MATH 331: TEST 1   NO NOTECARDS, BOX ANSWERS, SHOW WORK, THANKS !

Problem 1 [10pts] Complete the following statements: (don't just give me a synonym, I'm looking for a sentence or formula which captures the meaning)

a. We say $\lim_{z \to z_o} f(z) = L \in \mathbb{C}$ iff

For each $\epsilon > 0$, $\exists \delta > 0$ s.t. $z \in \mathbb{C}$ with $0 < |z - z_o| < \delta$
$\Rightarrow |f(z) - L| < \epsilon$.

b. We say $f : dom(f) \subseteq \to \mathbb{C}$ is **complex-differentiable** at z_o iff

$\lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o}$ exists in $\mathbb{C}$.

c. We say $D \subseteq \mathbb{C}$ is a **domain** iff

D is an open, connected, set.

d. We say $z \in U \subseteq \mathbb{C}$ is an **interior point** of U iff

$\exists \delta > 0$ such that $\underbrace{\{w \in \mathbb{C} \,|\, |w - z| < \delta\}}_{D(w,\delta)} \subseteq U$

e. We say $U \subseteq \mathbb{C}$ is open iff

each point in U is an interior point.

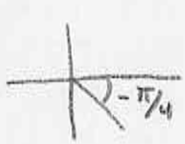
Problem 2 [30pts] Calculate the cartesian forms of

a. $\frac{3}{2-i}\left(\frac{2+i}{2+i}\right) = \frac{6+3i}{4+1} = \boxed{\frac{6}{5} + \frac{3}{5}i}$

$777 = 8(97) + 1$

$$\begin{array}{r} 97 \\ 8 \overline{)777} \\ \underline{72} \\ 57 \\ \underline{56} \\ 1 \end{array}$$

b. $(7-7i)^{777} = \left(7\sqrt{2}\, e^{-i\pi/4}\right)^{777} = \left(7\sqrt{2}\right)^{777} e^{-\frac{i777\pi}{4}}$



$= \left(7\sqrt{2}\right)^{777} \exp\left(\frac{-i\pi}{4}\right)$

$2\pi k i$ period of e^z

$= \boxed{\frac{\left(7\sqrt{2}\right)^{777}}{\sqrt{2}} - \frac{i\left(7\sqrt{2}\right)^{777}}{\sqrt{2}}}$

c. $Log(2i)$

$= \ln|2i| + i\,Arg(2i)$

$= \boxed{\ln 2 + i\frac{\pi}{2}}$

Problem 3 [5pts] Show that $\cos(z) = \cosh(iz)$. Differentiate to find a similar identity for sine and hyperbolic sine.

$$\cosh(iz) = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos(z).$$

$\frac{d}{dz}$

$$i\sinh(iz) = -\sin z \implies \boxed{\sinh(iz) = i\sin z}$$

$$\left(\begin{array}{l} \cos(iz) = \cosh(i(iz)) = \cosh(-z) = \cosh z. \\ \sin(iz) = -i\sinh(i(iz)) = -i\sinh(-z) = i\sinh z. \end{array}\right)$$

Problem 4 [10pts] Derive addition formulas for cosine, sine, cosh and sinh. To do this merely show

$$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

Your proof may assume the identity $e^{a+b} = e^a e^b$ for $a, b \in \mathbb{C}$ and should explicitly involve the definitions of sine and cosine in terms of the complex exponential. Once that is settled, write the other 3 addition rules for $\cos(z + w), \cosh(z + w), \sinh(z + w)$ by using the identity or technique of the previous problem.

