Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 49 Your signature below indicates you have:

(a.) I have read Sections 4.1 – 4.8 of Gamelin: ____________________.

(b.) I have read Cook's Guide to Sections 4.1 – 4.4 of Gamelin::________________.

Problem 50 #4 of section $IV.1$

Problem 51 #5 of section $IV.1$

Problem 52 #2 of section $IV.2$

Problem 53 #1 of section $IV.3$

Problem 54 #4 of section $IV.4$

Problem 55 #1a, c, f, g, h of section $IV.4$ (pg. 116)

Problem 56 #3 of section $IV.4$

Problem 57 #2 of section $IV.5$

Problem 58 #4 of section $IV.5$ ← NOT SOLVED (YET) CAN DO LATER FOR BONUS !?

Problem 59 #1, 4 of section $IV.8$

Problem 60 #10 of section $IV.8$

PROBLEM 50 (§ 4.1 # 4, pg. 106) Show $\int_{\partial D} \bar{z}\,dz = 2i\,\text{Area}(D)$.

$$\int_{\partial D} \bar{z}\,dz = \int_{\partial D} (x-iy)(dx+idy) = \int_{\partial D} x\,dx + y\,dy + i\int_{\partial D} x\,dy - y\,dx$$

$$= \cancel{\int_{\partial D} d\left(\tfrac{1}{2}x^2+\tfrac{1}{2}y^2\right)}^{\,0} + i\int_{\partial D} x\,dy - y\,dx$$

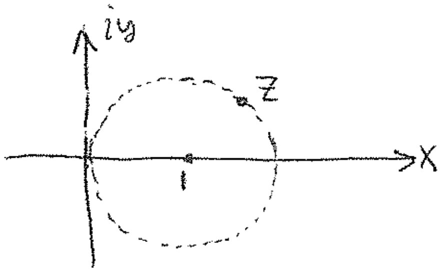
$$= i\iint_D \left(\frac{\partial}{\partial x}[x] - \frac{\partial}{\partial y}[-y]\right) dA$$

Green's Theorem.

$$= 2i\iint_D dA$$

$$= \underline{2i\,\text{Area}(D)}.$$

PROBLEM 51 (§4.1 #5, p. 106) Show that $\left| \oint_{|z-1|=1} \frac{e^z}{z+3}\,dz \right| \le 2\pi e^2$



this picture shows $|y| \le 1$ and $|x-1| \le 1$
$-1 \le y \le 1$, $0 \le x \le 2$.

didn't use this.

$$|z+3| = |z-1+4| \ge \big||z-1| \mp 4\big|$$

$$|z+3| \ge \big||z|-3\big| = \begin{cases} |z|-3 & : |z|>3 \\ 3-|z| & : |z| \le 3 \end{cases}$$

For $|z-1|=1$ it is clear that $|z| \le 2$

Hence $|z+3| \ge 3-|z| \ge 1$ ($z=2$ gives equality here)

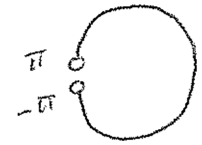
$$|e^z| = |e^x e^{iy}| = |e^x| \le e^2 \text{ for } |z-1|=1.$$

$|z+3| \ge 3-|z| \ge 3-|2| = 1$. Thus, for $|z-1|=1$,

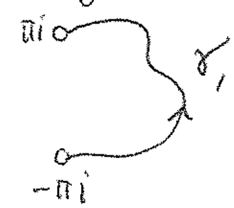
$$\left|\frac{e^z}{z+3}\right| \le \frac{e^2}{1} = M \text{ then apply } M\text{-}L \text{ ~~theorem~~}$$

estimate, $\left| \oint_{|z-1|=1} \frac{e^z}{z+3}\,dz \right| \le e^2(2\pi) = \underline{2\pi e^2}$.

