Same instructions as Mission 1. Enjoy!

Problem 25 Your signature below indicates you have:

(**a.**) I have read Sections 2.4-3.3 of Gamelin: ________________.

(**b.**) *I have read Cook's Guide to Sections 2.4-3.3 of Gamelin:*________________.

Problem 26 Prove the chain-rule for composition of complex valued functions of a complex variable. Use the criteria of Caratheodory as indicated in the guide in Section 2.2.

Problem 27 Work out the chain-rule for composition of $\gamma : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. Assume γ is a smooth map and assume f is holomorphic near the image of γ. I begin by reminding you from calculus III that if $f = (u, v)$ and $\gamma = (\gamma_1, \gamma_2)$ then

$$\frac{d}{dt}(f \circ \gamma) = \left(\frac{d}{dt}(u \circ \gamma),\ \frac{d}{dt}(v \circ \gamma)\right), \qquad \& \qquad \frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}\right).$$

where

$$\frac{d}{dt}(u \circ \gamma) = \nabla u(\gamma(t)) \bullet \frac{d\gamma}{dt} \qquad \& \qquad \frac{d}{dt}(v \circ \gamma) = \nabla v(\gamma(t)) \bullet \frac{d\gamma}{dt}.$$

In our current notation $(u, v) = u + iv$ hence the multivariate calculus above translates to:

$$\frac{d}{dt}(f \circ \gamma) = \left(\frac{d}{dt}(u \circ \gamma) + i\frac{d}{dt}(v \circ \gamma)\right) \qquad \& \qquad \frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}.$$

Given these results, **show** $\dfrac{d}{dt}(f \circ \gamma) = \dfrac{df}{dz}(\gamma(t))\dfrac{d\gamma}{dt}$.

Problem 28 Show $f(z) = \cosh^2 z - \sinh^2 z$ is constant on $\mathbb{C}$. Find the constant.

Problem 29 §2.3#5 (fun with CR-equations)

Problem 30 §2.3#6 (geometry of CR-equations)

Problem 31 §1.8#5 (fun with inverse tangent)

Problem 32 §2.4#4 (more fun with inverse tangent)

Problem 33 §2.5#1*d*, *e* (find harmonic conjugate)

Problem 34 §2.5#8 (use polar CR-equations to find harmonic conjugate)

Problem 35 §2.6#1*b* (level curves of conformal map appreciation)

Problem 36 §2.6#5 (find conformal map)

MISSION 3 SOLUTION

PROBLEM 26 Suppose $f \circ g$ is well-defined near p and $g'(p)$ and $f'(g(p))$ both exist then $(f \circ g)'(p) = f'(g(p))\, g'(p)$

By the theorem of Caratheodory $\exists\, \phi_f$ continuous near $u = g(p)$ such that $f(u) = f(g(p)) + \phi_f(u)(u - g(p))$, Likewise, $\exists\, \phi_g$ continuous near p such that $g(z) = g(p) + \phi_g(z)(z - p)$.
Moreover, $\lim_{u \to g(p)} \phi_f(u) = f'(g(p))$ and $\lim_{z \to p} \phi_g(z) = g'(p)$.
Consider the composite function $f \circ g$ for z

$$
\begin{aligned}
(f \circ g)(z) &= f(g(z)) \\
&= f(g(p)) + \phi_f(g(z))\,(g(z) - g(p)) \\
&= f(g(p)) + \phi_f(g(z))\, \phi_g(z)\,(z - p)
\end{aligned}
$$

Thus $\phi_{f \circ g}(z) = \phi_f(g(z))\, \phi_g(z)$ by inspection of the above algebra and we note by continuity of g and ϕ_f, ϕ_g,

$$\lim_{z \to p} \left[\phi_{f \circ g}(z) \right] = \phi_f(g(p))\, \phi_g(p) = \underline{f'(g(p))\, g'(p)} \; //$$

PROBLEM 27 Assume $f'(z)$ exists near each $z \in \gamma(\mathbb{R})$ where $\gamma : \mathbb{R} \to \mathbb{C}$ is a smooth path.