$$\sin z\cos w + \cos z\sin w = \tfrac{1}{2i}(e^{iz} - e^{-iz})\tfrac{1}{2}(e^{iw} + e^{-iw}) + \tfrac{1}{2}(e^{iz} + e^{-iz})\tfrac{1}{2i}(e^{iw} - e^{-iw})$$

$$= \tfrac{1}{4i}\Big[e^{i(z+w)} + e^{i(z-w)} - e^{i(w-z)} - e^{-i(z+w)} + e^{i(z+w)} - e^{i(z-w)} + e^{i(w-z)} - e^{-i(z+w)}\Big]$$

$$= \tfrac{1}{2i}\left(e^{i(z+w)} - e^{-i(z+w)}\right)$$

$$= \sin(z+w).$$

Differentiate w.r.t. z

(**a.**) $\cos(z + w) =$ $\cos z\cos w - \sin z\sin w.$

(**b.**) $\cosh(z + w) =$ $\cosh z\cosh w + \sinh z\sinh w.$

(**c.**) $\sinh(z + w) =$ $\sinh z\cosh w + \cosh z\sinh w.$

$$\cosh(z+w) = \cos(iz + iw)$$
$$= \cos iz\cos iw - \sin iz\sin iw$$
$$= \cosh z\cosh w - i^2\sinh z\sinh w$$
$$= \cosh z\cosh w + \sinh z\sinh w$$

$$\sinh(z+w) = \tfrac{d}{dz}(\cosh z+w)$$

Problem 5 [10pts] **Find all** $z \in \mathbb{C}$ **such that** $\sin(z) = 0$. You are allowed to assume knowledge of the zeros of the real-valued sine and cosine. Please provide evidence to support your claim.

Let $z = x + iy$ as is our usual custom,

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$
$$= \sin(x)\cosh(y) + i\cos x\sinh(y)$$

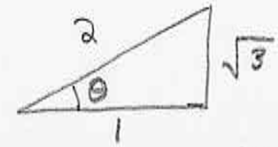
Hence $\sin(x+iy) = 0$ gives us two conditions

1.) $\sin x\cosh y = 0 \implies \sin x = 0$ as $\cosh y \geq 1$.

2.) $\cos x\sinh y = 0 \implies \sinh y = 0$ since $\sin x = 0 \implies \cos x \neq 0$.

Hence $\underbrace{\sin x = 0}_{x = n\pi,\ n \in \mathbb{Z}}$ and $\underbrace{\sinh y = 0}_{y = 0}$ $\therefore$ $\boxed{z = n\pi,\ n \in \mathbb{Z}}$

Problem 6 [15pts] Find all solutions of $z^4 = 8 + i8\sqrt{3}$.



$$z^4 = \sqrt{64 + 3(64)}\; e^{i\pi/3} = 16 e^{i\pi/3}$$

$\cos\theta = \frac{1}{2}$

$\theta = \pi/3$.

$$z \in \left(16 e^{i\pi/3}\right)^{1/4} \qquad \omega_0 = e^{\frac{2\pi i}{4}} = e^{\frac{\pi i}{2}} = i$$

principle 4^{th} root of unity.

$$z \in \left\{ 2e^{i\pi/12} i,\ 2e^{i\pi/12} i^2,\ 2e^{i\pi/12} i^3,\ 2e^{i\pi/12} i^4 \right\}$$

$$\boxed{z \in \left\{ \pm 2i e^{i\pi/12},\ \pm 2 e^{i\pi/12} \right\}}$$

Problem 7 [20pts] Let $f(z) = e^z$. Show that f is complex-differentiable at each $z = x + iy \in \mathbb{C}$ and the formula is simply $f'(z) = e^z$. *Hint: use the CR-equations...*