$\boxed{\text{PROBLEM 52}}$ (§4.2 # 2) evaluate $\int_\gamma \frac{dz}{z}$ for path γ that travels from $-\pi i$ to πi in right half plane

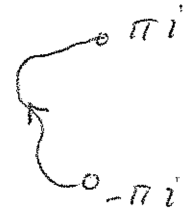


oops, no, he means



γ_1

and then γ_2 :



. In both cases use FTC for complex integral and an appropriate primitive $\left(F_1'(z) = \frac{1}{z} \text{ for } Re(z) > 0 \right.$ and $\left. F_2'(z) = \frac{1}{z} \text{ for } Re(z) < 0 \right)$

(I added a bit of insight to problem statement)

For γ_1 we are free to use $Log(z)$ as $\frac{d}{dz} Log(z) = \frac{1}{z}$ for all $z \in \mathbb{C}^-$ which includes $z \in \mathbb{C}$ s.t. $Re(z) > 0$.

$$\int_{\gamma_1} \frac{dz}{z} = Log(\pi i) - Log(-\pi i)$$

$\pi i = \pi e^{\frac{\pi i}{2}}$

$-\pi i = \pi e^{-\frac{\pi i}{2}}$

$$= \cancel{\ln \pi} + i\frac{\pi}{2} - \cancel{\ln \pi} + \frac{\pi i}{2}$$

$$= \boxed{\pi i}$$

We use $Log_0(z) = \ln|z| + i\, Arg_0(z)$ for γ_2 recall $Arg_0(z) \in arg(z) \cap (0, 2\pi]$. Hence,

$$\int_{\gamma_2} \frac{dz}{z} = Log_0(\pi i) - Log_0(-\pi i)$$

$$= Log_0(\pi e^{\frac{\pi i}{2}}) - Log_0(\pi e^{\frac{3\pi i}{2}})$$

$$= \ln \pi + \frac{\pi i}{2} - \left(\ln \pi + \frac{3\pi i}{2}\right)$$

$$= \boxed{-\pi i}$$

PROBLEM 53 (#1 of §4.3 p. 111)

By integrating $\exp(-z^2/2)$ around rectangle with vertices $\pm R$, $it \pm R$ and sending $R \to \infty$, show that:

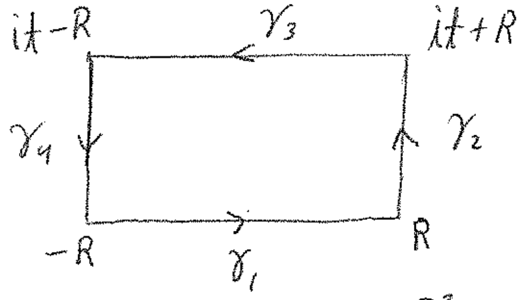
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx}\, dx = e^{-t^2/2}, \quad \infty < t < \infty$$

Use the known value of the integral for $t = 0$.

(this shows $e^{-x^2/2}$ has $\mathcal{F}(e^{-x^2/2})(t) = (e^{-x^2/2})(t)$.)

Fourier transform.

Let ∂D be the rectangle path below



Cauchy's Thm gives $\displaystyle\int_{\partial D} e^{-z^2/2}\, dz = 0.$

γ_2 : $x = R$, $y = u$ for $0 \le u \le t$

$dx = 0$, $dy = du$; $dz = i\, du$

$$\exp(-z^2/2) = \exp\left(\frac{-(R+iu)^2}{2}\right) = \exp\left(\frac{-R^2 - 2Riu + u^2}{2}\right)$$

$$|e^{-z^2/2}| = \left|\exp\left(\frac{-R^2}{2} + \frac{u^2}{2}\right)\right| \le e^{t^2/2} e^{-R^2/2}$$

$$L(\gamma_2) = t$$

$$\left|\int_{\gamma_2} e^{-z^2/2} dz\right| \le e^{t^2/2} e^{-R^2/2}\, t \longrightarrow 0 \quad \text{as } R \to \infty.$$

(holding t fixed)

Likewise $\displaystyle\int_{\gamma_4} e^{-z^2/2} dz \longrightarrow 0$ as $R \to \infty$ by similar ML-estimate for γ_4 path. Continuing

