Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 73 Your signature below indicates you have:

(a.) I have read Sections 5.5 – 5.7 of Gamelin: ________________.

(b.) I have read Cook's Guide to Sections 5.5 – 5.7 of Gamelin:________________.

Problem 74 §V.4 # 1d, 1e, 1f (page 147),

Problem 75 §V.4 # 2 (page 147),

Problem 76 §V.4 # 8 (page 148),

Problem 77 §V.4 # 10 (page 148),

Problem 78 §V.4 # 12 (page 149),

Problem 79 §V.5 # 1 (page 151),

Problem 80 §V.6 # 1 (page 153),

Problem 81 §V.6 # 5 (page 154),

Problem 82 §V.7 # $1a, 1b, 1c$ (page 157),

Problem 83 §V.7 # $2a, 2b, 2c$ (page 157),

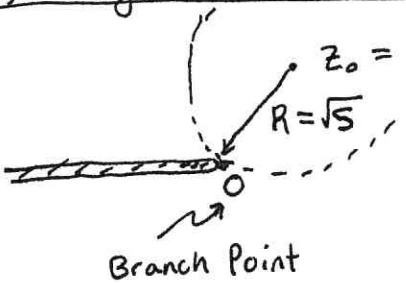
Problem 84 §V.8 # 2 (page 162),

Bonus: prove Theorem 5.2.7 to establish the equivalence between the standard formulation of uniform convergence and Gamelin's version. (this is worth 5pts if done neatly)

Mission 7:

$\boxed{P74}$ §V.4 #1d, 1e, 1f (pg. 147). Find the radius of convergence, given $f(z)$ and center at z_0.

(1d.) Log(z) about $z_0 = 1+2i$



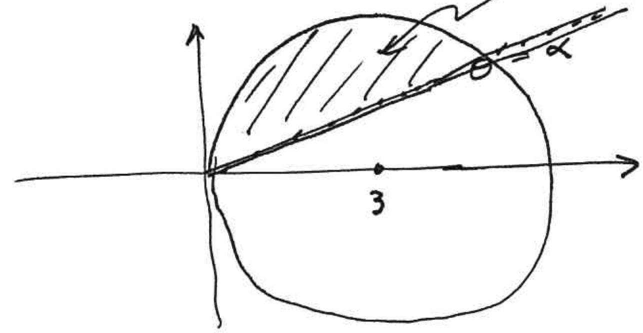
Here $z=0$ is nearest singularity, thus $\text{Log}(z) = \sum_{n=0}^{\infty} a_n (z-1-2i)^n$ will converge for $|z-1-2i| < \sqrt{5}$

$$\boxed{R = \sqrt{5}}$$

(1e.) $f(z) = z^{3/2}$ about $z_0 = 3$

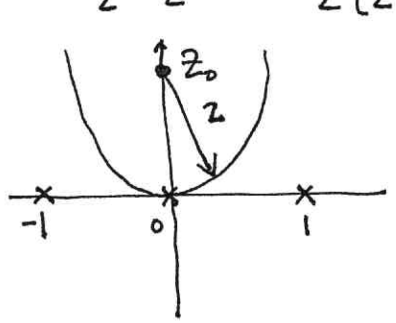
Well, depending on which branch we use the power series about 3 may or may not represent $f(z)$ to the branch point of $z=0$. That said, the power series defined by $\sum \frac{f^{(n)}(3)}{n!}(z-3)^n$ will converge on a disk up to the branch point $\therefore \boxed{R=3}$

$$f(z) = \exp\left(\tfrac{3}{2}\text{Log}_\alpha(z)\right)$$



not same branch as $f(z)$, this would be the continuation of $f(z)$ to another branch.