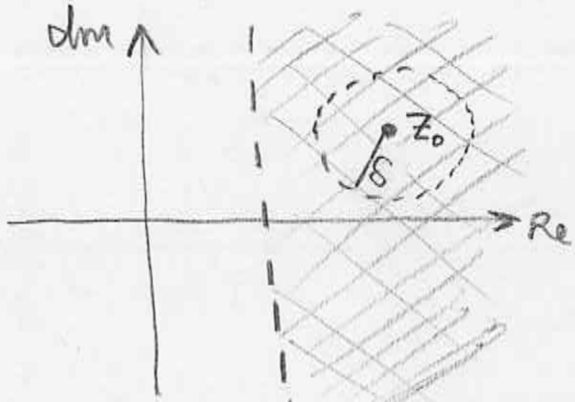
$$f(x+iy) = e^{x+iy} = e^x e^{iy} = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$$

$$u_x = v_y = e^x \cos y$$

$u_y = -v_x = -e^x \sin y$, moreover u_x, u_y, v_x, v_y continuous on $\mathbb{C}$ hence CR-eq$^{\underline{s}}$ & continuous diff. $\Rightarrow f'(z)$ exists for each $z \in \mathbb{C}$. And, by our previous work in lecture,

$$\begin{aligned} f'(z) &= u_x + i v_x \\ &= e^x \cos y + i e^x \sin y \\ &= e^z. \end{aligned}$$

Problem 8 [15pts] Let $U = \{z \in \mathbb{C} \mid Re(z) > 1\}$. Prove U is open.



Let $z_0 \in U$. Notice $z_0 = x_0 + iy_0$ has $Re(z_0) = x_0 > 1$. Let us note also $x_0 - 1 > 0$ thus $\delta = \frac{x_0 - 1}{2} > 0$. Consider $z \in D(z_0, \delta)$, we seek to show $z \in U$.

$$z \in D\left(z_0, \tfrac{x_0-1}{2}\right) \Rightarrow |z - z_0| < \frac{x_0-1}{2}$$

But $z = x + iy$ and $|x - x_0| \leq |z - z_0| < \frac{x_0 - 1}{2}$

$$\Rightarrow -\left(\frac{x_0-1}{2}\right) < x - x_0 \Rightarrow -x_0 + 1 < 2x - 2x_0 \Rightarrow x > \frac{1+x_0}{2}$$

But, $x_0 > 1 \Rightarrow \frac{1+x_0}{2} > \frac{1+1}{2} = 1 \quad \therefore x > 1 \Rightarrow z \in U \Rightarrow D(z_0, \delta) \subseteq U$

hence z_0 interior $\therefore U$ open.

Problem 9 [15pts] Calculate $\int_C \bar{z}\,dz$ where C is the positively oriented unit-circle.

$$C: z = e^{i\theta} \Rightarrow \frac{dz}{d\theta} = ie^{i\theta}$$

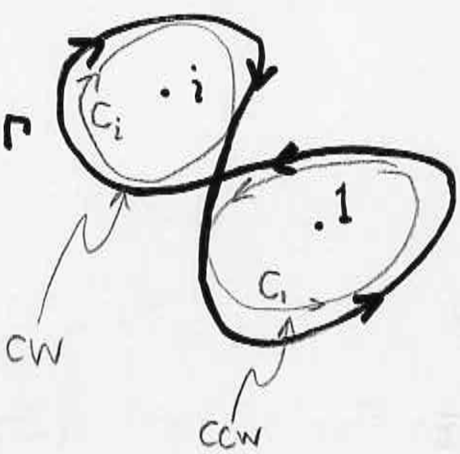
$$\int_C \bar{z}\,dz = \int_0^{2\pi} e^{-i\theta}\, ie^{i\theta} d\theta = \int_0^{2\pi} i\,d\theta = \boxed{2\pi i}$$

Problem 10 [20pts] Let Γ denote the figure-eight curve pictured below. Calculate:

"$g(z)$"

$$\int_\Gamma \left(\frac{z^2+7}{z-1} + \frac{z^4}{(z-i)^3}\right) dz = \int_\Gamma \frac{z^2+7}{z-1}\,dz + \int_\Gamma \frac{z^4\,dz}{(z-i)^3}$$

the integrands are analytic on C_i & C_1 respective and these terms $\longrightarrow 0$.