PROBLEM S3 continued

$-\gamma_3$: $dz = dx$ as $z = x + it$ for $-R \leq x \leq R$

$$\int_{\gamma_3} e^{-z^2/2}\, dz = \int_{-R}^{R} \exp\left(\tfrac{-1}{2}(x+it)^2\right) dx$$

$$= \int_{-R}^{R} \exp\left(\tfrac{-1}{2}(x^2 + 2ixt - t^2)\right) dx$$

$$= \int_{-R}^{R} e^{-x^2/2} e^{-itx} e^{t^2/2}\, dx$$

$$= e^{t^2/2} \int_{-R}^{R} e^{-x^2/2} e^{-itx}\, dx$$

γ_1 : $dz = dx$ as $z = x$ for $-R \leq x \leq R$

$$\int_{\gamma_1} e^{-z^2/2}\, dz = \int_{-R}^{R} e^{-x^2/2}\, dx$$

Thus, by $\int_{\partial D} e^{-z^2/2} dz = 0 \Longrightarrow \int_{\gamma_1} e^{-z^2/2} dz = \int_{-\gamma_3} e^{-z^2/2} dz$ we obtain,

$$\int_{-R}^{R} e^{-x^2/2}\, dx = e^{t^2/2} \int_{-R}^{R} e^{-x^2/2} e^{-itx}\, dx$$

As $R \to \infty$ a standard result of calculus III gives the LHS $\to \sqrt{2\pi}$ thus

$$\sqrt{2\pi} = e^{t^2/2} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx}\, dx \Longrightarrow \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx}\, dx = e^{-t^2/2}}$$

//

$\boxed{\text{PROBLEM 54}}$ (§4.4 #4, pg. 117) Let D be a bounded domain, with smooth boundary ∂D and let $z_0 \in D$. Using Cauchy Integral formula, show that $\exists$ constant C s.t.

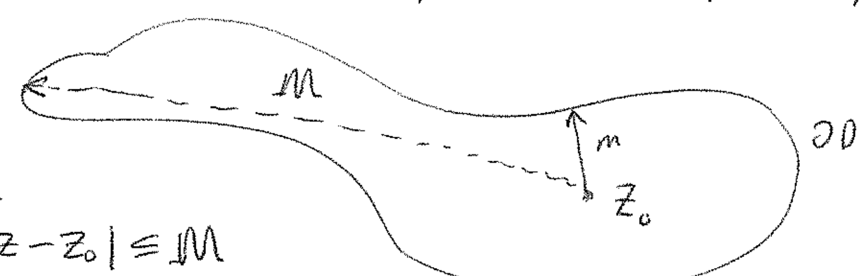
$$|f(z_0)| \leq C \sup\{|f(z)| : z \in \partial D\}$$

for any function $f(z)$ analytic on $D \cup \partial D$. By applying this estimate to $[f(z)]^n$, taking n^{th} roots, and letting $n \to \infty$, show that the estimate holds with $C = 1$.

Begin by applying Cauchy's $\int$-formula to $f(z)$ analytic on $D \cup \partial D$ hence for $z_0 \in D$ we have

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)\,dz}{z - z_0}$$

Consider, as D is bounded, and $z_0 \in D$, $\exists\ m, M$



for which

$$m \leq |z - z_0| \leq M$$

Thus, apply M-L estimate, (well, note $|f(z)| \leq \sup\{|f(z)| : z \in \partial D\}$

$$\left| \int_{\partial D} \frac{f(z)}{z - z_0}\,dz \right| \leq \frac{L}{m} \sup\{|f(z)| : z \in \partial D\}$$

$$\Rightarrow \quad |f(z_0)| \leq \frac{L}{2\pi m} \sup\{|f(z)| : z \in \partial D\}$$

If D is a disk and z_0 is its center then $L = 2\pi m$ and we obtain $|f(z_0)| \leq \sup\{|f(z)| : z \in \partial D\}$.

(I'll stop here, I'm interested to see if any of you found their way through what Gamelin sketches)

$\boxed{\text{PROBLEM 54}}$ other option: Prove that polynomial in z w/o zeros is constant, using Cauchy's theorem along the following lines: If $P(z)$ is nonconstant polynomial then write $P(z) = P(0) + zQ(z)$ and divide by $zP(z)$ and integrate around large circle. This will lead to $\rightarrow\leftarrow$ that $P(z)$ has no zeros.