(1f.) $f(z) = \dfrac{z-i}{z^3-z} = \dfrac{z-i}{z(z^2-1)} = \dfrac{z-i}{z(z+1)(z-1)}$ about $z_0 = 2i$



$\boxed{R=2}$ since $z=0$ is closest singular pt. to $z_0 = 2i$

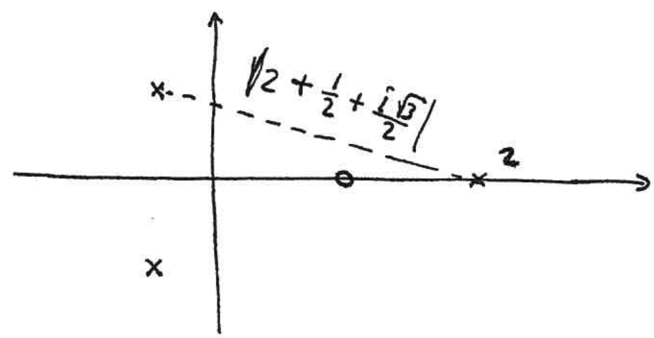
P75 §V.4 #2 Show $R = \sqrt{7}$ for the power series expansion of $f(z) = \dfrac{z^2-1}{z^3-1}$ about $z = 2$.

$$f(z) = \frac{(z+1)(z-1)}{(z+1)(z^2+z+1)} = \frac{z-1}{(z+\frac{1}{2})^2 + \frac{3}{4}}$$

($z = -1$ is removable singularity)

$$f(z) = \frac{z-1}{\left(z+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(z+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}$$

$\underbrace{z_0 = \frac{-1}{2} \pm \frac{i\sqrt{3}}{2}}$ Simple poles of $f(z)$. (nearest genuine singularities, both $\sqrt{7}$ from $z = 2$)



$$\left|2 + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right| = \frac{1}{2}\left|5 + i\sqrt{3}\right| = \frac{1}{2}\sqrt{25+3} = \frac{\sqrt{28}}{2} = \boxed{\sqrt{7}}.$$

P76 §V.4 #8, pg. 148

Expand $\mathrm{Log}(1+iz)$ and $\mathrm{Log}(1-iz)$ in power series about $z = 0$. Then by comparison with the power series for $\mathrm{Tan}^{-1}(z)$ to establish the identity

$$\mathrm{Tan}^{-1}(z) = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right) \quad ☆$$

(this is actually #7 and #8 $sol^{\underline{n}}$)

$f(z) = \mathrm{Tan}^{-1}(z) \hookrightarrow w = \mathrm{Tan}^{-1}(z)$

$\therefore \tan(w) = z \Rightarrow \dfrac{dw}{dz} = \dfrac{1}{\sec^2 w} = \dfrac{1}{1+z^2}$

Thus, $\dfrac{df}{dz} = \dfrac{1}{1+z^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n z^{2n}$. Integrate term-by-term and note $f(0) = 0$ hence, $\mathrm{Tan}^{-1}(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$.

PROBLEM 76 continued

$$f(z) = Log(1+cz) \qquad f(0) = Log(1) = 0$$

$$f'(z) = \frac{c}{1+cz} \qquad f'(0) = c$$

$$f''(z) = \frac{-c^2}{(1+cz)^2} \qquad f''(0) = -c^2$$

$$f'''(z) = \frac{(-2)(-1)\,c^3}{(1+cz)^3} \qquad f'''(0) = \frac{2!\,c^3(-1)^{3-1}}{1}$$

$$f^{(4)}(z) = \frac{(-3)(-2)(-1)\,c^4}{(1+cz)^4} \qquad f^{(4)}(0) = \frac{3!\,c^4(-1)^{4-1}}{1}$$

Thus, $f^{(n)}(0) = (n-1)!\,c^n(-1)^{n-1} \Rightarrow \dfrac{f^{(n)}(0)}{n!} = c^n\dfrac{(-1)^{n-1}}{n}$

By Taylor's Th$^{\underline{m}}$, $Log(1+cz) = \left(\displaystyle\sum_{n=1}^{\infty} \frac{c^n(-1)^{n-1}}{n} z^n\right)$

$$Log(1+iz) = \sum_{n=1}^{\infty} \frac{i^n(-1)^{n-1}}{n} z^n = \sum_{j=1}^{\infty} \frac{i^{2j}(-1)^{2j-1}}{2j} z^{2j} + \overline{\sum_{j=1}^{\infty} \frac{i^{2j-1}}{2j-1} z^{2j-1}}$$

$$Log(1-iz) = \sum_{n=1}^{\infty} \frac{(-i)^n(-1)^{n-1}}{n} z^n = \sum_{j=1}^{\infty} \frac{(-i)^{2j}(-1)^{2j-1}}{2j} z^{2j} + \underbrace{\sum_{j=1}^{\infty} \frac{(-i)^{2j-1}}{2j-1} z^{2j-1}}$$