$$
\begin{aligned}
\frac{d}{dt}\left[f(\gamma(t)) \right] &= \frac{d}{dt}\left[f(x(t), y(t)) \right] \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \quad : \text{ where } \gamma = x + iy \\
&= \frac{df}{dz}\frac{dx}{dt} + i\frac{df}{dz}\frac{dy}{dt} \quad : \underline{\text{CR-eq'n give } \frac{df}{dz} = \frac{\partial f}{\partial x} \text{ \& } \frac{df}{dz} = \frac{1}{i}\frac{\partial}{\partial}} \\
&= \frac{df}{dz}\left(\frac{dx}{dt} + i\frac{dy}{dt} \right) = \frac{df}{dz}\frac{dz}{dt} = \underbrace{f'(\gamma(t))\frac{d\gamma}{dt}}_{\text{to be less notationally ambiguous}}
\end{aligned}
$$

☺

<u>Note</u>: $\frac{df}{dz} = \underbrace{u_x + iv_x}_{\frac{\partial f}{\partial x}} = v_y - iu_y = \underbrace{\frac{1}{i}(u_y + iv_y)}_{\frac{1}{i}\frac{\partial f}{\partial y}}$

$\boxed{\text{PROBLEM 28}}$

$$f(z) = \cosh^2 z - \sinh^2 z$$

$$\frac{df}{dz} = 2\cosh z \sinh z - 2\sinh z \cosh z = 0$$

Thus $f'(z) = 0 \quad \forall z \in \mathbb{C} \leftarrow$ connected domain.

$\therefore \quad f(z) = \text{constant} \quad \forall z \in \mathbb{C}$

However, $f(0) = (\cosh(0))^2 - (\sinh(0))^2 = 1 \quad \therefore \boxed{\cosh^2 z - \sinh^2 z = 1}$ for all $z \in \mathbb{C}$.

Remark: there is a less elegant sol$^{\underline{n}}$ to this. Just work out $\cosh z = \frac{1}{2}(e^z + e^{-z})$ and $\sinh z = \frac{1}{2}(e^z - e^{-z})$ squared and take difference... algebra happens and out pops 1.

$\boxed{\text{PROBLEM 29}}$ (§2.3#5 p. 50) If $f = u + iv$ is analytic, then $|\nabla u| = |\nabla v| = |f'|$

Given $f = u + iv$ holomorphic $\Rightarrow u_x = v_y$ and $u_y = -v_x$

thus $\nabla u = \langle u_x, u_y \rangle = \langle u_x, -v_x \rangle \therefore |\nabla u| = \sqrt{(u_x)^2 + (v_x)^2}$

also $\nabla v = \langle v_x, v_y \rangle = \langle v_x, u_x \rangle \therefore |\nabla v| = \sqrt{(v_x)^2 + (u_x)^2}$

However, $f'(z) = u_x + iv_x \Rightarrow |f'(z)| = \sqrt{u_x^2 + v_x^2} = |\nabla u| = |\nabla v|$.

$\boxed{\text{PROBLEM 30}}$ (§2.3#6)

If $f = u + iv$ is holomorphic on D then ∇v is obtained by rotating ∇u by 90°. In particular ∇u and ∇v are orthogonal

Note, $\nabla u \cdot \nabla v = \langle u_x, -v_x \rangle \cdot \langle v_x, u_x \rangle = u_x v_x - v_x u_x = 0$ hence $\nabla u \perp \nabla v$.

Moreover, the 90° is measure in positive sense as

$$\underbrace{e^{i\pi/2}}_{i}(u_x + iu_y) = iu_x - u_y = iv_y + v_x = \nabla v$$

Thus $\nabla u = u_x + iu_y$ rotated by $e^{i\pi/2}$ yields ∇v.

PROBLEM 31 (§1.8 #5)

Let S denote $[i, i\infty]$ and $[-i, -i\infty]$ the vertical slits along y-axis. Show $\frac{1+iz}{1-iz} \in (-\infty, 0]$ iff $z \in S$

