Same as Mission1. Enjoy!

Problem 13 Your signature below indicates you have:

(a.) I have read Sections 2.1 – 2.3 of Gamelin: ________________.

(b.) I have read Cook's guide to Sections 2.1 – 2.3 of Gamelin:________________.

Problem 14 Let $S_\alpha = \mathbb{C} - e^{i\alpha}[0,\infty)$; this is $\mathbb{C}$ with the ray at $\theta = \alpha$ removed. Furthermore, define $\text{Log}_\alpha(z) = \ln|z| + i\text{Arg}_\alpha(z)$ where we define $\text{Arg}_\alpha(z)$ to be the single element found in $arg(z) \cap (\alpha, \alpha+2\pi]$. Calculate the discontinuity at $\theta = \alpha$ by following any circle $\gamma(t) = Re^{it}$ with parameter values $t \in (\alpha, \alpha+2\pi]$. Compare $\text{Log}_\alpha(\gamma(\alpha+2\pi))$ and $\lim_{t\to\alpha^+} \text{Log}_\alpha(\gamma(t))$.

Problem 15 Show $\sin(z+w) = \cos(z)\sin(w) + \sin(z)\cos(w)$ for all $z, w \in \mathbb{C}$. Then use the relation of $\sinh(z), \cosh(z)$ to $\sin(z), \cos(z)$ to derive a corresponding identity for $\sinh(z+w)$.

Problem 16 Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\}$.

(a.) show with a sketch $1+i$ is an interior point of $\mathbb{H}$.

(b.) show every point in $\mathbb{H}$ is an interior point of $\mathbb{H}$. Your argument may be guided by a sketch, but it should also be supported with explicit inequality arguments.

Problem 17 Let $a_n = z^n$ for $n \in \mathbb{N}$.

(a.) for which $z \in \mathbb{C}$ is a_n a bounded ?

(b.) for which $z \in \mathbb{C}$ does a_n form a convergent sequence ?

Problem 18 Show $\mathbb{C}^+ = \mathbb{C} - [0,\infty)$ is star-centered by proving any $p \in (-\infty, 0)$ serves as a star center for $\mathbb{C}^+$. *Your solution may be graphical, but, please explain your picture.*

Problem 19 Let $f(z) = 1/z^2$.

(a.) Show $f'(z) = -2/z^3$ by calculating the limit of the difference quotient.

(b.) Use the formulas given on page 36 of the guide to show $f'(z) = -2/z^3$

Problem 20 Let $g : dom(g) \subseteq \mathbb{C} \to \mathbb{C}$. **Prove the following:**
If $\lim_{h\to 0} g(h)/|h| = 0$ then $\lim_{h\to 0} g(h)/h = 0$.

Problem 21 Show $f(z) = z + \bar{z}$ is not complex differentiable at any point in $\mathbb{C}$.

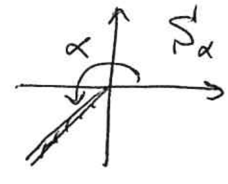
Problem 22 Use the CR-equations to show $\frac{d}{dz}\sin z = \cos z$.

Problem 23 Give a proof similar in style to Examples in 2.2 of the guide to show $\dfrac{d}{dz}z^3 = 3z^2$.

Problem 24 We wish to show $\mathbb{C}$ is complete. The heart of the claim follows from $|\mathfrak{Re}(w)| \leq |w|$ and $|\mathfrak{Im}(w)| \leq |w|$ paired with the fact we **assume** $\mathbb{R}$ is complete. **Show $\mathbb{C}$ is complete.** *To sketch the solution: assume $z_n = x_n + iy_n$ is a Cauchy sequence in $\mathbb{C}$. Apply the given inequalities to argue x_n and y_n are Cauchy real sequences and hence converge to x and y respective. Finish the argument by showing $z_n \to x + iy$.*

MISSION 2 : COMPLEX ANALYSIS

PROBLEM 14 Let $S_\alpha = \mathbb{C} - e^{i\alpha}[0,\infty)$



$$\mathrm{Log}_\alpha(z) = \ln|z| + i\underbrace{\mathrm{Arg}_\alpha(z)} \text{ for } z \in \mathbb{C}^\times$$

$$\mathrm{Arg}_\alpha(z) \in \underbrace{\arg(z) \cap (\alpha, \alpha+2\pi]}_{\text{singleton.}}$$

