MISSION 6:

PROBLEM 62 (§ V.1 #7)

Show $\sum' a_n$ converges $\Longleftrightarrow \sum_{k=m}^{n}{}' a_k \longrightarrow 0$ as $m,n \longrightarrow \infty$.

$\Rightarrow$] $\sum_{k=0}^{\infty}{}' a_k$ converges $\Rightarrow S_n = \sum_{k=0}^{n}{}' a_k$ is convergent

Hence S_n is Cauchy sequence (Th^{m} 2.1.13) thus for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $m,n > N$ with $m < n$ implies $|S_n - S_m| < \epsilon$. However, this gives us:

$$\left| \sum_{k=0}^{n}{}' a_k - \sum_{k=0}^{m}{}' a_k \right| = \left| \sum_{k=m+1}^{n}{}' a_k \right| < \varepsilon$$

we find as $m,n \longrightarrow \infty$ the quantity $\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$ for ε as small as we wish thus it follows that $\sum_{k=m}^{n}{}' a_k \longrightarrow 0$ as $m,n \longrightarrow \infty$.

$\Leftarrow$] If $\sum_{k=m}^{n}{}' a_k \longrightarrow 0$ then $\left| \sum_{k=0}^{n}{}' a_k - \sum_{k=0}^{m+1}{}' a_k \right| < \epsilon$ for $m,n > N$ for some $N \in \mathbb{N}$. But, this shows the partial sums of $\sum' a_n$ are Cauchy sequences. Thus by Th^{m} 2.1.13 we find the partials sums converge ($\mathbb{C}$ is complete). Thus $\sum' a_n$ converges. //

PROBLEM 63 (§ V.2 #1, pg. 137)

Show $f_k(x) = \dfrac{x^k}{k+x^{2k}}$ converges uniformly on $[0,\infty)$.
Hint. Determine the worst-case estimator by calculus.

We wish to calculate $\sup\{f_k(x) \mid 0 \le x < \infty\}$.

Notice,
$$\frac{df_k}{dx} = \frac{(kx^{k-1})(k+x^{2k}) - x^k(2kx^{2k-1})}{(k+x^{2k})^2}$$
$$= \frac{k^2x^{k-1} + kx^{3k-1} - 2kx^{3k-1}}{(k+x^{2k})^2}$$
$$= \frac{k^2x^{k-1} - kx^{3k-1}}{(k+x^{2k})^2}$$
$$= \frac{kx^{k-1}(k - x^{2k})}{(k+x^{2k})^2}$$

Observe $kx^{k-1}, (k+x^{2k})^2 > 0$ for $x > 0$ and $k \ge 1$.
However, $k - x^{2k}$ is positive for small x then negative for large x. The sol^{n} of $k - x^{2k} = 0$ gives the critical pt at the local max for $f_k(x)$

Oh, $k = x^{2k} \Rightarrow x = \sqrt[2k]{k} = k^{1/2k}$ for $k \ge 1$

If $k = 0$ then $f_0(x) = 0$, for $x > 0$, In summary
$$f_k(k^{1/2k}) = \frac{(k^{\frac{1}{2k}})^k}{k + (k^{\frac{1}{2k}})^{2k}} = \frac{\sqrt{k}}{2k} = \frac{1}{2\sqrt{k}}$$

Thus $\displaystyle\sup_{[0,\infty)}\{f_k(x)\} = \frac{1}{2\sqrt{k}} \longrightarrow 0$ as $k \to \infty$

As we discussed, the supremum will serve as a worst case estimator and thus we have shown $\{f_k\}$ converges uniformly on $[0,\infty)$.

$\boxed{\text{PROBLEM 64}}$ §V.2 #8 (pg. 138)

Show that $\sum_{k=1}^{\infty} z^k/k^2$ converges uniformly for $|z|<1$

Observe that $|z|<1 \Rightarrow \left|\frac{z^k}{k^2}\right| < \frac{1}{k^2} = M_k$. Note also $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ hence by Weierstrauss M-TEST we find $\sum_{k=1}^{\infty} z^k/k^2$ converges uniformly on $|z|<1$.

