\cancel{e^{i\theta}}\,d\theta = i\theta\Big|_0^{2\pi} = \boxed{2\pi i}$$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2+y^2} \longleftrightarrow \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle = \vec{F}$$

$$\oint_c \left(\frac{x-iy}{x^2+y^2}\right)(dx + i\,dy) = \underbrace{\oint_c \frac{x\,dx + y\,dy}{x^2+y^2}}_{\text{flux of } \vec{F} \text{ through } c} + i\underbrace{\oint_c \frac{-y\,dx + x\,dy}{x^2+y^2}}_{\text{circulation (or work) of } \vec{F} \text{ along } c.}$$

(You could have just cited the results I derived on 90d here)

In any event, we see $n = -1$ corresponds to a vector field $\vec{F}$ which does work 2π around c and has $\Phi_{\vec{F}} = 0$.

symbol for flux.

PROBLEM 52 continued

Remark: it's convenient to pause to find the form of $\vec{F}$ for z^n. Note, for $z \neq 0$

$$z^n = \frac{z^{n+1}}{z} = \frac{z^{n+1}\bar{z}}{z\bar{z}} = \frac{r^{n+2} e^{i(n+1)\theta - i\theta}}{r^2}$$

and so we find $z^n = r^n \exp(ni\theta)$

$$z^n = \underbrace{r^n \cos(n\theta)}_{Re(z^n)} + i \underbrace{r^n \sin(n\theta)}_{Im(z^n)}$$

Given 90d discussion,

$$z^n \rightsquigarrow \vec{F}_n = \langle r^n \sin(n\theta), r^n \cos(n\theta) \rangle$$

Continuing, of course $r^n = 1$ for $|z| = 1$ on C,

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b.) $n = -2$, $\int_C z^{-2} dz = \int_0^{2\pi} e^{-2i\theta} ie^{i\theta} d\theta = i\int_0^{2\pi} e^{-i\theta} d\theta$

$$= \frac{i}{-i} e^{-i\theta} \Big|_0^{2\pi}$$

$$= -(e^{-2\pi i} - 1)$$

$$= 0.$$

Thus $\vec{F}_{n=-2} = \frac{1}{x^2+y^2}\langle -\sin(2\theta), \cos(2\theta)\rangle$

does zero work and has net flux zero through C.

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c.) $\int_C z^0 dz = \int_0^{2\pi} ie^{i\theta} d\theta = e^{i\theta}\Big|_0^{2\pi} = 0$

Hence the constant vector field $\vec{F}_0 = \langle 1, 1\rangle$ does no work $\oint_C \vec{F}_0 \cdot d\vec{r} = 0$ and $\Phi_{\vec{F}_0} = 0$.

(this is easy to believe)

$\vec{F}_0 = -\nabla U$ where $U(x,y) = -x-y \Leftarrow$ PE fnc.

PROBLEM S2

(d.) well, my approach has not illuminated much. Perhaps the error of the exposition is in supposing we're that familiar with $\oint \vec{F} \cdot d\vec{r}$ or $\Phi_{\vec{F}}$. Perhaps polar coordinates are the way to go:

$$z^n = r^n e^{ni\theta}$$

$$\int_{C_R} z^n dz = \int_0^{2\pi} R^n e^{ni\theta} iRe^{i\theta}\, d\theta = iR^{n+1}\int_0^{2\pi} e^{(n+1)i\theta} d\theta$$