Consider then, the circle centered at $z=0$ parametrized by, $R>0$, $t \mapsto Re^{it}$ for $t \in (\alpha, \alpha+2\pi]$. Notice that $\mathrm{Arg}_\alpha(Re^{it}) = t$ by our choice of parameter domain. Consequently, as $|Re^{it}| = |R|\,|e^{it}| = R$.

$$\mathrm{Log}_\alpha(Re^{it}) = \ln R + it$$

Hence, using $\gamma(t) = Re^{it}$ as in problem statement,

$$\mathrm{Log}_\alpha(\gamma(\alpha+2\pi)) = \mathrm{Log}_\alpha(Re^{i(\alpha+2\pi)}) = \underbrace{\ln R + i(\alpha+2\pi)}_{\text{I}}$$

However,

$$\lim_{t\to\alpha^+}(\mathrm{Log}_\alpha(\gamma(t))) = \lim_{t\to\alpha^+}(\ln R + it) = \ln R + i\alpha \quad \text{II}.$$

Thus $\text{I} - \text{II} = \ln R + i(\alpha + 2\pi) - [\ln R + i\alpha] = \underline{\underline{2\pi i}}$

<u>THE POINT:</u> no matter which branch we define Log_α for (select from log if you like to take the set of values $\log(z)$ as basic) there <u>must</u> be some $2\pi i$ – jump along some ray. It's not a quirk of Log it's ubiquitous for all choices of Log_α.

$\boxed{\text{PROBLEM 15}}$ Recall $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ and $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ by def$^{\underline{n}}$ whereas $\sinh z = \frac{1}{2}(e^{z} - e^{-z})$ and $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$

Notice, $e^{z} = \cosh z + \sinh z$ whereas we have $e^{iz} = \cos z + i \sin z$ by almost the same <u>algebra</u>!

$$\boxed{\begin{array}{l} \underline{\textit{Lemma}}: \quad \dfrac{\sinh(iz)}{i} = \sin(z) \quad \text{and} \quad \cosh(iz) = \cos(z) \\ \text{and likewise} \quad \sin(iz) = i\sinh(z) \quad \cos(iz) = \cosh(z). \end{array}}$$

<u>Proof</u>: I'll do one, the rest are proved similarly,

$$\sin(iz) = \frac{1}{2i}\left(e^{i(iz)} - e^{-i(iz)}\right) = \frac{1}{2i}\left(e^{-z} - e^{z}\right) = i\underbrace{\tfrac{1}{2}(e^{z} - e^{-z})}_{\sinh(z)}$$

where I used $\frac{1}{i} = -i$ in last step. //

- With all of the above in mind I attach the given problem

$$\sin(z+w) = \frac{1}{2i}\left(e^{i(z+w)} - e^{-i(z+w)}\right) \quad \text{(I.)}$$

Likewise, consider,

$$\cos z \sin w + \sin z \cos w = \frac{1}{2}(e^{iz} + e^{-iz})\frac{1}{2i}(e^{iw} - e^{-iw}) + \frac{1}{2i}(e^{iz} - e^{-iz})\frac{1}{2}(e^{iw} + e^{-iw})$$

$$= \frac{1}{4i}\Big[e^{iz}e^{iw} - \cancel{e^{iz}e^{-iw}} + \cancel{e^{-iz}e^{iw}} - e^{-iz}e^{-iw} + e^{iz}e^{iw} + \cancel{e^{iz}e^{-iw}} - \cancel{e^{-iz}e^{iw}} - e^{-iz}e^{-iw} \Big]$$

$$= \frac{1}{2i}\left[e^{iz}e^{iw} - e^{-iz}e^{-iw} \right]$$

$$= \frac{1}{2i}\left[e^{i(z+w)} - e^{-i(z+w)} \right]$$

$$= \sin(z+w) \quad \text{see (I)}$$

appreciate this step uses adding $\angle$'s f-las for sine & cosine as well as usual Law of Exponents for real exponentials!