Suppose $z \in S$ then $z = i + it$ for $t \geq 1$ or $z = -i - it$ for $t \geq 1$. Consider case $z = i + it$ for $t \in [1, \infty)$,

$$\frac{1+iz}{1-iz} = \frac{1+i(i+it)}{1-i(i+it)} = \frac{1-1-t}{1+1+t} = \frac{-t}{2+t} < 0 \quad \text{as } t \geq 1,$$

thus, $\frac{1+iz}{1-iz} \in (-\infty, 0]$. Next case $z = -i - it$ for $t \in [1, \infty)$,

$$\frac{1+iz}{1-iz} = \frac{1+i(-i-it)}{1-i(-i-it)} = \frac{1+1+t}{-t} = \frac{2+t}{-t} < 0 \quad \text{as } t \geq 1,$$

thus $\frac{1+iz}{1-iz} \in (-\infty, 0]$. Hence we've shown $\Leftarrow$ of the iff $= \Leftrightarrow$

Continued $\rightarrow$

Remark: Some students attacked the 1st part, but, I think no one quite appreciated what was asked in 2nd part of this problem. I hope my sol^n helps.

PROBLEM 31 continued (proof of the $\Rightarrow$ in the iff)

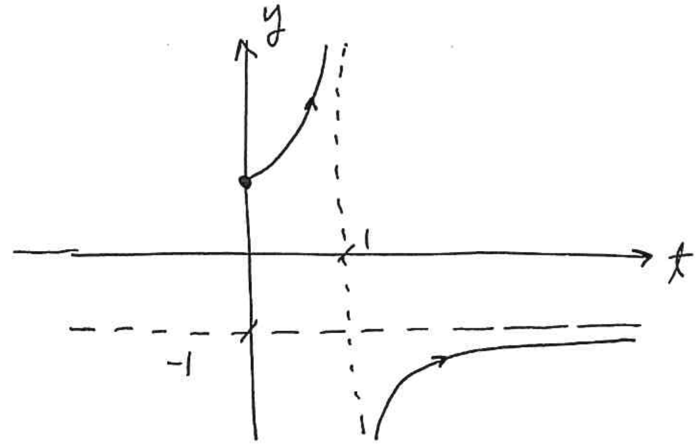
$$\frac{1+iz}{1-iz} = -t \quad \text{for} \quad t \geq 0$$

$$\Rightarrow \quad 1 + iz = izt - t$$

$$\Rightarrow \quad z(i - it) = -1 - t$$

$$\Rightarrow \quad -iz = \frac{-1-t}{t-1}$$

$$\Rightarrow \quad z = i\left[\frac{-1-t}{t-1}\right]$$

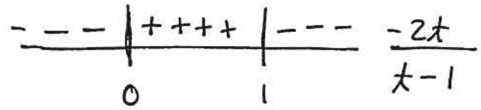


$y = \dfrac{-1-t}{t-1} \longrightarrow -1$ as $t \to \infty$

Observe, for $0 \leq t < 1$ we find $t - 1 < 0$ and $-1-t < 0$ hence $\left[\frac{-1-t}{t-1}\right] > 0$. Furthermore as $t \longrightarrow 1^-$ we see $\dfrac{-1-t}{t-1} \longrightarrow \dfrac{-2}{-0} = \infty$.

To see $\left[\frac{-1-t}{t-1}\right] \geq 1$ (apart from the graph) for $0 \leq t \leq 1$

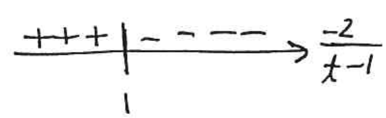
Note $\dfrac{-1-t}{t-1} - 1 \geq 0 \iff \dfrac{-1-t-t+1}{t-1} \geq 0 \iff \dfrac{-2t}{t-1} \geq 0$



thus $0 \leq t \leq 1 \Rightarrow z \in [i, i\infty)$.

Likewise if $t > 1$ then we may argue $\left[\frac{-1-t}{t-1}\right] \leq -1$ as

$$\frac{-1-t}{t-1} \leq -1 \iff \frac{-1-t}{t-1} + 1 \leq 0 \iff \frac{-1-t+t-1}{t-1} \leq 0 \iff \frac{-2}{t-1} \leq 0$$



thus $t > 1 \Rightarrow z \in [-i, -i\infty)$

Therefore, we find $z \in S$ in all cases for $\dfrac{1+iz}{1-iz} \in (-\infty, 0]$.

PROBLEM 31 continued, Show $f(z) = \mathrm{Tan}^{-1}(z) = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right)$ is a bijection from $\mathbb{C} - S'$ to $HS = \{w \in \mathbb{C} \mid -\pi < \mathrm{Im}(w) < \pi\}$

one-to-one and onto

Upto this point, we've merely showed $\frac{1+iz}{1-iz} \in \mathbb{C}_m^-$ provided $z \notin S'$.