Notice, $P(z) = P(0) + P'(0)z + \frac{1}{2}P''(0)z^2 + \cdots + \frac{1}{n!}P^{(n)}(0)z^n$ can be easily derived just as in the real case, simply evaluate $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ at $z=0$ to obtain $P(0) = a_0$ then $P'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$ hence $P'(0) = a_1$ etc... $a_n = \frac{1}{n!}P^{(n)}(0)$. We have

$$P(z) = P(0) + \Big(\underbrace{P'(0) + \cdots \frac{1}{n!}P^{(n)}(0)z^{n-1}}_{\text{call this } Q(z).}\Big)z$$

Now, divide by $zP(z)$,

$$\frac{P(z)}{zP(z)} = \frac{P(0)}{zP(z)} + \frac{Q(z)}{P(z)}$$

Hence, $\frac{1}{z} = \frac{P(0)}{zP(z)} + \frac{Q(z)}{P(z)}$. Now, integrate around $|z| = R$ for $R \gg 0$. (call this circle C_R)

$$\underbrace{\int_{C_R} \frac{dz}{z}}_{\substack{\| \\ 2\pi i}} = \underbrace{\int_{C_R} \frac{P(0)\,dz}{zP(z)}}_{\substack{\text{will tend} \\ \text{to zero as} \\ R \to \infty}} + \underbrace{\int_{C_R} \frac{Q(z)\,dz}{P(z)}}_{0} = \int_{C_R} \frac{P(0) + zQ(z)}{zP(z)}\,dz$$

if $P(z) \neq 0$ $\forall z \in \mathbb{C}$ since $\frac{Q(z)}{P(z)}$ holomorphic on C_R in such a case.

PROBLEM 54 continued

Assume $P(z) \neq 0 \quad \forall z \in \mathbb{C}$ where

$P(z) = a_0 + a_1 z + \cdots + a_n z^n$. Write $P(z) = P(0) + z\,Q(z)$

and observe, for $z \neq 0$,

$$\frac{1}{z} = \frac{P(z)}{zP(z)} = \frac{P(0) + z\,Q(z)}{zP(z)} = \frac{P(0)}{z(P(z))} + \frac{Q(z)}{P(z)}.$$

Notice as $z \to \infty$ clearly $|zP(z)| \to \infty$ hence

$\left| \frac{P(0)}{zP(z)} \right| \leq \frac{M}{R^{n+1}}$ for $R \gg 0$ and $|z| \geq R$. For

such $R > 0$ we observe, ($Q(z)/P(z)$ holomorphic on $\mathbb{C}$ by assumption $P(z) \neq 0$)

$$\int_{C_R} \frac{dz}{z} = \int_{C_R} \frac{P(0)\,dz}{zP(z)} + \int_{C_R} \frac{Q(z)}{P(z)}\,dz$$

$$\Rightarrow \quad |2\pi i| \leq \frac{2\pi R}{R^{n+1}} + 0$$

Yet as $R \to \infty$ we find $2\pi \leq 0$ $\rightarrow\!\leftarrow$.

Hence, $P(z) \neq 0 \quad \forall z \in \mathbb{C}$ is an impossible truth.

It must be $\exists\, z_0 \in \mathbb{C}$ for which $P(z_0) = 0$.

Remark: details for showing $\frac{P(0)}{zP(z)}$ are given in great detail in the previous mission 4 PROBLEM 44 concerning §3.5 #3. (I stole an argument from Churchill p. 131 of 6^{th} Ed. where a nice argument is offered)

PROBLEM 55 / § 4.4 # 1a, c, f, g, h (p. 116)

1a.) $\displaystyle\oint_{|z|=2} \frac{z^n\,dz}{z-1} = 2\pi i\,(z^n)\Big|_{z=1} = \boxed{2\pi i}$ ($n>0$ given)

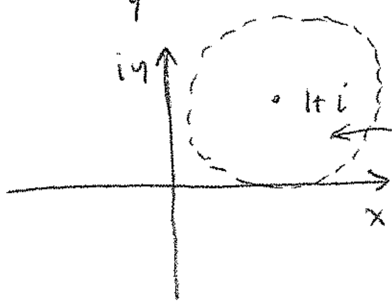
$$\int \frac{f(z)\,dz}{(z-z_0)^m} = \frac{2\pi i\, f^{(m-1)}(z_0)}{(m-1)!}$$

this is often what we need to solve typical problems.

