Math 332 Tour 2: implicit & inverse mapping theorems, Tangent & Normal Spaces

Same instructions as Tour 1. Thanks!

Problem 17 Suppose $x_1, \dots, x_n$ are coordinates of a normed linear space V with respect to the basis $\beta = \{v_1, \dots, v_n\}$. Let $F, G : V \to \mathbb{R}$ be differentiable functions on V and $h : \mathbb{R} \to \mathbb{R}$ a differentiable function on $\mathbb{R}$. Show: for $c \in \mathbb{R}$ and for $i = 1, \dots, n$,

$$\frac{\partial}{\partial x_i}\left[cF(x) + G(x)\right] = c\frac{\partial F}{\partial x_i} + \frac{\partial G}{\partial x_i} \qquad \& \qquad \frac{\partial}{\partial x_i}\left[h(F(x))\right] = h'(F(x))\frac{\partial F}{\partial x_i}.$$

Problem 18 add problem in lecture here.

Problem 19 If $x^2+y^2+z^2+w^2 = 1$ and $xywz = 1$ then calculate $\frac{\partial z}{\partial x}\Big|_y$. That is, take z, w to be dependent variables and calculate the derivative of z with respect to x while holding y-fixed.

Problem 20 Let $G(x, y, a, b) = (x^2 - y^2 - ax + by, 2xy - xb - ya)$. Suppose $M = G^{-1}(2, 1)$.

(a.) Solve for a, b as functions of x, y

(b.) use the implicit function theorem to show where it is possible to solve for a, x as functions of b, y (no need to actually solve it, demonstration of existence suffices)

(c.) use the implicit function theorem to show where it is possible to solve for a, y as functions of b, x (no need to actually solve it, demonstration of existence suffices))

(d.) use the implicit function theorem to show where it is possible to solve for x, y as functions of a, b. (no need to actually solve it, demonstration of existence suffices))

note: I don't expect you to analyze the subtle question of if it is still possible to solve where there implicit function theorem breaks down. I merely wish for you to find the low-hanging fruit which the implicit function theorem provides

Problem 21 Let $F(x, y, z, w) = (e^x \cosh y, e^x \sinh y, e^z \cos w, e^z \sin w)$ for all $(x, y, z, w) \in \mathbb{R}^4$. Show this mapping is locally invertible. Prove that no global inverse exists.

Problem 22 Define $F(x, y, z) = (x/y, y/z, z)$ for $y, z \neq 0$. Calculate J_F and determine where F can be F is locally invertible. Calculate $F^{-1}(a, b, c)$.

Problem 23 Let $F(x, y) = (x^3 - 3xy^2, 3x^2y - y^3)$ for all $(x, y) \in \mathbb{R}^2$. Show F is locally invertible at all points in the plane except one. Find the inverse for F restricted to the sector $-\pi/3 < \theta < \pi/3$ for $r > 0$ (I use the usual polar coordinates in the plane)

Problem 24 Consider $\gamma(t) = (t, t^2/2, 4, -t)$ for $t \in \mathbb{R}$. Let $C = \gamma(\mathbb{R})$. Find the tangent and normal space to C at $\gamma(2)$.

Problem 25 Consider $F(x, y, z, w) = (x^2 + y^2, z^2 - w^2)$. Define $M = F^{-1}(5, -7)$.

(a.) Find the tangent space and normal space to M at the point $p = (1, 2, 3, 4)$.

(b.) Find a parametrization of M near $p = (1, 2, 3, 4)$ and find T_pM via a calculation involving the parametrization

Problem 26 Consider $F(x, y, z, t) = x^2 + y^2 + z^2 - t^2$. Let $M = F^{-1}(0)$ and $p = (\ 1, \sqrt{2}, \sqrt{3}, \sqrt{6}\ )$.

(a.) Find the normal space to M at p,

(b.) Find a parametrization of M and use it to calculate T_pM.

Problem 27 Let $S_R(x_o, y_o)$ be the circle of radius R centered at (x_o, y_o).

(a.) Find a parametrization of $M = S_R(x_o, y_o) \times S_A(x_1, y_1) \subseteq \mathbb{R}^4$. Find the tangent space at an arbitrary point in M

(b.) Express $M = S_R(x_o, y_o) \times S_A(x_1, y_1) \subseteq \mathbb{R}^4$ as the level-set of an appropriate function. Find the normal space to M at an arbitrary point on M.

Problem 28 Use the method of Lagrange multipliers to find the distance between the unit-circle $x^2 + y^2 = 1$ and the line $x + y = 4$.

Problem 29 Find the highest and lowest points on the ellipse of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$.

Problem 30 Use the method of Lagrange multipliers to find the minimum distance from the origin to the curve of intersection of the surfaces $z^2 = x^2 + y^2$ and $x - 2z = 3$.

Problem 31 Let A be a symmetric matrix; $A^T = A$. Define $Q(x) = x^T Ax$ for each $x \in \mathbb{R}^n$. Apply the method of Lagrange multipliers to find the condition for min/max of Q restricted to $S_{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (here I use $\|x\|^2 = x^T x$, that is $\|x\|$ is the Euclidean norm).

Problem 32 add problem in lecture here.