Let $w = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right)$ and solve for $z \in \mathbb{C} - S'$

$$\Rightarrow 2iw = \mathrm{Log}\left(\frac{1+iz}{1-iz}\right)$$

$$\Rightarrow e^{2iw} = \frac{1+iz}{1-iz} \longrightarrow (1-iz)e^{2iw} = 1+iz$$

$$\longrightarrow e^{2iw} - 1 = iz + ize^{2iw}$$

$$\longrightarrow \frac{e^{2iw}-1}{i(1+e^{2iw})} = z$$

I claim $g(w) = \frac{e^{2iw}-1}{i(1+e^{2iw})}$ is an inverse function to $f(z)$. You can check

$f(g(w)) = w$ for each $w \in HS$

$g(f(z)) = z$ for each $z \in \mathbb{C} - S'$

In view of this 1-1 and onto are simple,

$$f(z_1) = f(z_2) \Rightarrow g(f(z_1)) = g(f(z_2)) \Rightarrow z_1 = z_2.$$

For $w \in HS$ choose $g(w) \in \mathbb{C} - S'$ and observe $f(g(w)) = w$ thus f is surjection (onto) HS.

PROBLEM 32) § 2.4 #4

Recall $\operatorname{Tan}^{-1}(z) = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iz}{1-iz}\right)$, $z \notin (-i\infty, -i] \cup [i, i\infty)$

Find derivative of $\operatorname{Tan}^{-1}(z)$. Find derivative of $\tan^{-1}(z)$ for any holomorphic branch function def^{n} on domain D.

$$\frac{d}{dz}\left[\operatorname{Tan}^{-1}(z)\right] = \frac{1}{2i}\frac{d}{dz}\left[\operatorname{Log}\left(\frac{1+iz}{1-iz}\right)\right]$$

$$= \frac{1}{2i}\cdot\frac{1-iz}{1+iz}\cdot\frac{d}{dz}\left[\frac{1+iz}{1-iz}\right]$$

$\frac{d}{dz}\operatorname{Log}(z) = \frac{1}{z}$ & chain-rule.

$$= \left(\frac{1}{2i}\right)\left(\frac{1-iz}{1+iz}\right)\left(\frac{i(1-iz)-(-i)(1+iz)}{(1-iz)^2}\right)$$

quotient rule.

$$= \frac{1}{2i}\left(\frac{1\cdot(i+\cancel{z}+i-\cancel{z})}{(1+iz)(1-iz)}\right)$$

$$= \frac{1}{(1+iz)(1-iz)}$$

$$= \frac{1}{1+z^2}.$$

Since $\tan^{-1}(z)$ differs by $\operatorname{Tan}^{-1}(z)$ by a constant on any branch $\Rightarrow \frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$.

Essentially, same story with $\operatorname{Log}_\alpha(z)$ we also have $\frac{d}{dz}\operatorname{Log}_\alpha(z) = \frac{1}{z}$.

PROBLEM 33 | §2.5 # 1d & 1e (harmonic conjugate hunting)

§2.5 #1d) $u = e^{x^2-y^2}\cos(2xy)$

Observe $e^{z^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy)$

Hence $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ for

$v = \text{Im}(e^{z^2})$ that is : $\boxed{v(x,y) = e^{x^2-y^2}\sin(2xy)}$

Remark : I also derive this in the section of the guide on harmonic forms ... see Ex. 3.3.3.

§2.5 #1e) $u = \tan^{-1}(y/x)$.

Oh, I'll behave, first show $u_{xx} + u_{yy} = 0$.

$$u_x = \frac{1}{1+y^2/x^2}\,\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$u_{xx} = \frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right] = -y\left(\frac{-2x}{(x^2+y^2)^2}\right) = \frac{2xy}{(x^2+y^2)^2}$$

$$u_y = \frac{1}{1+y^2/x^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$u_{yy} = \frac{\partial}{\partial y}\left[\frac{x}{x^2+y^2}\right] = x\left(\frac{-2y}{(x^2+y^2)^2}\right) = \frac{-2xy}{(x^2+y^2)^2}$$

Thus $u_{xx} + u_{yy} = 0$ hence u is harmonic on $\mathbb{C} - [-i\infty, i\infty]$ ($\mathbb{C}$ with $x=0$ removed)

To derive v solve $u_x = v_y \Rightarrow \boxed{\dfrac{-y}{x^2+y^2} = \dfrac{\partial v}{\partial y}}$

and $-u_y = v_x \Rightarrow \boxed{\dfrac{-x}{x^2+y^2} = \dfrac{\partial v}{\partial x}}$ simultaneously.