Upon adding $Log(1+iz)$ and $Log(1-iz)$
we find & cancel as $i^{2j-1} = -(-i)^{2j-1}$.
Hence, noting $(-1)^{2j-1} = -1$,

$$Log(1+iz) + Log(1-iz) = -2\sum_{j=1}^{\infty} \frac{i^{2j}}{2j} z^{2j}, \qquad i^{2j} = (i^2)^j = (-1)^j$$

$$= -2\sum_{n=0}^{\infty} \frac{(-1)^j}{2j} z^{2j}$$

oh, but, we want <u>odd</u> powers so <u>subtract</u> as to keep &

$$Log(1+iz) - Log(1-iz) = \sum_{j=1}^{\infty}\left(\frac{i^{-1}i^{2j}}{2j-1} - \frac{(-1)^{2j-1}\,i^{-1}i^{2j}}{2j-1}\right) z^{2j-1}$$

$$= 2i^{-1}\sum_{j=1}^{\infty} \frac{(-1)^j}{2j-1} z^{2j-1} = \frac{2}{i}\,Tan^{-1}(z).$$

the identity ☆ follows.

PROBLEM 76 continued for $1 \pm iz \notin (-\infty, 0]$,

$$\mathrm{Tan}^{-1}(z) = \frac{i}{2}\left[\mathrm{Log}(1+iz) - \mathrm{Log}(1-iz)\right] = \frac{i}{2}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right)$$

(this is not quite ✱, I must have a sign-error in here some where, I can't find it at the moment.)

PROBLEM 77) § V.4 #10 (p. 148)

Define, for $\alpha \in \mathbb{C}$, for $n \in \mathbb{N}$,

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

Find the R.O.C. for binomial series $\sum_{n=0}^{\infty}\binom{\alpha}{n} z^n$.

Also, show the binomial series represents principal branch of $(1+z)^\alpha$. For which α does the series truncate to polynomial

Use Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\left[\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\right]\left[\frac{(n+1)!}{\underbrace{\alpha(\alpha-1)\cdots}_{\text{cancel out}}\underbrace{(\alpha-(n+1)+1)}_{\alpha-n}}\right]\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{\alpha-n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1+1/n}{-1+\alpha/n}\right|$$

$$= \boxed{1.}$$ ← Radius of Convergence for Binomial Series. (independent of α)

The

$$f(z) = (1+z)^\alpha \qquad f(0) = 1$$

$$f'(z) = \alpha(1+z)^{\alpha-1} \qquad f'(0) = \alpha$$

$$f''(z) = \alpha(\alpha-1)(1+z)^{\alpha-2} \qquad f''(0) = \alpha(\alpha-1)$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(z) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+z)^{\alpha-n} \qquad f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-n+1)$$

Hence, $(1+z)^\alpha = \sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}z^n + 1$ which shows that the Binomial Series rep. $(1+z^\alpha)$. Finally, when $\alpha \in \mathbb{N}$ we have $\binom{\alpha}{m} = 0$ for $m \geq \alpha. \Rightarrow$ Binomial Series truncates at order α.

PROBLEM 77 continued

Once again to get polynomial we need $\binom{\alpha}{n} = 0$ for $n \geq N$.

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} = 0$$

$$\Rightarrow \alpha(\alpha-1)\cdots(\alpha-n+1) = 0$$

$$\Rightarrow \alpha = 0,\ \alpha = 1,\ \ldots,\ \alpha = n-1$$

The only sol^{n}'s for $\binom{\alpha}{n} = 0$ are found in $\mathbb{N}$. Moreover, for fixed $n \in \mathbb{N}$ we have $\binom{n}{m} = 0$ for all $m > n$, Thus only $\alpha \in \mathbb{N}_0$ gives the Binomial Series as a mere polynomial. Otherwise, for $\alpha \in \mathbb{C} - (\mathbb{N} \cup \{0\})$.