$\boxed{\text{PROBLEM 65}}$ §V.2 #10 (p. 138)

Suppose $\{f_n(x)\}$ converges uniformly on E_j for $1 \le j \le n$
show $\{f_n(x)\}$ converges uniformly on $E_1 \cup E_2 \cup \cdots \cup E_n$

Uniform convergence on E_j means for each $\epsilon > 0$ there exists $N_j \in \mathbb{N}$ such that $n > N_j \Rightarrow |f_n(x) - \bar{f}_j(x)| < \epsilon$ for all $x \in E_j$. Let $\epsilon > 0$ and choose $N = \max\{N_1, N_2, \dots, N_n\}$. If $n > N$ and $x \in E_1 \cup \cdots \cup E_n$ then $x \in E_j$ for some $j \in \{1, 2, \dots, n\}$ and

$$|f_n(x) - \bar{f}_j(x)| < \epsilon \qquad (\text{as } N > N_j).$$

Thus $\{f_n(x)\}$ converges uniformly on $E_1 \cup E_2 \cup \cdots \cup E_n$ to $f(x)$ where

$$f(x) = \begin{cases} \bar{f}_1(x) & : x \in E_1 \\ \bar{f}_2(x) & : x \in E_2 \\ \vdots & \\ \bar{f}_n(x) & : x \in E_n \end{cases}$$

we should explain why f is single-valued, in particular, if $x \in E_1 \cap E_2$ then why does $\bar{f}_1(x) = \bar{f}_2(x)$?

~~I believe the answer is that the limit fcn is unique.~~

$\boxed{\text{PROBLEM 66}}$ §V.3 # 1a, 1b, 1c (pg. 143)

Find the radius of convergence (r.o.c.) of

(a.) $\sum_{k=0}^{\infty} 2^k z^k$, (b.) $\sum_{k=0}^{\infty} \frac{k}{6} z^k$, (c.) $\sum_{k=1}^{\infty} k^2 z^k$

Use ratio test (see pg. 141)

(a.) $$R = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \left|\frac{2^k}{2^{k+1}}\right| = \lim_{k\to\infty}\left(\frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

(b.) $$R = \lim_{k\to\infty} \left|\frac{k/6}{k+1/6}\right| = \lim_{k\to\infty}\left(\frac{k}{k+1}\right) = \boxed{1}$$ (typo in the key here)

(c.) $$R = \lim_{k\to\infty} \left|\frac{k^2}{(k+1)^2}\right| = \lim_{k\to\infty}\left(\frac{1}{(1+1/k)^2}\right) = \boxed{1.}$$

$\boxed{\text{PROBLEM 67}}$ §V.3 # 2a, 2b, 2f (p. 143)

Find z for which the following series converges

(a.) $\sum_{k=1}^{\infty} (z-1)^k$, (b.) $\sum_{k=10}^{\infty} \frac{(z-i)^k}{k!}$, (f.) $\sum_{n=3}^{\infty} \frac{2^n}{n^2}(z-2-i)^n$

(a.) $$\left(\sum_{k=0}^{\infty} (z-1)^k\right)(z-1) = \frac{z-1}{1-(z-1)}$$ for $|z-1| < 1$ geometric series result.

Thus $\boxed{\{ z \in \mathbb{C} \mid |z-1| < 1 \}}$ is where the given series converges.