PROBLEM 15 continued:

We've show $\underline{\sin(z+w) = \cos z \sin w + \sin z \cos w}$. (III)

In the Lemma, I observed

$$i \sinh(\beta) = \sin(i\beta) \implies \underline{\sinh\beta = -i\sin(i\beta)}.$$

Apply to $\beta = z+w$ and use the other facts $\cos(iz) = \cosh z$ etc.

$$\begin{aligned}
\sinh(z+w) &= -i\sin(i(z+w)) \\
&= -i\sin(iz+iw) \qquad \text{using (III)} \\
&= -i\left[\cos(iz)\sin(iw) + \sin(iz)\cos(iw)\right] \qquad \text{Lemma} \\
&= -i\left[\cosh(z)\cdot(i\sinh(w)) + (i\sinh z)\cosh(w)\right] \\
&= \underline{\cosh z \sinh w + \sinh z \cosh w}.
\end{aligned}$$

Naturally, a proof w/o use of Lemma by direct appeal to $\cosh z = \frac{1}{2}(e^z + e^{-z})$ & $\sinh z = \frac{1}{2}(e^z - e^{-z})$ and law $e^z e^w = e^{z+w}$ is also possible, but I asked for you to think about the Lemma.

(grader: be cool, let the no-Lemma sol^n also count)

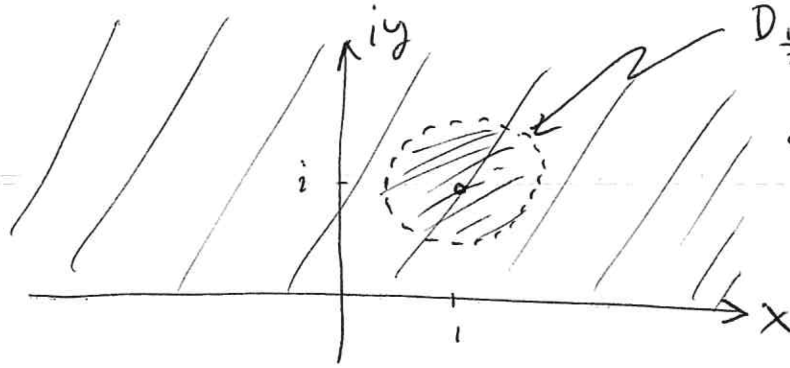
$$\begin{aligned}
\cosh z \sinh w + \sinh z \cosh w &= \frac{1}{4}\left[(e^z + e^{-z})(e^w - e^{-w}) + (e^z - e^{-z})(e^w + e^{-w})\right] \\
&= \frac{1}{4}\left(2e^z e^w - 2e^{-z}e^{-w}\right) \\
&= \frac{1}{2}\left(e^{z+w} - e^{-(z+w)}\right) \\
&= \sinh(z+w). \quad \text{(E)}
\end{aligned}$$

Remark: you could also prove (E) to derive (III). There are many interelations here...

PROBLEM 16

Let $\mathbb{H} = \{ z \in \mathbb{C} \,/\, \operatorname{Im}(z) > 0 \}$

(a.)

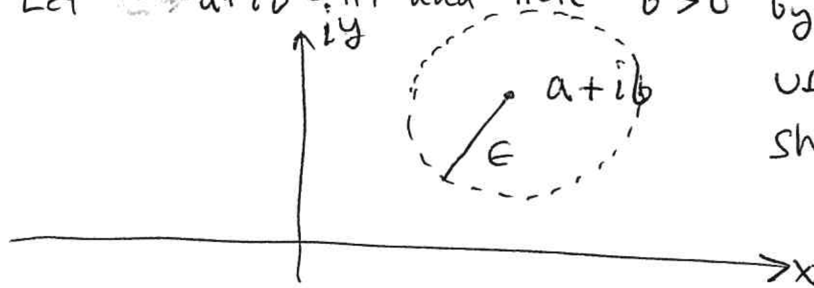


$D_{\frac{1}{2}}(1+i)$ = open disk in $\mathbb{H}$ centered at $1+i$

$\Leftarrow \mathbb{H}$ the upper half-plane.

this shows $1+i$ is an <u>interior point</u> of $\mathbb{H}$. other choices for δ in $D_\delta(1+i)$ are possible!

(b.) Let $a+ib \in \mathbb{H}$ and <u>note</u> $b>0$ by def$^{\underline{n}}$ of $\mathbb{H}$.



use $\epsilon = b/2$. should work.

Let $z \in D_{b/2}(a+ib)$ then $|z-(a+ib)| < \frac{b}{2}$.

