Copying answers and steps is strictly forbidden. Same instructions as Mission 1. Do not fold. Thanks!

Problem 21 Show $f(z) = z + \overline{z}$ is not complex differentiable at any point in $\mathbb{C}$.

Problem 22 Let $f = u + iv$ where $f(z) = \sin z$. Show u, v are continuously differentiable by explicitly calculating their formulas in terms of x, y and then show $\frac{d}{dz} \sin z = \cos z$ via the CR-equations written in the form $f'(z) = u_x + iv_x$.

Problem 23 Let $R = \{z \in \mathbb{C} \mid 1 < |z| < 2,\ \mathfrak{Re}(z) > 0,\ \mathfrak{Im}(z) > 0\}$. Describe the shape of the transformed region $f(R)$ given:

(a.) $f(z) = iz$

(b.) $f(z) = z^2$

(c.) $f(z) = 1/z$

Problem 24 Find a sharp bound for $f(z) = e^z$ for $z = x + iy$ with $0 \leq x, y \leq 1$.

Problem 25 Find the harmonic conjugate v of $u(x, y) = -e^x y \sin(y) + e^x x \cos(y)$. Also, find $f = u + iv$ and write f as a function of z.

Problem 26 Suppose $f = u + iv$ is a holomorphic function on $\mathbb{C}$ and $u_x(x, y) = x^2 + y$ everywhere. Is this possible? If so, find all $f(z)$ for which $u_x(x, y) = x^2 + y$.

Problem 27 Suppose $f = u + iv$ is holomorphic at a point $z_o = x_o + iy_o$.

(a.) $|f'(z_o)| = ||(\nabla u)(x_o, y_o)|| = ||(\nabla v)(x_o, y_o)||$
(here we compare the length of the complex number $f'(z_o)$ with the lengths of the two-dimensional vectors $(\nabla u)(x_o, y_o)$ and $(\nabla v)(x_o, y_o)$

(b.) Show that $(\nabla u)(x_o, y_o)$ is orthogonal to $(\nabla v)(x_o, y_o)$.

Problem 28 Show $f : R \to T$ defined by $f(z) = \dfrac{1}{2i}\text{Log}\left(\dfrac{1+iz}{1-iz}\right)$ defines a bijection where

$$R = \{z \in \mathbb{C} \mid z \notin [i, i\infty] \cup [-i\infty, -i]\}$$

and

$$T = \{w \in \mathbb{C} | -\pi < \mathfrak{Im}(w) < \pi\}$$

Also, show $f(z) = \text{Tan}^{-1}(z)$.

Problem 29 The beauty of a holomorphic mapping has many facets. In particular, if $f = u + iv$ is holomorphic it is very neat to examine the level curves of u and v. Graph level curves for u and v of $f(z) = z^2 e^z$. Where is the mapping conformal ? What does this mean geometrically about the level curves of u and v where they intersect? (obviously you should use Mathematica, Maple or some technology to make these graphs, I used Desmos.)

Problem 30 We wish to show $\mathbb{C}$ is complete. The heart of the claim follows from $|\mathfrak{Re}(w)| \leq |w|$ and $|\mathfrak{Im}(w)| \leq |w|$ paired with the fact we **assume** $\mathbb{R}$ is complete. **Show** $\mathbb{C}$ **is complete**. *To sketch the solution: assume $z_n = x_n + iy_n$ is a Cauchy sequence in $\mathbb{C}$. Apply the given inequalities to argue x_n and y_n are Cauchy real sequences and hence converge to x and y respective. Finish the argument by showing $z_n \to x + iy$.*