(b.) $$R = \lim_{k\to\infty} \left(\frac{1/k!}{1/(k+1)!}\right) = \lim_{k\to\infty}\left(\frac{(k+1)k!}{k!}\right) = \infty \quad \therefore \quad \boxed{\mathbb{C}}$$

(series converges everywhere)

(f.) $$R = \lim_{n\to\infty} \left(\frac{2^n}{n^2} \cdot \frac{(n+1)^2}{2^{n+1}}\right) = \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right)^2 \frac{1}{2}\right)$$
$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^2 \frac{1}{2}\right]$$
$$= \frac{1}{2}.$$

Hence z s.t. $|z-2-i| < \frac{1}{2}$ give a convergent series for (f.).

$\boxed{\text{PROBLEM 68}}$ § IV.3 #5 (p. 143)

What functions are represented by the following series

(a.) $\sum_{k=1}^{\infty} kz^k$, (b.) $\sum_{k=1}^{\infty} k^2 z^k$

(a.) Observe
$$f(z) = \sum_{k=1}^{\infty} kz^k = z\left(\sum_{k=1}^{\infty} kz^{k-1}\right)$$
$$= z\frac{d}{dz}\left(\sum_{k=0}^{\infty} z^k\right)$$
$$= z\frac{d}{dz}\left(\frac{1}{1-z}\right)$$
$$= \boxed{\frac{z}{(1-z)^2}}.$$

(b.)
$$g(z) = \sum_{k=1}^{\infty} k^2 z^k$$

notice at $k=1$ this is zero thus we must subtract to be fair ☺

$$= z^2\sum_{k=1}^{\infty} k(k-1)z^{k-2} + \sum_{k=1}^{\infty} kz^k - z$$
$$= z^2\frac{d^2}{dz^2}\left(\sum_{k=0}^{\infty} z^k\right) + \frac{z}{(1-z)^2} - z \text{ by (a.)}$$
$$= z^2\frac{d}{dz}\left[\frac{d}{dz}\left(\frac{1}{1-z}\right)\right] + \frac{z}{(1-z)^2} - z$$
$$= z^2\frac{d}{dz}\frac{1}{(1-z)^2} + \frac{z}{(1-z)^2} - z$$
$$= \boxed{\frac{2z^2}{(1-z)^3} + \frac{z}{(1-z)^2} - z}$$

PROBLEM 69 § V.3 #6 (p.144)

Show $\sum a_k z^k$, $\sum k a_k z^{k-1}$, $\sum \frac{a_k}{k+1} z^{k+1}$ all have the same radius of convergence.

$$\lim_{k\to\infty}\left[\frac{k a_k}{(k+1)a_{k+1}}\right] = \lim_{k\to\infty}\left[\frac{a_k}{(1+1/k)a_{k+1}}\right] = \lim_{k\to\infty}\left(\frac{a_k}{a_{k+1}}\right)\lim_{k\to\infty}\left(\frac{1}{1+1/k}\right)$$

$$\lim_{k\to\infty}\left[\frac{a_k/k+1}{a_{k+1}/k+2}\right] = \lim_{k\to\infty}\left[\frac{(k+2)a_k}{(k+1)a_{k+1}}\right] = \lim_{k\to\infty}\left(\frac{k+2}{k+1}\right)\lim_{k\to\infty}\left(\frac{a_k}{a_{k+1}}\right)$$

Thus either the limits exist and are equal, or they all diverge to ∞. In either case all three series share the same radius of convergence.

PROBLEM 70 § V.4 #1a, 1b, 1c (p.147)

Find the r.o.c. for the power series for the fncts about the given points:

(a.) $\frac{1}{z-1}$ about $z=i$ (b.) $\frac{1}{\cos z}$ about $z=0$, (c.) $\frac{1}{\cosh z}$ about $z=0$

In each case we simply look for closest singular point to the given center.

a.) $f(z) = \frac{1}{z-1}$ has singularity at $z=1$ thus

$R = |1-i| = \boxed{\sqrt{2}}$. (Could also expand via geometric to derive directly)

b.) $f(z) = \frac{1}{\cos z}$ has singularities at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

the closest to $z=0$ is either $\pm\frac{\pi}{2} \Rightarrow \boxed{R = \pi/2}$

c.) As $\cos(iz) = \cosh(z)$ we see $iz = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

give singularities of $f(z) = \frac{1}{\cosh z} \Rightarrow R = \left|\frac{i\pi}{2} - 0\right| = \boxed{\frac{\pi}{2}}$.