If $x+iy = z$ we find explicitly,

$$|(x-a)+i(y-b)| < \frac{b}{2}$$

At this point, it is useful to observe as $|\operatorname{Im}(w)| \leq |w|$,

$$|y-b| \leq |x-a+i(y-b)| < \frac{b}{2}$$

thus $|y-b| < b/2$. Unwrapping this yields,

$$-\frac{b}{2} < y-b < \frac{b}{2} \implies \frac{b}{2} < y < \frac{3b}{2} \implies y > 0$$

as $b>0$. Consequently, $x+iy \in \mathbb{H}$ and we have shown $D_{b/2}(a+ib) \subseteq \mathbb{H} \implies a+ib$ is an interior point of $\mathbb{H}$. However, as $a+ib$ was arbitrary this shows <u>every</u> point in $\mathbb{H}$ is interior thus $\mathbb{H}$ is open. //

$\boxed{\text{PROBLEM 17}}$ Let $a_n = z^n$ for $n \in \mathbb{N}$

(a.) observe $|z^n| = |(|z|e^{i\theta})^n| = ||z|^n e^{in\theta}| = |z|^n$

we know $|z|^n \in \mathbb{R}^+$ and for $0 \leq r < \infty$

$$r^n \longrightarrow \begin{cases} 0 & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

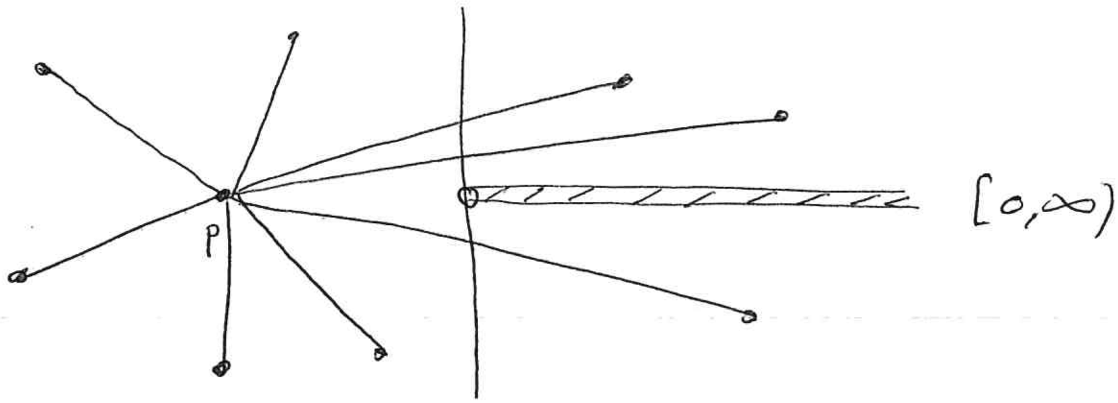
thus, $a_n = z^n$ is bounded iff $|z| \leq 1$.
(for z on the unit-disk)

(b.) If $|z| = 1$ then $z = e^{i\theta}$ and so
$z^n = e^{in\theta}$, As $n \to \infty$ this oscillates
unless, $\theta = 2\pi k$ for some $k \in \mathbb{Z}$. In
other words, only $z = 1$ converges for those
z with $|z| = 1$. On the other hand, if
$|z| < 1$ then $|a_n| = |z|^n \longrightarrow 0$ as $n \to \infty$
hence a_n converges <u>absolutely</u> to zero
hence $a_n \longrightarrow 0$.

<u>Summary</u>:

$$z^n = a_n \longrightarrow \begin{cases} \text{d.n.e (diverges)} & |z| > 1 \\ \text{diverges} & |z| = 1,\ z \neq 1 \\ 1 & z = 1 \\ 0 & |z| < 1. \end{cases}$$

PROBLEM 18



clearly $[p, z] \subset \mathbb{C}^+ = \mathbb{C} - [0,\infty)$ for each $z \in \mathbb{C}^+$ hence $p \in (-\infty, 0)$ is a <u>star</u> - <u>center</u> for $\mathbb{C}^+$.

PROBLEM 19

Remark: I didn't require details of algebra for PROBLEM 18. But, to show $[p, z] \in \mathbb{C}^+$ for each $z \in \mathbb{C}^+$ it suffices to show $\gamma(t) = p + t(z-p) \in \mathbb{C}^+$ for all $t \in [0,1]$. It's not hard to work out such details.