1c.) $\displaystyle\oint_{|z|=1} \frac{\sin z}{z}\,dz = 2\pi i \sin(0) = \boxed{0.}$

1f.) $\displaystyle\int_{|z-1-i|=\frac{5}{4}} \frac{\mathrm{Log}(z)}{(z-1)^2}\,dz = 2\pi i\,\frac{d}{dz}\Big[\mathrm{Log}(z)\Big]\Big|_{z=1} = 2\pi i\,\frac{1}{z}\Big|_1 = \boxed{2\pi i}$

$|z-1-i| = \frac{5}{4} = 1.25 < \sqrt{2} \approx 1.414$



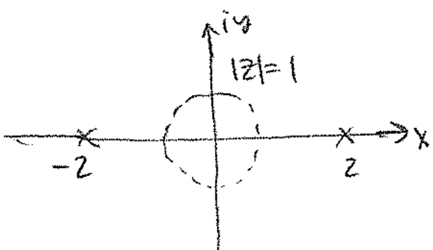
Log(z) holomorphic in and on the given contour.

1g.) $\displaystyle\oint_{|z|=1} \frac{dz}{z^2(z^2-4)e^z} = \oint_{|z|=1} \frac{\left(\frac{e^{-z}}{z^2-4}\right)}{z^2}\,dz = 2\pi i\,\frac{d}{dz}\Big|_{z=0}\left(\frac{1}{e^z(z^2-4)}\right)$

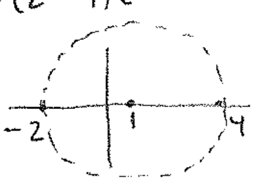
$\displaystyle = 2\pi i\left(\frac{-1}{[e^z(z^2-4)]^2}\left(e^z(z^2-4)+2ze^z\right)\right)\Big|_{z=0}$

$\displaystyle = 2\pi i\left(\frac{-1}{16}(-4)\right)$

$\displaystyle = \frac{\pi i}{2}$



1h.) $\displaystyle\oint_{|z-1|=3} \frac{dz}{z(z^2-4)e^z} =$



sorry folks, this has singularity on $|z-1|=3$ at $z=-2$ where $z^2-4=0$. in appropriate at this juncture.

PROBLEM 56 (§ 4.4 # 3, p. 117)

Use Cauchy's $\int$-f'la to derive $u(z_0) = \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \frac{d\theta}{2\pi}$ for $z_0 \in D$ where $u(z)$ is harmonic on D and the closed disk $|z - z_0| \leq \rho$ is within D.

Consider γ parametrized as $\gamma(t) = z_0 + \rho e^{it}$ for $0 \leq t \leq 2\pi$. We assume $[\gamma] \subseteq D$ and $u(z)$ is harmonic on D. By Gamelin, pg. 83, we have Th^m that $\exists\, v$ harmonic on z such that $|z - z_0| \leq \rho$ (indeed, on a larger disk containing the ρ-disk thus $f = u + iv$ is holomorphic on and near γ)

Apply Cauchy's Integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)\, dz}{z - z_0}$$

using $t = \theta$ as we often do for circles.

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i \rho e^{i\theta} d\theta$$

$$= \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})\, d\theta}{2\pi}$$

$$= \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \frac{d\theta}{2\pi} + i \int_0^{2\pi} v(z_0 + \rho e^{i\theta}) \frac{d\theta}{2\pi}$$

But, $f(z_0) = u(z_0) + iv(z_0)$ hence we obtain,

$$u(z_0) = \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \frac{d\theta}{2\pi} \quad //$$