$\boxed{\text{PROBLEM 78}}$ Suppose $f(z)$ is analytic with $f(z) = \sum' a_n z^n$

(1.) if f is even then $f(z) = f(-z)$ by def^{n}.
Furthermore, $f'(z) = -f'(-z) \Rightarrow f'(0) = -f'(0) \therefore f'(0) = 0$.
and $f^{(n)}(z) = (-1)^n f^{(n)}(-z) \Rightarrow f^{(n)}(0) = (-1)^n f^{(n)}(0)$
for even n we obtain no condition. However, as we already saw for $n=1$, when n odd, $f^{(n)}(0) = -f^{(n)}(0)$ hence $f^{(n)}(0) = 0$ $\therefore$ $\boxed{f(z) = \sum_{j=0}^{\infty}{}' \frac{f^{(2j)}(0)}{(2j)!} z^{2j}}$.

(2.) If f is odd then $f(z) = -f(-z)$ hence
$f'(z) = f'(-z) \Rightarrow f'(0) = f'(0)$ which is not particularly restrictive. Continuing, $f^{(n)}(z) = f^{(n)}(-z)(-1)^{n+1}$. For example, $f''(z) = -f''(-z)$, $f'''(z) = f'''(-z)$, ...
thus $f^{(n)}(0) = (-1)^{n+1} f^{(n)}(0) \Rightarrow f^{(2j)}(0) = -f^{(2j)}(0)$
whereas $f^{(2j+1)}(0) = f^{(2j+1)}(0)$ $\therefore$ $\boxed{f(z) = \sum_{j=0}^{\infty}{}' \frac{f^{(2j+1)}(0)}{(2j+1)!} z^{2j+1}}$

Remark: can improve my sol^{n} via explicit ~~induction argument~~!

$\boxed{\text{PROBLEM 79}}$ § V.5 #1, pg. 151 : Expand $f(z)$ about ∞

(a.) $$f(z) = \frac{1}{z^2+1} = \frac{1}{z^2}\left(\frac{1}{1+1/z^2}\right)$$

$$= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n$$ geometric series !

$$= \underline{\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n+2}} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} + \cdots}.$$

(b.) $$f(z) = \frac{z^2}{z^3-1} = \frac{z^2}{z^3}\left(\frac{1}{1-1/z^3}\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z^3}\right)^n$$

$$= \underline{\sum_{n=0}^{\infty}\frac{1}{z^{3n+1}} = \frac{1}{z} + \frac{1}{z^4} + \frac{1}{z^7} + \cdots}.$$

(c.) $$f(z) = e^{\frac{1}{z^2}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z^2}\right)^n = \underline{\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^{2n}} = 1 + \frac{1}{z^2} + \cdots}.$$

(d.) $$z\sinh\left(\frac{1}{z}\right) = z\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1}$$

$$= \underline{\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\frac{1}{z^{2n}}} = 1 + \frac{1}{\;}.$$

(§ V.7 #1)

$\boxed{\text{PROBLEM 80}}$ Find the zeros and their order for the following,

(a.) $$\frac{z^2+1}{z^2-1} = \frac{1}{z^2-1}(z+i)(z-i) \longrightarrow \boxed{z_0 = \pm i \text{ both are } \underline{\text{simple}} \text{ zeros, order 1.}}$$

(d.) ~~(b.)~~ $f(z) = \cos z - 1 = 0 \implies \cos z = 1$

↑ oops.

$$\implies \boxed{z = 2\pi k \text{ for } k \in \mathbb{Z}, \text{ order two}}$$

Observe $f'(z) = -\sin(z)$ and $f''(z) = -\cos(z)$

thus $f'(2\pi k) = -\sin(2\pi k) = 0$ whereas $f''(2\pi k) = -1 \neq 0$

P8a continued

(b.) $f(z) = \frac{1}{z} + \frac{1}{z^5} = \frac{z^4+1}{z^5} = 0$ when $z^4+1=0$

Hence $z_0 \in (-1)^{1/4} = e^{i\pi/4}\{1, i, -1, -i\}$

Or $\boxed{z_0 = e^{i\pi/4},\ ie^{i\pi/4},\ -e^{i\pi/4},\ -ie^{i\pi/4}}$

are all zeros which are <u>simple</u> (order 1)

<u>Why simple?</u> $f'(z) = \frac{-1}{z^2} - \frac{5}{z^6}$ clearly nonzero at each z_0.