$$= \frac{iR^{n+1}}{(n+1)i} e^{(n+1)i\theta}\Big|_0^{2\pi}$$

$$= \frac{R^{n+1}}{n+1}\left(e^{2\pi(n+1)i} - 1\right)$$

$$= 0 \quad \text{for } n \neq -1.$$

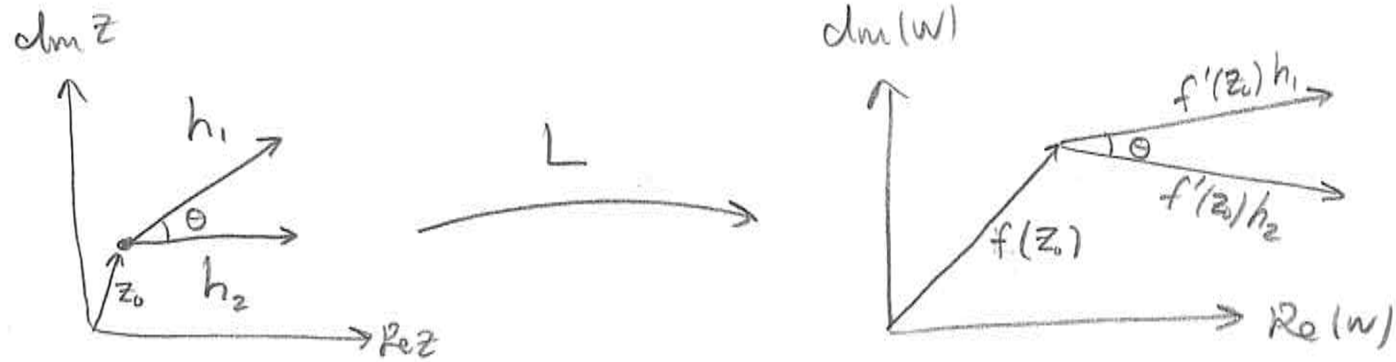
Remark: this is not yet the answer I'm hoping for... I want to better understand geometrically why $n=-1$ is so special. Maybe the answer I want is here and I just don't see it yet...

$\boxed{\text{PROBLEM 53}}$ I should think someone ought to have asked if $\exists$ a typo here. I said show <u>linearization</u> is conformal... sorry my hint misleading. On the other hand, I've alread proved f is conformal in the notes... so look there for that. My intent here was for you to discover the following:

$$\underbrace{L(z_0+h) = f(z_0) + f'(z_0)h}$$

- takes h and multiplies by $f'(z_0)$
  But, this is rescaling (or dilation) by $|f'(z_0)|$ and rotation by $\exp(i\,\mathrm{Arg}(f'(z_0)))$. Followed by translation by $f(z_0)$

- If we took h_1 and h_2 then they'd be dilated the same and rotated the same angle $\mathrm{Arg}(f'(z_0))$. Consequently,
  $\angle(h_1, h_2) = \angle(f'(z_0)h_1, f'(z_0)h_2)$



<u>Alternatively:</u>

$$(L\circ\alpha)'(0) = \frac{d}{dt}\Big[f(z_0) + f'(z_0)(\alpha(t) - z_0)\Big]\Big|_{t=0}$$
$$= f'(z_0)\frac{d\alpha}{dt}\Big|_{t=0}$$
$$= f'(z_0)\,\alpha'(0)$$

Likewise $(L\circ\gamma)'(0) = f'(z_0)\,\gamma'(0)$ and we can use the argument in my notes to show $\angle$ same.

$\boxed{\text{PROBLEM 54}}$ Find a transformation which maps the unit disk $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ onto the right half-plane $W = Re(w)$ where $z = -i \longmapsto w = 0 = f(-i)$. Find $f(z)$.

We wish to find $a, b, c, d \in \mathbb{C}$ such that

$$f(z) = \frac{az+b}{cz+d} = \frac{(az+b)\overline{(cz+d)}}{|cz+d|^2}$$

$$= \frac{a\bar{c}\, z\bar{z} + a\bar{d} z + b\bar{c}\bar{z} + b\bar{d}}{|cz+d|^2}$$

Let $a = a_1 + ia_2$, $b = b_1 + ib_2$ etc... $z = x+iy$,

$$f(z) = \frac{(a_1+ia_2)(c_1-ic_2)(x^2+y^2) + (a_1+ia_2)(d_1-id_2)(x+iy)}{|cz+d|^2}$$

$$\frac{(b_1+ib_2)(c_1-ic_2)(x-iy) + (b_1+ib_2)(d_1-id_2)}{|cz+d|^2}$$

$$= \Big[(a_1c_1 + a_2c_2)(x^2+y^2) + (a_1d_1+a_2d_2)x - (a_2d_1 - a_1d_2)y +$$

$$+ (b_1c_1+b_2c_2)x - (b_2c_1 - b_1c_2)y + b_1d_1 + b_2d_2\Big]\frac{1}{|cz+d|^2} +$$

$$+ i\Big[(a_2c_1 - a_1c_2)(x^2+y^2) + (a_2d_1 - a_1d_2)x + (a_1d_1 + a_2d_2)y +$$

$$+ (b_2c_1 - b_1c_2)x - (b_1c_1 + b_2c_2)y + b_2d_1 - b_1d_2\Big]\frac{1}{|cz+d|^2}$$

Thus $f(z) = Re(f(z))$ yields $i[\cdots\cdots] = 0$
the relation above is to hold for most $z \in D$ so we can diff. w.r.t x, y and evaluate at $z=0$ to extract eq^{n}'s.