Let

$f(z) = 1/z^2$

Show $\frac{df}{dz} = \frac{-2}{z^3}$

by diff. quotient limit

(a.) $$f'(z) = \lim_{h\to 0}\left[\frac{\frac{1}{(z+h)^2} - \frac{1}{z^2}}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{z^2 - (z+h)^2}{hz^2(z+h)^2}\right]$$

$$= \lim_{h\to 0}\left[\frac{\cancel{z^2} - (\cancel{z^2} + 2zh + h^2)}{hz^2(z+h)^2}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2z - h}{z^2(z+h)^2}\right]$$

$$= \frac{-2z - 0}{z^2(z+0)^2} = \frac{-2z}{z^3} = \underline{\frac{-2}{z^2}}. //$$

PROBLEM 1.9 continued

(b.) on pg. 36 we learn $f'(z) = u_x + i v_x$

$$f(z) = \frac{1}{z^2} = \frac{\bar{z}\bar{z}}{(z\bar{z})^2} = \frac{(x-iy)^2}{(x^2+y^2)^2} = \frac{x^2-y^2-2ixy}{(x^2+y^2)^2}$$

$$u = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow u_x = \frac{2x(x^2+y^2)^2-(x^2-y^2)2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$\Rightarrow u_x = \frac{2x(x^2+y^2)-4x(x^2-y^2)}{(x^2+y^2)^3}$$

$$\Rightarrow u_x = \frac{-2x^3+6xy^2}{(x^2+y^2)^3} = -2\left[\frac{x^3-3xy^2}{(x^2+y^2)^3}\right]$$

$$v = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow v_x = \frac{-2y(x^2+y^2)^2+2xy(x^2+y^2)2(2x)}{(x^2+y^2)^4}$$

$$= \frac{-2y(x^2+y^2)+8x^2y}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y-2y^3+8x^2y}{(x^2+y^2)^3}$$

$$= -2\left[\frac{+y^3-3x^2y}{(x^2+y^2)^3}\right] = -2\left[\frac{y^3-3x^2y}{(x^2+y^2)^3}\right]$$

$$u_x + iv_x = -2\left[\frac{x^3-3xy^2+iy^3-3ix^2y}{(x^2+y^2)^3}\right]$$

$$= -2\left[\frac{x^3+3x^2(-iy)+3x(-iy)^2+(-iy)^3}{(x^2+y^2)^3}\right]$$

$$= -2\left[\frac{(x-iy)^3}{(x^2+y^2)^3}\right]$$

diff. quotient wins!

$$= -2\left[\frac{\bar{z}^3}{(z\bar{z})^3}\right] = \frac{-2\bar{z}^3}{z^3\bar{z}^3} = \frac{-2}{z^3} \quad //$$

$\boxed{\text{PROBLEM 20}}$ Let $g : dom(g) \subseteq \mathbb{C} \longrightarrow \mathbb{C}$.

Prove $\displaystyle\lim_{h \to 0} \left(\frac{g(h)}{|h|} \right) = 0 \Rightarrow \lim_{h \to 0} \left[\frac{g(h)}{h} \right] = 0$.

<u>Proof</u>: Assume $\displaystyle\lim_{h \to 0} \left[\frac{g(h)}{|h|} \right] = 0$. Let $\epsilon > 0$

then note $\exists \delta > 0$ s.t. $0 < |h| < \delta$ implies

$\left| \frac{g(h)}{|h|} - 0 \right| < \epsilon$ since we know $\displaystyle\lim_{h \to 0} \left[\frac{g(h)}{|h|} \right] = 0$.

Observe $\left| \frac{g(h)}{|h|} \right| = \frac{|g(h)|}{||h||} = \frac{|g(h)|}{|h|} = \left| \frac{g(h)}{h} \right|$

thus for $0 < |h| < \delta$ we have $\left| \frac{g(h)}{h} - 0 \right| < \epsilon$

Therefore, by def$^{\underline{n}}$ of limit, $\displaystyle\lim_{h \to 0} \left[\frac{g(h)}{h} \right] = 0$. //

$\boxed{\text{PROBLEM 21}}$

Show $f(z) = z + \bar{z}$ is not complex diff. at any pt. in $\mathbb{C}$.

$f(x+iy) = x + iy + (x - iy) = 2x \rightarrow u = 2x, \; V = 0$

Hence $\left. \begin{array}{ll} u_x = 2, & u_y = 0 \\ V_x = 0, & V_y = 0 \end{array} \right\}$ CR - eq^{ns} $u_x = V_y$ & $u_y = -V_x$

FAIL ON $\mathbb{C}$

$\therefore f(z)$ not complex diff. anywhere in $\mathbb{C}$.