(c.) $f(z) = z^2 \sin z$

$= z^2\left(z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \cdots\right)$

$= z^3 - \frac{1}{6}z^5 + \frac{1}{120}z^7 + \cdots$

$= z^3\left(1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 + \cdots\right) \Rightarrow$ $\boxed{z_0 = 0 \text{ has order } 3}$

However, we also have $\sin(n\pi) = 0$ for all $n \in \mathbb{Z}$ and $f'(z) = 2z\sin z + z^2\cos(z)$

thus $f'(n\pi) = n^2\pi^2\cos(n\pi) = (-1)^n n^2\pi^2 \neq 0$ for $n \neq 0$

Hence $\boxed{z_0 = n\pi \text{ for } n \in \mathbb{Z} \text{ with } n \neq 0 \text{ are simple zeros.}}$

(e.) $\frac{\cos z - 1}{z} = \frac{1}{z}\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \cdots - 1\right) = \underbrace{\frac{1}{2}z + \frac{1}{24}z^3 + \cdots}$

$\boxed{\Rightarrow \text{clear } z_0 = 0 \text{ is simple zero.}}$

Alternatively, $z_0 = 2\pi k$ for $k \in \mathbb{Z}$, $k \neq 0$ then

$f(z) = \frac{\cos z - 1}{z} \longrightarrow f'(z) = \frac{-(\sin z)z - (\cos z - 1)}{z^2}$

Thus, $f'(2\pi k) = 0$. But, $f''(z) = \frac{1}{z^4}\left[(-\sin z - \cos z + \sin z)z^2 + 2z\left((\sin z)z - (\cos z - 1)\right)\right]$

and we see $f''(2\pi k) \neq 0$ thus

$\boxed{z_0 = 2\pi k,\ k \in \mathbb{Z},\ k \neq 0 \text{ are double zeros}}$

P 8a continued

(f.) $$\underset{g(z)}{\overset{\shortparallel}{\frac{\cos z - 1}{z^2}}} = \frac{1}{z^2}\left[\frac{z^2}{2} - \frac{z^4}{4!} + \cdots\right] = \underbrace{\frac{1}{2} - \frac{z^2}{4!}}_{z=0 \text{ not a zero.}} + \cdots$$

However, $$\frac{d}{dz}\left[\frac{\cos z - 1}{z^2}\right] = \frac{-(\sin z)z^2 - 2z(\cos z - 1)}{z^4}$$

and $$\frac{d^2}{dz^2}\left[\frac{\cos z - 1}{z^2}\right] = \frac{\left[-2z\sin z - z^2\cos z - 2(\cos z - 1) - 2z(-\sin z)\right]z^4}{z^8} - \frac{4z^3\left[-(\sin z)z^2 - 2z(\cos z - 1)\right]}{z^8}$$

Hence $z_o = 2\pi k$ for $k \neq 0$ has $g''(2\pi k) \neq 0$ yet $g'(2\pi k) = 0$ and $g(2\pi k) = 0$ thus

$\boxed{z_o = 2\pi k,\ k \neq 0,\ k \in \mathbb{Z} \text{ are double zeros.}}$

You know, maybe trig. is easier,

$$\frac{\cos(z) - 1}{z^2} = \frac{\cos(2\pi k + z - 2\pi k) - 1}{z^2}$$

$$= \frac{\cos(z - 2\pi k)\cos(2\pi k) - \sin(z - 2\pi k)\sin(2\pi k) - 1}{z^2}$$

$$= \frac{\cos(z - 2\pi k) - 1}{z^2}$$ (ah, duh!) 2π-periodic!

$$= \frac{1}{z^2}\left(-\frac{1}{2}(z - 2\pi k)^2 + \frac{1}{4!}(z - 2\pi k)^4 + \cdots\right)$$

$$= (z - 2\pi k)^2 \left[\underbrace{\frac{-\frac{1}{2} + \frac{1}{4!}(z - 2\pi k)^4 + \cdots}{z^2}}\right]$$

clearly non zero at $z = 2\pi k$ hence this shows $z = 2\pi k \neq 0$ is double zero.