$\boxed{\text{PROBLEM 57}}$ §4.5 #2, p. 119

Suppose there exists a disk $D = \{ z \in \mathbb{C} \mid |z - z_0| \leq \varepsilon \}$ for which the <u>entire</u> function $f(z)$ takes on no value; that is, $f(z) \notin D \quad \forall z \in \mathbb{C}$. Observe that

$$g(z) = \frac{1}{f(z) - z_0}$$

is bounded as $f(z) \neq z_0 \quad \forall z \in \mathbb{C}$. Moreover, as $f'(z)$ exists for all $z \in \mathbb{C}$ we have $g'(z) = \dfrac{-f'(z)}{(f(z) - z_0)^2}$ $\forall z \in \mathbb{C}$ thus $g(z)$ is entire <u>and</u> bounded. By Liouville's Th$^{\underline{m}}$ $g(z) = k \Rightarrow k = \dfrac{1}{f(z) - z_0}$ and we may derive $f(z) = z_0 + \dfrac{1}{k}$ (which is constant).

Remark: Notice, this little argument suggests entire functions take <u>most</u> values in $\mathbb{C}$, you can anticipate $\sin z = 2$ has a sol$^{\underline{n}}$, we can't delete a disk of values $\Rightarrow$ it's quite improbable to pick a value which an entire fnct does not attain. The Th$^{\underline{m}}$ of Casorati - Weierstrauss is in some sense dual to this remark (see p. 175 where we learn an "<u>essential</u>" isolated singularity approaches <u>all</u> values in $\mathbb{C}$ for various sequences approaching the singularity.

PROBLEM 59 continued

I just used that in the notation of §4.8 the CR-eq^{e}'s can be expressed as $\frac{\partial f}{\partial \bar{z}} = 0$ and $\frac{\partial f}{\partial z} = 0 \Rightarrow f$ antiholomorphic. Let me expand on that observation (about $\frac{\partial f}{\partial \bar{z}} = 0$)

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial_x + i\partial_y}{2}\right)(u+iv) = \underbrace{\tfrac{1}{2}(u_x - v_y) + \tfrac{i}{2}(v_x + u_y) = 0}_{u_x = v_y \ \& \ u_y = -v_x.}$$

(d.) h harmonic, then any m^{th} order partial derivative of h is linear combination of $\frac{\partial^m h}{\partial z^m}$ and $\frac{\partial^m h}{\partial \bar{z}^m}$

This claim follows from the identities we observed at the outset,

$$\partial_x = \partial_z + \partial_{\bar{z}} \quad \& \quad \partial_y = i(\partial_z - \partial_{\bar{z}})$$

Since these operators commute, the binomial Th$^{\underline{m}}$ applies and we can calculate

$$\partial_x^m \partial_y^n = \sum_{k=0}^{m}\sum_{j=0}^{n} i^n \binom{m}{k}\binom{n}{j} \partial_z^{m-k} \partial_{\bar{z}}^{k} \partial_z^{n-j} \partial_{\bar{z}}^{j} (-1)^j$$

$$= \sum_{k=0}^{m}\sum_{j=0}^{n} (-1)^j (i)^n \binom{m}{k}\binom{n}{j} \partial_z^{m+n-k-j} \partial_{\bar{z}}^{k+j}$$

observe that the $(m+n)$-th partial derivative in x, y is a linear combination of $(m+n-k-j) + (k+j) = m+n$ order derivatives in $z, \bar{z}$. (I changed m to $m+n$ but this certainly suffices)

PROBLEM 59 § 4.8 # 1, 4 pg.

1.) $\underbrace{\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \;\&\; \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}_{\text{def}^{n} \text{ given by Gamelin on p. 124.}}$

$$\frac{\partial}{\partial z}(z) = \frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right)(x+iy) = \frac{1}{2}\partial_x x + \frac{i}{2i}\partial_y y = 1.$$

$$\frac{\partial}{\partial \bar{z}}(z) = \frac{1}{2}\left(\partial_x + i\partial_y\right)(x+iy) = \frac{1}{2}\partial_x x + \frac{i^2}{2}\partial_y y = \frac{1}{2} - \frac{1}{2} = 0.$$

$$\frac{\partial}{\partial z}(\bar{z}) = \frac{1}{2}\left(\partial_x - i\partial_y\right)(x-iy) = \frac{1}{2}\partial_x x + \frac{i^2}{2}\partial_y y = \frac{1}{2} - \frac{1}{2} = 0.$$

$$\frac{\partial}{\partial \bar{z}}(\bar{z}) = \frac{1}{2}\left(\partial_x + i\partial_y\right)(x-iy) = \frac{1}{2}\partial_x x - \frac{i^2}{2}\partial_y y = \frac{1}{2} + \frac{1}{2} = 1.$$

4.) Convert $\partial_x^2 + \partial_y^2$ to $\partial_z \partial_{\bar{z}}$ notation.