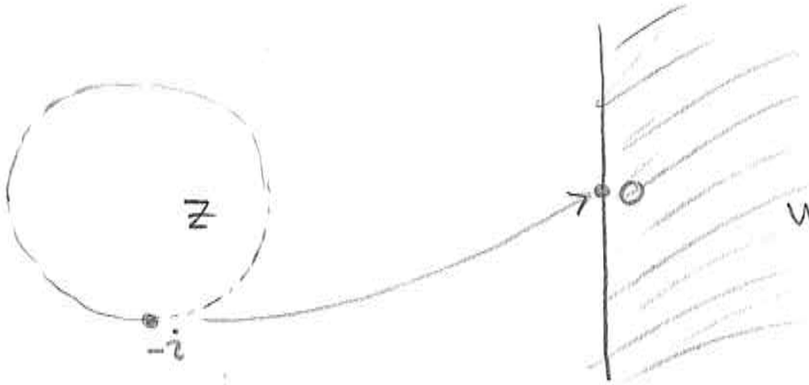
- x: $a_2d_1 - a_1d_2 + b_2c_1 - b_1c_2 = 0$ $\quad$ ($\frac{\partial}{\partial x}$ followed $\big|_{z=0}$)
- y: $a_1d_1 + a_2d_2 - b_1c_1 - b_2c_2 = 0$ $\quad$ ($\frac{\partial}{\partial y}$ followed by $z=0$
- Const : $b_2d_1 - b_1d_2 = 0$ $\quad$ ($z=0$)
- x^2 : $a_2c_1 - a_1c_2 = 0$ $\quad$ ($\partial^2/\partial x^2$ followed by $z=0$

Thus $a_1c_2 = a_2c_1$ and $b_1d_2 = b_2d_1$ etc...

<u>Remark</u>: I include this to show shared suffering.

PROBLEM 54 the approach of the last page is cumbersome. Notice I only asked to <u>find</u> a map. I didn't ask for a derivation so I'm working too hard. Change gears a bit and <u>guess</u>:

$$z = -i \longmapsto w = 0 \implies (z+i) \propto \text{ numerator of } f(z)$$



$w = f(z)$

I read that Mobius trans. take "circles to circles" with the understanding that lines in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ are circles on the Riemann Sphere

$\implies$ some point on $|z| = 1$ must map to ∞.

By symmetry it better be $z = i \longrightarrow$ ½ by zero

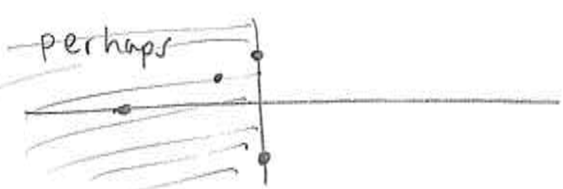
$$f(z) = e^{i\theta}\left[\frac{z+i}{z-i}\right]$$

Consider $g(z) = \frac{z+i}{z-i}$ this map has $g(i) = \infty$, $g(-i) = 0$

but $g(1) = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{2} = i$, $g(-1) = \frac{-1+i}{-1-i} = -i$

$g(1/2) = \frac{1/2 + i}{1/2 - i} = \frac{(1/2+i)^2}{5/4} = \frac{4}{5}\left(-\frac{3}{4} + i\right) = \frac{-3}{5} + \frac{4i}{5}$

$g(i/2) = \frac{i/2 + i}{i/2 - i} = \frac{3i/2}{-i/2} = -3$



$$\boxed{f(z) = \frac{z+i}{i-z}}$$

<u>Remark</u>: the next page is the best.

PROBLEM 54 Concluded! (for now)

$$W = \frac{z+i}{i-z} \quad \text{where} \quad \underbrace{z = re^{i\theta} \quad \text{for } 0 \le r \le 1}$$

this is how to think about a <u>disk</u>

$$W = \left(\frac{re^{i\theta}+i}{i-re^{i\theta}}\right)\left(\frac{-i-re^{-i\theta}}{-i-re^{-i\theta}}\right)$$

$$= \frac{-ire^{i\theta}+1-r^2e^{i\theta}e^{-i\theta}-ire^{-i\theta}}{1+r^2}$$

$$= \frac{-ir\left(e^{i\theta}+e^{-i\theta}\right)+1-r^2}{1+r^2}$$

$$= \frac{1-r^2}{1+r^2} + i\,\frac{2r}{1+r^2}\,\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$$