We solve both by $u = x^2+y^2$ substitution to obtain

$\boxed{v(x,y) = -\ln\sqrt{x^2+y^2}}$

§2.5 #1e Comment on PROBLEM 33

$$\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$$

$$\Rightarrow i\,\text{Log}(z) = i\ln|z| - \text{Arg}(z)$$

$$\Rightarrow \underbrace{-i\,\text{Log}(z)}_{f} = \underbrace{-i\ln|z|}_{iv} + \underbrace{\text{Arg}(z)}_{u}$$

Identify we're given $u = \text{Arg}(z)$ in this problem and we found $v = -\ln|z| = -\ln\sqrt{x^2+y^2}$.

This is often how these problems are made, take your favorite holomorphic fnct. $f(z)$ multiply by some complex # then let the students play...

PROBLEM 34 $\S 2.5 \# 81$

$$u(re^{i\theta}) = \theta \log(r) = \theta \ln(r)$$

Show harmonic via checking Laplace Eq^{n} in polar coordinates $\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0}$ Then find harmonic conjugate via polar CR-eq^{n}'s.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \ \& \ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$u_\theta = \ln(r), \quad u_{\theta\theta} = 0$$

$$u_r = \frac{\theta}{r} \Rightarrow r u_r = \theta \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] = \frac{1}{r}\frac{\partial}{\partial r}[\theta] = 0$$

Thus $\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ is true and we find u is harmonic. (award some credit for deriving Laplace's Eq^{n} in polars here, it's the chain-rule and some persistence.)

Continuing,

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \longrightarrow \frac{\partial v}{\partial \theta} = \theta \longrightarrow v = \frac{1}{2}\theta^2 + C_1(r)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \Rightarrow \ln(r) = -r\frac{dC_1}{dr}$$

$$\Rightarrow \frac{dC_1}{dr} = \frac{-\ln(r)}{r}$$

$$\Rightarrow C_1 = \int \frac{-\ln(r)\,dr}{r} = \int -w\,dw \ : \ w = \ln(r)$$

$$= -\frac{1}{2}w^2 + C_2$$

$$= -\frac{1}{2}(\ln(r))^2 + C_2$$

$$\therefore \boxed{v = \frac{1}{2}\theta^2 - \frac{1}{2}[\ln(r)]^2}$$

continued

PROBLEM 34 continued

$$u + iv = \theta \ln(r) + \frac{i}{2}(\theta^2 - \ln^2(r))$$
$$= \frac{-i}{2}\left(\ln^2(r) + 2i\theta\ln(r) - \theta^2\right)$$
$$= -\frac{i}{2}\left[(\ln(r))^2 + 2i\theta\ln(r) + (i\theta)^2\right]$$
$$= \frac{-i}{2}\left[\ln(r) + i\theta\right]^2$$
$$= \underline{-\frac{i}{2}\left[\log(z)\right]^2}.$$

Remark: let us derive Laplace's Eq^{n} in Polar coordinates for completeness of this sol^{n}. The key is simply: $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta} \qquad \left[\begin{array}{l} \frac{\partial r}{\partial x} = \frac{2x}{2r} = \frac{x}{r} \\ \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} \end{array}\right.$$
$$= \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta}$$
$$= \underbrace{\left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right]}_{\partial_x} u$$

Likewise,

$$\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial \theta} = \frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta} = \underbrace{\left[\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right]}_{\partial_y} u$$

