Working together is encouraged, share ideas not calculations. Explain your steps. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. No fuzzies thanks. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 Problem from Lecture to be added here.

Problem 2 Let $\beta = \{v_1, \dots, v_n\}$ be a basis for a normed linear space V such that if $x = \sum_{i=1}^n x_i v_i$ then $|x_i| \leq \|x\|$. Suppose $T : V \to V$ is a linear transformation. Show $\lim_{x \to a} T(x) = T(a)$ for $a \in V$ by an $\varepsilon\delta$-argument.

Problem 3 Problem from Lecture to be added here.

Problem 4 Let $\eta(A, B) = \text{trace}(A^T B)$ for all $A, B \in \mathbb{R}^{n \times n}$. Show that η is an inner product on $\mathbb{R}^{n\times n}$. This shows that $||A|| = \sqrt{\text{trace}(A^T A)}$ is a norm as it is simply the norm induced from η. Also, calculate $||A||$ for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Problem 5 Let $A, B \in \mathbb{R}^{n\times n}$ and $|| \cdot ||$ denote the Frobenius norm. Show that $||AB|| \leq ||A||\,||B||$. *Hint:* $\|A\|^2 = \|col_1(A)\|^2 + \cdots + \|col_n(A)\|^2$.

Problem 6 Show that $\text{SL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$ is topologically closed. *It's also closed as a subgroup of the general linear group of invertible matrices in $\mathbb{R}^{n\times n}$. In fact, for reasons we will eventually appreciate, $SL(n, \mathbb{R})$ is a Lie group, that is a group which is also a smooth manifold in the natural sense.*

Problem 7 Show $\mathbb{R}^{m\times n}$ is a complete space. Assume it is already known that $\mathbb{R}^{m\times n}$ is a normed linear space with respect to the Frobenius norm $||A|| = \sqrt{\text{trace}(A^T A)}$.

Problem 8 Define $\sin(A) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$ for $A \in \mathbb{R}^{n\times n}$. Show that $\sin(A)$ exists for any $A \in \mathbb{R}^{n\times n}$. *Hint: in the notes we prove e^A exists for any A.*

Problem 9 Let $F(A) = A^3$ for $A \in \mathbb{R}^{n\times n}$. Prove F is differentiable on $\mathbb{R}^{n\times n}$ by proposing a formula for $dF_A(H)$ and showing your proposed differential is linear in H and satisfies the needed limit. *Please do not use partial differentiation (yet).*

Problem 10 Let $F(A) = A^3$. Calculate $\frac{\partial F}{\partial X_{ij}}(A)$ with respect to the standard matrix basis. Explain why $dF_A(H) = A^2H + AHA + HA^2$. *(explain both the existence of dF_A as well as the formula as it can be ciphered from your partial derivatives, cite an appropriate theorem from my notes, do not use the previous problem in your argument)*

Problem 11 Let $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by $F(x, y) = x \bullet y$. Calculate $dF_{(a,b)}(h, k)$.

Problem 12 Use chain-rule for $f(x, y) = \sqrt[y]{x}$ composed with $\gamma(t) = (t, t)$ to calculate $\frac{d}{dt}[f(\gamma(t))]$. *Thus, in view of the fact $\sqrt[t]{t} = f(\gamma(t))$ you have calculated $\frac{d}{dt}\left[\sqrt[t]{t}\right]$.*

Problem 13 Find the Jacobian matrix for the following maps from $\mathbb{R}^n$ to $\mathbb{R}^n$:

(a.) $F(x, y) = (x^2 - y^2, 2xy)$

(b.) $G(x, y, z) = (x^2 + 2yz, z^2 + 2xy, y^2 + 2xz)$

(c.) $H(x, y) = \dfrac{1}{x^2 - y^2}(x, -y)$

Problem 14 Let V, W be finite-dimensional normed linear spaces. Show that if $F : V \to W$ is Frechet differentiable at $a \in V$ then F is continuous at a.

Problem 15 Let V be a real NLS with basis $\beta = \{v_1, \dots, v_n\}$. Define the **$i$-th coordinate function** $x_i : V \to \mathbb{R}$ by:

$$x_i(a_1v_1 + \cdots + a_nv_n) = a_i$$

for $i = 1, 2, \dots, n$. Prove the following:

(a.) x_i is differentiable and hence continuous on V,

(b.) $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ for $1 \le i, j \le n$. Recall $\delta_{ij} = \begin{cases} 1 & for\ i = j \\ 0 & for\ i \neq j \end{cases}$

Problem 16 Problem to be announced in lecture here.