PROBLEM 71 §V.4#3 (p.147)

Find power series expansion of $\mathrm{Log}(z)$ about the point $z_0 = i-2$. Show the r.o.c. $R = \sqrt{5}$. Explain why this does not contradict the discontinuity at $z=-2$ of $\mathrm{Log}(z)$

There are two easy ways to do this.

(1.) Taylor Series,

$$f(z) = \mathrm{Log}(z) \, , \quad f'(z) = \frac{1}{z} \, , \quad f''(z) = \frac{-1}{z^2}$$

$$f'''(z) = \frac{-2(-1)}{z^3} \, , \cdots \quad f^{(n)}(z) = \frac{(n-1)!\,(-1)^{n-1}}{z^n}$$

Hence as $z_0 = i-2$ we have,

$$\frac{f^{(n)}(i-2)}{n!} = \frac{(n-1)!\,(-1)^{n-1}}{n!\,(i-2)^n} = \frac{(-1)^{n-1}}{n}\frac{1}{(i-2)^n}$$

$$\therefore \quad \mathrm{Log}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(i-2)^n}(z-i+2)^n + \mathrm{Log}(i-2)$$

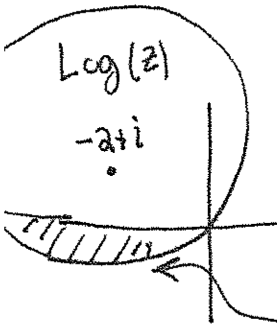
$$R = \lim_{n\to\infty}\left|\frac{(n+1)(i-2)^{n+1}}{n(i-2)^n}\right| = \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)(i-2)\right| = \sqrt{5}$$

2.) $\mathrm{Log}(z) = \mathrm{Log}(z-i+2+i-2) = f(z)$

$$\frac{df}{dz} = \frac{1}{z} = \frac{1}{i-2+z-i+2} = \frac{1}{i-2}\left[\frac{1}{1+\frac{z-i+2}{i-2}}\right]$$

$$\therefore \quad \frac{df}{dz} = \frac{1}{i-2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(i-2)^n}(z-i+2)^n$$

$$\Rightarrow \quad f(z) = \mathrm{Log}(i-2) + \sum_{n=0}^{\infty}\frac{(-1)^n\,(z-i+2)^{n+1}}{(n+1)(i-2)^{n+1}}$$



$\mathrm{Log}_0(z)$

both Log and Log_0 are branches of log(z) we've continued past the branch cut to another branch.

PROBLEM 72 §V.4 #7 (p. 147)

Find the power series expansion of the principal branch $\text{Tan}^{-1}(z)$ about $z=0$. What is the r.o.c. of the series? Hint: use geom. series.

Observe, $w = \text{Tan}^{-1}(z) \Rightarrow \tan(w) = z$

thus $\sec^2(w)\frac{dw}{dz} = 1 \Rightarrow \frac{dw}{dz} = \frac{1}{\sec^2 w} = \frac{1}{1+\tan^2 w}$

we find $\frac{d}{dz}\left(\tan^{-1}(z)\right) = \frac{1}{1+z^2}$ for any branch of $\tan^{-1}(z)$.

Let $f(z) = \text{Tan}^{-1}(z) \Rightarrow \frac{df}{dz} = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

$\therefore f(z) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$, but $f(0) = \text{Tan}^{-1}(0) = 0$.

thus $\boxed{f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}}$. The

geometric series held for $|-z^2| < 1 \Rightarrow |z| < 1$ and integration does not change r.o.c.

$\therefore \boxed{R = 1}$