Hence,

$$\partial_x\partial_x + \partial_y\partial_y = \left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right) +$$
$$+ \left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)$$
$$= (\cos^2\theta + \sin^2\theta)\,\partial_r^2 + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta)\,\partial_\theta^2 + \cdots$$

continued

PROBLEM 34 continued

$$(\partial_x^2 + \partial_y^2)u = \left(\cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)\left(\cos\theta\, u_r - \frac{1}{r}\sin\theta\, u_\theta\right) + \left(\sin\theta\,\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)\left(\sin\theta\, u_r + \frac{1}{r}\cos\theta\, u_\theta\right)$$

$$= \underline{\cos^2\theta\, u_{rr}} - \cos\theta\frac{\partial}{\partial r}\left[\frac{1}{r}\sin\theta\, u_\theta\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left[\cos\theta\, u_r\right] + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[\sin\theta\, u_\theta\right]$$

$$+ \underline{\sin^2\theta\, u_{rr}} + \sin\theta\frac{\partial}{\partial r}\left[\frac{1}{r}\cos\theta\, u_\theta\right] + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left[\sin\theta\, u_r\right] + \frac{\cos\theta}{r^2}\frac{\partial}{\partial\theta}\left[\cos\theta\, u_\theta\right]$$

$$= u_{rr} + \cancel{\frac{\cos\theta\sin\theta}{r^2}u_\theta} - \cancel{\frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}} + \underline{\underline{\frac{\sin^2\theta}{r}u_r}} - \cancel{\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r}}$$

$$+ \cancel{\frac{\sin\theta}{r^2}\cos\theta\, u_\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$+ \cancel{\frac{\sin\theta\cos\theta}{-r^2}u_\theta} + \cancel{\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}} + \underline{\underline{\frac{\cos^2\theta}{r}u_r}} + \cancel{\frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r}}$$

$$- \cancel{\frac{\cos\theta\sin\theta}{r^2}u_\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$= \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)(u)$$ ← also useful version

Note, $\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] = \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}$ thus

$$\underbrace{u_{xx} + u_{yy} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}}_{\text{used this earlier.}}$$

PROBLEM 35 | § 2.6 #16

Sketch families of level curves of u and v for $f(z) = \frac{1}{z^2}$. Comment on where $f(z)$ is conformal

Let $z = re^{i\theta}$ then $f(z) = \dfrac{1}{(re^{i\theta})^2} = \dfrac{e^{-2i\theta}}{r^2} = \dfrac{\cos(2\theta) - i\sin(2\theta)}{r^2}$

Hence $u = \dfrac{\cos(2\theta)}{r^2}$ and $v = \dfrac{-\sin(2\theta)}{r^2}$

and the level curves may be nicely plotted via polar-techniques.

Note: $f'(z) = \dfrac{-2}{z^3} \neq 0$ for $z \neq 0$

So we expect $f(z)$ conformal on $\mathbb{C}^\times$.

My crude polar sketches ↙ illustrate just that.

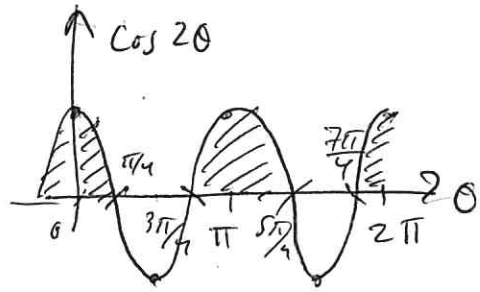
Remark: I gave you all permission to use technology on this if you asked so.... perhaps this wasn't too hard.

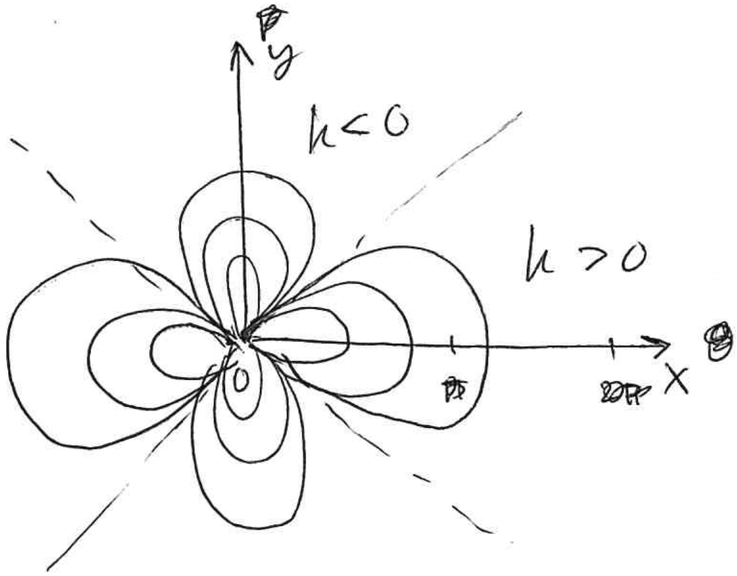
PROBLEM 35 : how to do the sketch sans-techno