P 8a continued

(g.) $e^z - 1 = \left(1 + z + \frac{1}{2}z^2 + \cdots\right) - 1 = z\left(1 + \frac{1}{2}z + \cdots\right)$

$\boxed{z = 0 \text{ is simple zero.}}$

(the exponential has $2\pi i \mathbb{Z}$ periodicity)

$$e^{2\pi i k} = 1$$

$$f(z) = e^{z - 2\pi i k + 2\pi i k} - 1 = \cancel{e^{2\pi i k}}^{\,1} e^{z - 2\pi i k} - 1$$

(again, duh.) $f(z) = e^{z - 2\pi i k} - 1$

Hence $e^z - 1 = \sum_{n=1}^{\infty} \frac{(z - 2\pi i k)^n}{n!} = (z - 2\pi i k)\left[1 + \frac{1}{2}(z - 2\pi i k) + \cdots\right]$

$\boxed{\text{Hence } z = 2\pi i k \text{ are simple zeros for all } k \in \mathbb{Z}.}$

Note:

$f'(z) = e^z$

$f'(2\pi i k) = e^{2\pi i k} = 1 \neq 0$

proves the zero is simple in simpler fashion here.

(h.) $\overbrace{\sinh^2 z + \cosh^2 z}^{f(z)} = 0$

$$\frac{1}{4}\left(e^{2z} - \cancel{2} + e^{-2z}\right) + \frac{1}{4}\left(e^{2z} + \cancel{2} + e^{-2z}\right) = 0$$

$$\frac{1}{2}\left(e^{2z} + e^{-2z}\right) = 0$$

$$\cosh(2z) = \cos(2iz) = 0 \qquad j \in \mathbb{Z}.$$

Hence, $2iz = (2j - 1)\frac{\pi}{2} \quad \therefore \boxed{z = \frac{(2j-1)\pi}{4i}}$

zeros of $f(z)$.

$$\begin{aligned} f'(z) &= 2\sinh z \cosh z + 2\cosh z \sinh z \\ &= 4 \sinh z \cosh z \\ &= (e^z - e^{-z})(e^z + e^{-z}) \\ &= e^{2z} - e^{-2z} \\ &= 2\sinh(2z). \end{aligned}$$

$\Longrightarrow$ the zeros found are simple. $f'\left(\left[\frac{2j-1}{4}\right]\frac{\pi}{i}\right) \neq 0$.

p 82 continued

(i.) $f(z) = \dfrac{\text{Log}(z)}{z}$ note $\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$ (for $z \neq 0$.)

Need $\ln|z| = 0$ and $\text{Arg}(z) = 0$.

thus $|z| = 1$ and $\theta = 0$ $\therefore$ $z = 1$.

$$f'(z) = \frac{1/z}{z} - \frac{\text{Log}(z)}{z^2} \Rightarrow f'(1) = 1 \neq 0.$$

thus $\boxed{z_0 = 1 \text{ is simple zero}}$

PROBLEM 81 §V.6 #5 p. 154 (sorry I got out of order)

Define E_n by $\dfrac{1}{\cosh z} = \sum_{n=0}^{\infty} \dfrac{E_n}{n!} z^n$. Find the radius of convergence and show $E_n = 0$ for n odd. Also find E_0, E_2, E_4, E_6.

$\dfrac{1}{\cosh z} = \dfrac{1}{\cos(iz)}$ has singularity at $iz = \pm\frac{\pi}{2}$ $\therefore z = \frac{\pm\pi i}{2}$

Hence the nearest singularity to $z=0$ is at $\pi i/2$ or $-\pi/2$ $\therefore$ $R = \pi/2$

The argument $E_n = 0$ for n odd was already given in PROBLEM 78 since $\dfrac{1}{\cosh(-z)} = \dfrac{1}{\cosh z}$.

$$\frac{1}{\cosh z} = \frac{1}{1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{1}{6!}z^6 + \cdots}$$

$$= 1 - \left(\tfrac{1}{2}z^2 + \tfrac{1}{24}z^4 + \tfrac{1}{720}z^6 + \cdots\right) + \left(\tfrac{1}{2}z^2 + \tfrac{1}{24}z^4\right)^2 - \left(\tfrac{1}{2}z^2\right)^3 + \cdots$$

$$= 1 - \tfrac{1}{2}z^2 + \left(\tfrac{-1}{24} + \tfrac{1}{4}\right)z^4 + \left(\tfrac{-1}{720} + \tfrac{1}{24} - \tfrac{1}{8}\right)z^6 + \cdots$$

$$= E_0 + \frac{E_2}{2}z^2 + \frac{E_4}{24}z^4 + \frac{E_6}{6!}z^6 + \cdots$$

Thus, $\boxed{E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61}$

(for example, $\frac{-1}{24} + \frac{1}{4} = \frac{E_4}{24} \Rightarrow E_4 = -1 + 6 = 5$.)

