Same instructions as Tour 1. Thanks!

Problem 33 Suppose $T : V \times V \times V^* \to \mathbb{R}$ is a multilinear map. If $\beta = \{e_1, \dots, e_n\}$ is basis for V and $\theta^1, \dots, \theta^n$ is the dual basis for V^* then find $C_{ij}{}^k \in \mathbb{R}$ for which $T = \sum_{i,j,k} C_{ij}{}^k \theta^i \otimes \theta^j \otimes e_k$.

Problem 34 Renteln Exercise 2.2 page 36. (contraction of symmetric and antisymmetric gives zero)

Problem 35 Renteln Exercise 2.5 page 41. ($\alpha \wedge \beta = (-1)^{|\alpha|} \beta \wedge \alpha$ see my notes for help)

Problem 36 Renteln Exercise 2.15 page 50. (Cartan's Lemma)

Problem 37 Renteln Exercise 2.17 page 50. (Hodge duality on $\mathbb{R}^n$ with Euclidean metric)

Problem 38 Let $\omega_{\langle a,b,c\rangle} = ae^1 + be^2 + ce^3$ and $\Phi_{\langle a,b,c\rangle} = ae^2 \wedge e^3 + be^3 \wedge e^1 + ce^1 \wedge e^2$ notice the Hodge dual with respect to the Euclidean metric on $\mathbb{R}^3$ gives $\star\omega_{\langle a,b,c\rangle} = \Phi_{\langle a,b,c\rangle}$ and $\star 1 = e^1 \wedge e^2 \wedge e^3$ and $\star^2 = 1$. Calculate the following:

(a.) $\omega_{\vec{A}} \wedge \omega_{\vec{B}}$ and write your answer in terms of Φ of a well-known vector product from Calculus III,

(b.) $\star(\omega_{\vec{A}} \wedge \omega_{\vec{B}} \wedge \omega_{\vec{C}})$ and explain the geometric significance of this quantity which hopefully you saw in Calculus III.

Problem 39 Problem from Lecture to be added here.

Problem 40 Let $\mathcal{I}(p, n)$ denote the set of all strictly increasing p-tuples of indices taken from $\{1, 2, \dots, n\}$. Show that if $\beta = \{e_1, \dots, e_n\}$ is a basis for V then $\Lambda^p V$ has basis $\beta^p = \{e_I \mid I \in \mathcal{I}(p, n)\}$ where the **multi-index** notation $I = (i_1, \dots, i_p)$ implicits $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$.

Problem 41 Let $F(x, y, z) = (\cos(x), \sin(xy))$. Calculate the push-forward of $X = a\partial_x + b\partial_y + c\partial_z$.

Problem 42 Observe $\chi = (\theta, \phi)$ gives a coordinate chart on $S_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. The inverse of this chart is given by the patch $\chi^{-1}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Calculate the push-forward under χ^{-1} of ∂_θ and ∂_ϕ in terms of the standard coordinate derivations $\partial_x, \partial_y, \partial_z$ the Cartesian coordinate system (x, y, z) for $\mathbb{R}^3$.

Problem 43 Renteln Exercise 3.19 page 74-75. (spherical frame from derivation viewpoint)

Problem 44 Renteln Exercise 3.20 page 77-78. (Lie bracket of vector fields)

Problem 45 Renteln Exercise 3.22 page 84. (derivative of vector field not tensorial)

Problem 46 Renteln Exercises 3.25 and 3.26 page 90. (exterior differentiation)

Problem 47 In $\mathbb{R}^4$ with metric $\eta = Diag(-1, 1, 1, 1)$ I describe in my notes how Hodge duality introduces certain signs. The basic idea is very much like the simpler context of $\mathbb{R}^3$. Because I use the metric which agrees with the euclidean metric on x, y, z components the work and flux-form correspondences naturally generalize: a general one-form on $\mathbb{R}^4_{txyz}$ space has the form:

$$\alpha = \alpha_0 dt + \alpha_1 dx + \alpha_2 dy + \alpha_3 dz = \alpha_0 dt + \omega_{\vec{\alpha}}.$$

Notice, I am encouraging the notation $\vec{\alpha}$ for the spatial vector piece of the one-form α. No such simple correspondence is possible for a generic two-form since it has six independent components:

$$\begin{aligned}\beta &= F_1 dt \wedge dx + F_2 dt \wedge dy + F_3 dt \wedge dz + G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy \\ &= dt \wedge \omega_{\vec{F}} + \Phi_{\vec{G}}\end{aligned}$$

Clearly the formula in terms of the work and flux-form correspondance will make it easier for us to follow calculus and algebra for β. Next, a three-form has the general form:

$$\begin{aligned}\gamma &= G_0 dx \wedge dy \wedge dz + G_1 dt \wedge dy \wedge dz + G_2 dt \wedge dz \wedge dx + G_3 dt \wedge dx \wedge dy \\ &= G_0 dx \wedge dy \wedge dz + dt \wedge \Phi_{\vec{\gamma}}\end{aligned}$$

where I am encouraging use of the notation $\vec{\gamma} = \langle G_1, G_2, G_3 \rangle$ to emphasize the correspondence between spatial 3-vectors and those components of γ. Continuting, there is just one 4-form:

$$\zeta = f dt \wedge dx \wedge dy \wedge dz.$$

Please notice that all the coefficients of the forms are in fact 0-forms on $\mathbb{R}^4$, that is, functions of t, x, y, z. This introduces time derivative terms in the formulas you are to find below. Use the notation given above to calculate:

(a.) df where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $d\alpha$

(c.) $d\beta$

(d.) $d\gamma$

(e.) $d\zeta$

Problem 48 Again, using the notation introduced in the previous problem, find the explicit (and as nice as possible) formulas for:

(a.) $\star f$ where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $\star\alpha$

(c.) $\star\beta$

(d.) $\star\gamma$

(e.) $\star\zeta$