$$\frac{\cos 2\theta}{r^2} = k$$

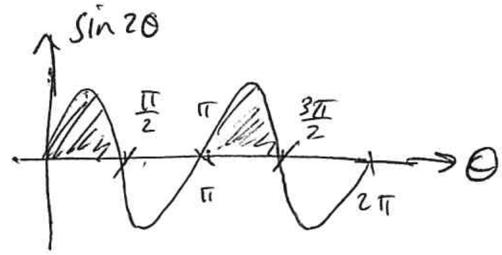
$$\frac{\cos 2\theta}{k} = r^2$$

$$r = \sqrt{\frac{\cos(2\theta)}{k}}$$



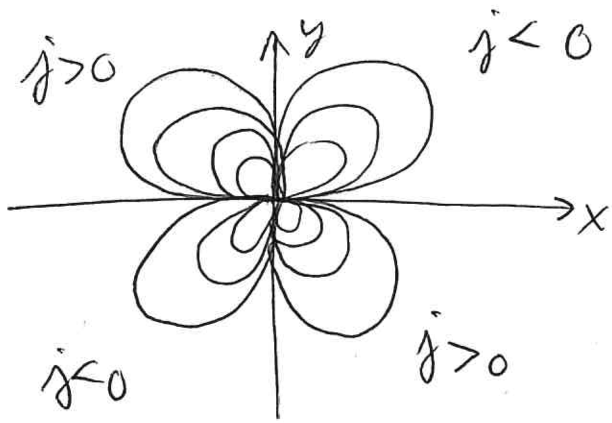


$$-\frac{\sin 2\theta}{r^2} = j$$



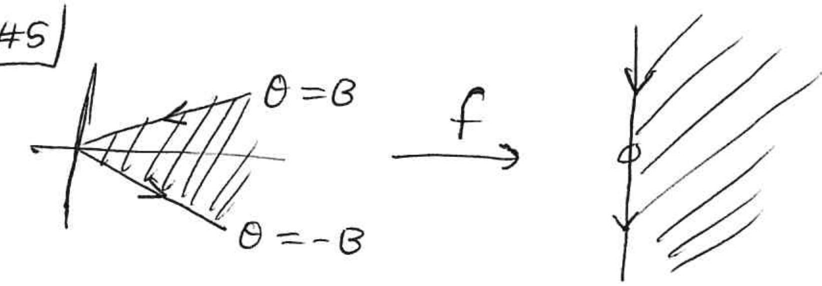
$$r = \sqrt{\frac{\sin 2\theta}{-j}}$$

Superimpose these to see orthogonal web at all points except $z=0$.



PROBLEM 36 §2.6 #5

Find map of



I envision $\theta = \beta \longmapsto \theta = \frac{\pi}{2}$

$\theta = -\beta \longmapsto \theta = -\frac{\pi}{2}$

I know from experience with z^n that

z^n maps $\frac{2\pi}{n} \longrightarrow 2\pi$ sector.

or $\frac{\pi}{n} \longrightarrow \pi$ sector

this suggests $2\beta = \frac{\pi}{n} \hookrightarrow n = \frac{\pi}{2\beta}$

So, use $\boxed{f(z) = z^{\frac{\pi}{2\beta}}}$

Note, $r>0$

$$\begin{cases} f(re^{i\beta}) = (re^{i\beta})^{\frac{\pi}{2\beta}} = r^{\frac{\pi}{2\beta}} e^{i\beta\frac{\pi}{2\beta}} = ir^{\frac{\pi}{2\beta}} \\ f(re^{-i\beta}) = r^{\frac{\pi}{2\beta}} e^{-\frac{i\pi}{2}} = -ir^{\frac{\pi}{2\beta}} \\ f(r) = r^{\frac{\pi}{2\beta}} \end{cases}$$

(this shows for sure the wedge maps to <u>right</u> half-plane)