$$= \frac{1-r^2}{1+r^2} + i\left[\frac{2r\cos\theta}{1+r^2}\right]$$

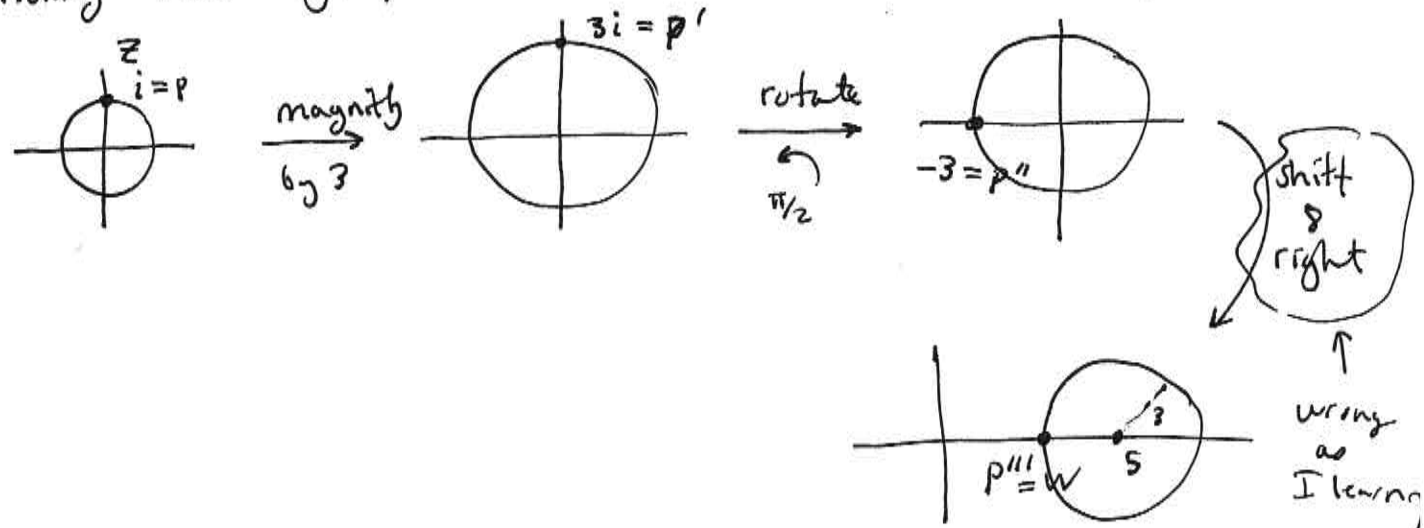
As $0 \le \theta \le 2\pi \Rightarrow -1 \le \cos\theta \le 1$ it is clear this map has $\operatorname{Im}(W) > 0$ and < 0.
However, $0 \le r \le 1 \Rightarrow 1-r^2 \ge 0$ hence $\frac{1-r^2}{1+r^2} \ge 0$ and thus $\operatorname{Re}(W) = \frac{1-r^2}{1+r^2} \ge 0$

Hence $f(z) = \frac{z+i}{i-z}$ takes values in the desired right half-plane.

PROBLEM 55

Find a linear transformation which maps circle $|z|=1$ to $|w-5|=3$ such that $z=i \longmapsto w=f(i)=2$.

We can think about this like we think about building new graphs from old in college algebra etc...



$$z \mapsto z' = 3z \longmapsto z'' = e^{i\pi/2} z' = 3iz \longmapsto z''' = w = 3iz + 8$$

Let's check our work, suppose $|z| = 1$ or $x^2+y^2 = 1$.

$w = 3iz + 8 = 3i(x+iy) + 8 = u + iv$

Hence $u = 8 - 3y$ and $v = 3x$

$$\begin{aligned}(u-5)^2 + v^2 &= (3-3y)^2 + (3x)^2 \\ &= 9 - 6y + 9y^2 + 9x^2 \\ &= 9 - 6y + 9 \\ &= 18 - 6y.\end{aligned}$$

duh! Shift $\boxed{5}$ right ! (see how I did this on purpose for the learning ü)