Notice $\partial_x = \partial_z + \partial_{\bar{z}}$ whereas $\partial_y = \frac{1}{i}(\partial_{\bar{z}} - \partial_z)$

or $\partial_y = i(\partial_z - \partial_{\bar{z}})$. Therefore,

$$\partial_x^2 + \partial_y^2 = (\partial_z + \partial_{\bar{z}})(\partial_z + \partial_{\bar{z}}) + i^2(\partial_z - \partial_{\bar{z}})(\partial_z - \partial_{\bar{z}})$$

$$= \cancel{\partial_z^2} + \partial_{\bar{z}}\partial_z + \partial_z\partial_{\bar{z}} + \cancel{\partial_{\bar{z}}^2} - (\cancel{\partial_z\partial_z} - \partial_z\partial_{\bar{z}} - \partial_{\bar{z}}\partial_z + \cancel{\partial_{\bar{z}}^2})$$

$$= 4\,\partial_{\bar{z}}\partial_z$$ as we may show $\partial_z\partial_{\bar{z}} = \partial_{\bar{z}}\partial_z$ follows from $\partial_x\partial_y = \partial_y\partial_x$.

(a.) h harmonic $\iff h_{xx} + h_{yy} = 0 \iff \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0.$

(b.) h harmonic $\iff \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0 \iff \frac{\partial}{\partial \bar{z}}\left(\frac{\partial h}{\partial z}\right) = 0$

$\iff h'(z)$ is holomorphic.

(c.) h harmonic $\iff \frac{\partial^2 h}{\partial z \partial \bar{z}} = 0 \iff \frac{\partial}{\partial z}\left(\frac{\partial h}{\partial \bar{z}}\right) = 0$

(I called conjugate analytic "antiholomorphic") $\iff \frac{\partial h}{\partial \bar{z}}$ is conjugate analytic

PROBLEM 60

$g(z)$ continuously diff. on $\mathbb{C}$ and

$g(z) = 0 \quad \forall z \notin D$ ← a compact set.

Show that:

$$g(w) = \frac{-1}{\pi} \iint_{\mathbb{C}} \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w} \, dx dy$$

Apply Pompeiu's formula to a disk $\tilde{D}$ which contains D hence $g|_{\partial\tilde{D}} = 0$ and we obtain

$$g(w) = \frac{1}{2\pi}\int_{\partial\tilde{D}} \cancel{\frac{g(z)\,dz}{w-z}}^{\,0} - \frac{1}{\pi}\iint_{\tilde{D}} \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w} \, dx dy$$

Of course the nontrivial part of $\iint$ stems from the nonzero values of g within $D \subset \tilde{D} \subset \mathbb{C}$. Hence we just add zero as we change $\tilde{D} \longrightarrow \mathbb{C}$ and we find

$$g(w) = \frac{-1}{\pi} \iint_{\mathbb{C}} \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w} \, dx dy .$$

<u>Remark</u>: Gamelin mentions by formal IBP,

$$g(w) = \frac{-1}{\pi}\iint_{\mathbb{C}} \frac{\partial g}{\partial \bar{z}} \frac{1}{z-w} \, dx dy = \underbrace{\frac{1}{\pi}\iint_{\mathbb{C}} g(z) \underbrace{\frac{\partial}{\partial \bar{z}}\left(\frac{1}{z-w}\right)} dx dy}_{}$$

behaves like $\delta(z-w)$ in the sense

$$\iint_{\mathbb{C}} g(z)\,\delta(z-w)\,dx dy = g(w)$$