$$= \int_{C_1} \frac{z^2+7}{z-1}\,dz + \int_{C_i} \frac{z^4\,dz}{(z-i)^3}$$

$$= 2\pi i (z^2+7)\Big|_{z=1} - \frac{2\pi i}{2} g''(i)$$

$$= (2\pi i)(8) - \pi i\,(12z^2)\big|_{z=i}$$

$$= 16\pi i - \pi i\,(-12)$$

$$= \boxed{28\pi i}$$

$$2\pi i\, g(z) = \int \frac{g(w)\,dw}{w-z} \Rightarrow \frac{2\pi i\, g^{(n)}(z)}{n!} = \int \frac{g(w)\,dw}{(w-z)^{n+1}}$$

Problem 11 [15pts] Establish the following result: if P is a polynomial of degree at least 2 and P has all its zeros inside the circle $|z| = r_o$ then if C_o denotes the CCW-oriented circle $|z| = r_o$ then $\int_{C_o} \frac{1}{P(z)} dz = 0$.

Observe $\frac{1}{P(z)}$ is analytic outside C_o and in particular between C_o and C_R hence $\int_{C_o} \frac{dz}{P(z)} = \int_{C_R} \frac{dz}{P(z)}$ by deformation Th$^{\underline{m}}$.

Furthermore, $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $n \geq 2$ can be bounded on $C_R : |z| = R$,

$$\left|\frac{1}{P(z)}\right| = \frac{1}{|a_0 + a_1 z + \cdots + a_n z^n|} \leq \frac{1}{\big||a_0| - |a_1 z + \cdots + a_n z^n|\big|}$$

$$\leq \underbrace{\frac{1}{\big||a_0| - ||a_1 R| - |\cdots |a_n R^n|\cdot||\big|}}_{f(R)}$$

Hence $\left|\frac{1}{P(z)}\right| \leq \frac{1}{f(R)}$

Bounding Th$^{\underline{m}}$ $\Rightarrow \left|\int_{C_R} \frac{dz}{P(z)}\right| \leq \frac{2\pi R}{f(R)} \longrightarrow 0$ as $R \to \infty$ since $n \geq 2$.

Problem 12 [15pts] Factor $f(z) = z^5 - 32$ completely.

(use of polar form fine, for example, can use $(z - \sqrt{2}e^{i\pi/4})$ and not simplify to $(z - 1 - i)$.)

$$f(z) = (z-2)(z-2\omega_0)(z-2\omega_0^2)(z-2\omega_0^3)(z-2\omega_0^4)$$

$$\omega_0 = e^{\frac{2\pi i}{5}}$$

Problem 13 [10pts] Suppose $u_x(p) = v_y(p)$ and $u_y(p) = -v_x(p)$. Given this information alone, does it follow that $f = u + iv$ complex differentiable at p? If not, provide a counter-example.

No, this does not suffice. Discontinuous u_x, u_y etc... $\Rightarrow f'(z)$ d.n.e. Many counter-examples exist, I often give

$$f(x,y) = \begin{cases} (1+i) & : xy = 0 \\ 0 & : xy \neq 0 \end{cases}$$

We have $u = 1$ and $v = 1$ along coord. axes yet u, v discontinuous hence $f'(z)$ d.n.e. (as it implies continuity)

Problem 14 [10pts] Prove that $\frac{d}{dz} z^c = cz^{c-1}$ where z^c is defined such that it is complex differentiable on $\mathbb{C}_-$.

$$z^c = \exp(c\,\mathrm{Log}(z)) \leftarrow \mathbb{C}_- = \mathbb{C} - \{x + i0 \mid x \leq 0\}.$$

$$\frac{d}{dz}(z^c) = \exp(c\,\mathrm{Log}(z)) \frac{d}{dz}[c\,\mathrm{Log}(z)]$$

$$= z^c \cdot \frac{c}{z}$$

$$= cz^{c-1}.$$