$$\boxed{w = 3iz + 5}$$

$w - 5 = 3iz$ hence $z\bar{z} = 1$

$\Rightarrow |w-5|^2 = |3iz|^2 = 9|z|^2 = 9.$

$\hookrightarrow \underline{|w-5| = 3}$. ✓

PROBLEM 56

If $\bar{z} = e^{i\theta} z$ then $\bar{z} = \bar{x} + i\bar{y}$ and $z = x + iy$ are related by a rotation [here I suspend the usual practice of $\bar{z} = x - iy$, if needed we'll say $z^* = x - iy$]
We transform the ellipse $x^2/a^2 + y^2/b^2 = 1$ to $\bar{x}, \bar{y}$-eq$^{=}$ by substitution: $z = e^{-i\theta}\bar{z} = (\cos\theta - i\sin\theta)(\bar{x} + i\bar{y})$

$$\Rightarrow \quad x + iy = (\cos\theta\,\bar{x} + \sin\theta\,\bar{y}) + i(\cos\theta\,\bar{y} - \sin\theta\,\bar{x})$$

Hence,

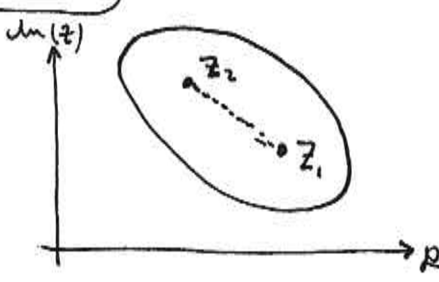
$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(\cos\theta\bar{x} + \sin\theta\bar{y})^2}{a^2} + \frac{(\cos\theta\,\bar{y} - \sin\theta\,\bar{x})^2}{b^2}$$

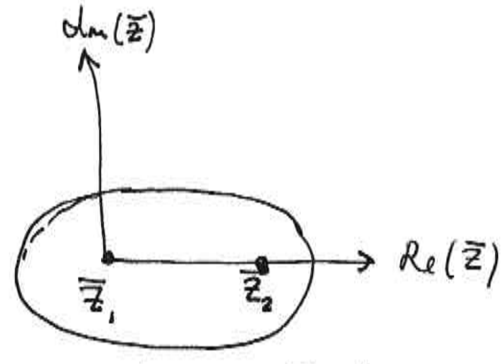
$$= \frac{\cos^2\theta\,\bar{x}^2 + 2\sin\theta\cos\theta\,\bar{x}\bar{y} + \sin^2\theta\,\bar{y}^2}{a^2} + \frac{\cos^2\theta\,\bar{y}^2 - 2\sin\theta\cos\theta\,\bar{y}\bar{x} + \sin^2\theta\,\bar{x}^2}{b^2}$$

$$= \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)\bar{x}^2 + 2\left(\frac{\sin\theta\cos\theta}{a^2} - \frac{\sin\theta\cos\theta}{b^2}\right)\bar{x}\bar{y} + \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)\bar{y}^2$$

$$\therefore \quad \boxed{1 = \left[\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right]\bar{x}^2 + 2\sin\theta\cos\theta\left[\frac{1}{a^2} - \frac{1}{b^2}\right]\bar{x}\bar{y} + \left[\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right]\bar{y}^2}$$

PROBLEM 57





I choose $z_1 \longmapsto 0 = \bar{z}_1 \Rightarrow$ use $t = -z_1$ hence the rule looks like $\bar{z} = (-z_1 + z)e^{i\theta}$. I also wish $Re(\bar{z}_2) = \bar{z}_2$
Note $z_2 \longmapsto (-z_1 + z_2)e^{i\theta}$. Let $\alpha = \angle(z_2 - z_1)$ or in more useful notation $\alpha = Arg(z_2 - z_1)$. I believe we can use $\theta = -\alpha$, but let's work it out

PROBLEM 57 continued

Consider $\bar{z} = (z - z_1) \exp(-i \operatorname{Arg}(z_2 - z_1))$

Clearly $\bar{z}_1 = (z_1 - z_1) \exp(-i\operatorname{Arg}(z_2 - z_1)) = 0$.

and $\bar{z}_2 = (z_2 - z_1) \exp(-i \operatorname{Arg}(z_2 - z_1))$

$$\Rightarrow \bar{z}_2 = |z_2 - z_1| \exp(i\operatorname{Arg}(z_2 - z_1)) \exp(-i\operatorname{Arg}(z_2 - z_1))$$

$$\therefore \bar{z}_2 = |z_2 - z_1| \in \mathbb{R}$$

$$\therefore Re(\bar{z}_2) = \bar{z}_2 .$$

Note than $z \in \mathbb{C}$ s.t. $|z - z_1| + |z - z_2| = k$ transforms to what?