PROBLEM 22

$$\tfrac{1}{2i}\left(e^{-y}-e^{y}\right) = i\,\tfrac{1}{2}\left(e^{y}-e^{-y}\right) = i\sinh(y)$$

Let $f(z) = \sin(z) = \sin(x+iy)$

$$= \sin(x)\cos(iy) + \cos(x)\sin(iy)$$

$$= \underbrace{\sin(x)\cosh(y)}_{u} + i\,\underbrace{\cos(x)\sinh(y)}_{v}$$

$$u_x = \cos(x)\cosh(y) \qquad v_x = -\sin(x)\sinh(y)$$

$$u_y = \sin(x)\sinh(y) \qquad v_y = \cos(x)\cosh(y)$$

thus $u_x = v_y$ and $u_y = -v_x$ so $f'(z)$ exists as u, v, are continuously differentiable. Moreover,

$$f'(z) = u_x + i v_x = \cos(x)\cosh(y) + i\left[-\sin(x)\sinh(y)\right]$$

$$= \cos(x)\cos(iy) - \sin(x)\sin(iy)$$

$$= \cos(x+iy)$$

$$= \cos(z). \quad \therefore \ \frac{d}{dz}\left[\sin(z)\right] = \cos(z)$$

PROBLEM 23 Let $f(z) = z^3$

then $f(z+h) - f(z) = (z+h)^3 - z^3 = z^3 + 3z^2h + 3zh^2 + h^3 - z^3$ yields $f(z+h) - f(z) = (3z^2 + 3zh + h^2)h$. Oops, I forgot, use $f(z) - f(a) = z^3 - a^3 = (z-a)(z^2 + az + a^2)$

I identify $\phi(z) = z^2 + az + a^2$ is a difference quotient fnct. for $f(z) = z^3$ as it is continuous and by construction

$$\phi(z) = \frac{f(z) - f(a)}{z - a} \quad \text{for } z \neq a$$

$$\begin{array}{r}
z^2 + az + a^2 \\
z - a \,\big)\, z^3 - a^3 \\
z^3 - az^2 \\
\hline
az^2 - a^3 \\
az^2 - a^2z \\
\hline
a^2z - a^3 \\
a^2z - a^3 \\
\hline
0
\end{array}$$

Hence by Caratheodory's Th^{m},

$$f'(a) = \lim_{z \to a} \phi(z) = a^2 + a^2 + a^2 = 3a^2$$

$$\therefore \ \frac{d}{dz}\left(z^2\right) = 3z^2.$$ //

PROBLEM 24 Show $\mathbb{C}$ is complete. We assume $\mathbb{R}$ is complete.

Assume $z_n = x_n + iy_n$ is a Cauchy sequence in $\mathbb{C}$. Thus, for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $m,n > N$ we find $|z_n - z_m| < \epsilon$. However, for the same choice of N we observe from $|\text{Re}(w)| \leq |w|$

$$|\text{Re}(z_n - z_m)| = |x_n - x_m| \leq |z_n - z_m| < \epsilon$$

Hence $n,m > N \Rightarrow |x_n - x_m| < \epsilon$ $\therefore \{x_n\}$ is a Cauchy seq. of real numbers. Likewise as $|\text{Im}(w)| \leq |w|$ apply this to $w = z_n - z_m$,

$$|\text{Im}(z_n - z_m)| = |y_n - y_m| \leq |z_n - z_m| < \epsilon$$

Thus, $n,m > N \Rightarrow |y_n - y_m| < \epsilon$ $\therefore \{y_n\}$ is Cauchy seq. in $\mathbb{R}$. But, we assume $\mathbb{R}$ is complete so Cauchy $\Rightarrow$ Convergent and $\exists x, y \in \mathbb{R}$ s.t. $x_n \longrightarrow x$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$.

We have a theorem (Th^m 2.1.10 of Guide) that $z_n \longrightarrow x + iy$ for $z_n = x_n + iy_n$ Iff $\begin{matrix} x_n \longrightarrow x \\ y_n \longrightarrow y \end{matrix}$.

Consequently the given real convergence of $x_n \longrightarrow x$ and $y_n \longrightarrow y$ implies $z_n \longrightarrow x + iy \in \mathbb{C}$.

Thus Cauchy sequences in $\mathbb{C}$ are convergent in $\mathbb{C}$ and we conclude $\mathbb{C}$ is COMPLETE. //