PROBLEM 80 §V.6 #1

Calculate the terms to order 7 for $\frac{1}{\cos z}$ about $z=0$.

$$\frac{1}{\cos z} = \frac{1}{1-\frac{1}{2}z^2+\frac{1}{24}z^4-\frac{1}{720}z^6+\cdots}$$

(just keep to order 6 since $\cos z$ even)

$$= 1 + \left(\tfrac{1}{2}z^2-\tfrac{1}{24}z^4+\tfrac{1}{720}z^6\right)+\left(\tfrac{1}{2}z^2-\tfrac{1}{24}z^4\right)^2+\left(\tfrac{1}{2}z^2\right)^3+\cdots$$

$$= 1 + \tfrac{1}{2}z^2 + \left(\tfrac{-1}{24}+\tfrac{1}{4}\right)z^4+\left(\tfrac{1}{720}-\tfrac{1}{24}+\tfrac{1}{8}\right)z^6+\cdots$$

$$= \boxed{1+\frac{1}{2}z^2+\frac{5}{24}z^4+\frac{61}{720}z^6+\cdots}$$

The idea here is to only keep terms which contribute to orders zero through 7 for $\frac{1}{\cos z}$. I know $\frac{1}{\cos z} = \sum_{n=0}^{\infty} a_n z^n$ as $a_{2n-1}=0$ for all n since $\cos(z) = \cos(-z)$ so I really am just working to find a_0, a_2, a_4, a_6 for this problem.

PROBLEM 83 §V.7 # 2a, 2b, 2c p. 157:

(a.) $$\underbrace{\frac{z^2+1}{z^2-1}}_{f(z)} = \frac{1+1/z^2}{1-1/z^2} = \left(1+\frac{1}{z^2}\right)\sum_{n=0}^{\infty}\left(\frac{1}{z^2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{2n}} + \sum_{n=0}^{\infty}\frac{1}{z^{2n+2}}$$

$$= 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^{2n}}$$

Hence $f(z)$ is analytic at ∞, however $f(\infty)=1\neq 0$.

(b.) $g(z) = \frac{1}{z}+\frac{1}{z^5} = \frac{1}{z}\left(1+\frac{1}{z^4}+\cdots\right) \hookrightarrow$ $g(\infty)=0$ and ∞ has order one.

(c.) $h(z) = z^2\sin(z) = z^3 - \frac{z^5}{3!}+\frac{z^7}{5!}+\cdots$

$h(z)$ is not analytic at ∞.

PROBLEM 84 | § V.8 # 2, page 162

Show $f(z) = \text{Log}(z) = (z-1) - \frac{1}{2}(z-1)^2 + \cdots$ has an analytic continuation around unit-circle $\gamma(t) = e^{it}$ $0 \leq t \leq 2\pi$. Determine explicitly f_t for each t. How is $f_{2\pi}$ related to f_0

$$f'(z) = \frac{1}{z}, \quad f''(z) = \frac{-1}{z^2}, \quad f^{(3)}(z) = \frac{(-2)(-1)}{z^3}$$

We see $f^{(n)}(z) = \dfrac{(n-1)!(-1)^{n-1}}{z^n}$ for $n \geq 1$

$$f^{(n)}(e^{it}) = (-1)^{n-1} e^{-nit} (n-1)!$$

Thus, $$f_t(z) = \sum_{n=1}^{\infty} \frac{(n-1)!(-1)^{n-1} e^{-nit}}{n!} (z - e^{it})^n + \text{Log}(e^{it})$$

$$\therefore \boxed{f_t(z) = it + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} e^{-nit}}{n} (z - e^{it})^n}$$

$$f_0(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n$$

$$f_{2\pi}(z) = 2\pi i + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (z-1)^n = 2\pi i + f_0(z)$$

$$\therefore \boxed{f_{2\pi} = f_0 + 2\pi i}$$

Remark: $\text{Log}(e^{it}) = \begin{cases} it & \text{for } 0 \leq t < \pi \\ i(t-2\pi) & \text{for } \pi < t < 2\pi \end{cases}$

So... it is not $\text{Log}(z)$ used for $\text{Log}(e^{it}) = it$.
Rather $(z-1) - \frac{1}{2}(z-1)^2 + \cdots$ something sneaky here...