$$\bar{z} = (z - z_1) e^{-i\operatorname{Arg}(z_2 - z_1)}$$

$$\bar{z} e^{i\operatorname{Arg}(z_2 - z_1)} = z - z_1$$

$$\therefore \underline{z = z_1 + \bar{z} \exp(i \operatorname{Arg}(z_2 - z_1))}.$$

Therefore,

$$|z - z_1| + |z - z_2| = k$$

$$\Rightarrow |\bar{z} e^{i\operatorname{Arg}(z_2 - z_1)}| + |z_1 - z_2 + \bar{z} \exp(i\operatorname{Arg}(z_2 - z_1))| = k$$

$$\Rightarrow |\bar{z}| + \left|-|z_2 - z_1| e^{i\operatorname{Arg}(z_2 - z_1)} + \bar{z} e^{i\operatorname{Arg}(z_2 - z_1)}\right| = k$$

$$\Rightarrow |\bar{z}| + \left|\bar{z} - |z_2 - z_1|\right| = k$$

$$|\bar{x} + i\bar{y}|^2 = \left(k - |\bar{z} - \overbrace{|z_2 - z_1|}^{d}|\right)^2$$

$$\bar{x}^2 + \bar{y}^2 = k^2 - 2k|\bar{z} - |z_2 - z_1|| + |\bar{z} - |z_2 - z_1||^2$$

$$\bar{x}^2 + \bar{y}^2 = k^2 - 2k\sqrt{(\bar{x} - d)^2 + \bar{y}^2} + (\bar{x} - d)^2 + \bar{y}^2$$

$$\cancel{\bar{x}^2} + \cancel{\bar{y}^2} = k^2 - 2k\sqrt{(\bar{x} - d)^2 + \bar{y}^2} + \cancel{\bar{x}^2} - 2\bar{x}d + d^2 + \cancel{\bar{y}^2}$$

$$2k\sqrt{(\bar{x} - d)^2 + \bar{y}^2} = k^2 - 2\bar{x}d + d^2$$

$$(\bar{x} - d)^2 + \bar{y}^2 = \frac{1}{4k^2}(k^2 - 2\bar{x}d + d^2)^2 = \frac{1}{4k^2}\left[(k^2 + d^2)^2 - 4\bar{x}d(k^2 + d^2) + 4\bar{x}^2 d^2\right] \;2$$

Problem 57 continued ($\bar{d} = d$ dash)

$$\bar{x}^2 - 2\bar{x}\bar{d} + \bar{d}^2 + \bar{y}^2 = \frac{1}{4k^2}\left[(k^2+d^2)^2 - 4\bar{x}d(k^2+d^2) + 4\bar{x}^2 d^2\right]$$

$$\bar{y}^2 + \bar{x}^2\left(1 - \frac{4d^2}{4k^2}\right) + \bar{x}\left(-2\bar{d} + \frac{4(k^2+d^2)d}{4k^2}\right) = \frac{(k^2+d^2)^2}{4k^2} - d^2$$

$$\bar{y}^2 + \bar{x}^2 + \bar{x}\underbrace{\left(\frac{d\left(-2 + \frac{k^2+d^2}{k^2}\right)}{1 - \frac{4d^2}{4k^2}}\right)}_{2B} = \underbrace{\frac{\frac{(k^2+d^2)^2}{4k^2} - d^2}{1 - d^2/k^2}}_{C}$$

$$\bar{y}^2 + (\bar{x} - B)^2 - B^2 = C$$

$$(\bar{x} - B)^2 + \bar{y}^2 = C + B^2$$

$$\frac{(\bar{x} - B)^2}{\underbrace{C + B^2}_{a^2}} + \frac{\bar{y}^2}{\underbrace{C + B^2}_{b^2}} = 1$$

$$\frac{(\bar{x} - B)^2}{a^2} + \frac{\bar{y}^2}{b^2} = 1$$

Need to translate real part to remove B.

$$\boxed{\bar{z} = (z - z_1)e^{-i\operatorname{Arg}(z_2 - z_1)} - B}$$

where $2B = d\left(\frac{d^2/k^2 - 1}{1 - d^2/k^2}\right) = -d \therefore B = \frac{-d}{2} = \frac{-|z_2 - z_1|}{2}$.