Please solve these problems on other paper and clearly label each problem in the order which they are given here. Staple your work in the upper left corner with a metal staple and do not fold. Please show work for full credit. I am interested in the steps. This instruction applies for the remainder of the problem sets and it is assumed you remember this. **Thanks!**

Please do your best with the time you have, I would rather get two solved problems then none. Also, notice some of these problems are very simple. I marked the ones that I suspect take deeper thinking with $*$. In this particular Problem Set, many of the problems are little more than understanding notation and doing a simple, short calculation. Also, remember your calculus III, to find vector from P to Q use $v = Q - P$ this is especially relevant to Problems 1 and 2.

Problem 1 Suppose a line $\mathcal{L}$ in $\mathbb{R}^3$ contains the points $(1,1,1)$ and $(2,4,6)$. Describe $\mathcal{L}$ as follows:

(a.) parametrically with parameter λ.

(b.) as the solution set of one or more equations in x, y, z.

Problem 2 Suppose a plane $\mathcal{P}$ contains the points $(1,1,0\dots,0)$, $(1,2,0\dots,0)$ and $(2,1,0\dots,0)$ in $\mathbb{R}^n$. Describe $\mathcal{P}$ as follows:

(a.) parametrically with parameters α, β

(b.) as the solution set of one of more equations in $x_1, x_2, \dots, x_n$.

Problem 3 Suppose $\beta = \{1, x-1, (x-1)^2\}$ is a basis for polynomials P_2. Find the coordinate vector of $f(x) = ax^2 + bx + c$ with respect to β. That is, find $\Phi_\beta(f(x))$. Furthermore, suppose $T : P_2 \to P_2$ is defined by $T(f(x)) = f''(x)$. Find the matrix of T w.r.t. the basis β; that is, calculate $[T]_{\beta,\beta}$.

Problem 4 Suppose X, Y are sets and $V \subseteq Y$ and $U \subseteq X$. Let $f : X \to Y$ be a function, we define the **inverse image of V under f** by:

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\},$$

likewise, define the **image of U under f** by:

$$f(U) = \{y \in Y \mid \text{there exists } x \in U \text{ with } f(x) = y\} = \{f(x) \mid x \in U\}.$$

Given the definitions above, calculate the images and inverse images given below (find the set which describes the image or inverse image and if possible identify it geometrically) :

(a.) if $F : \mathbb{R}^2 \to \mathbb{R}$ is defined by $F(x,y) = y - x^2 - 1$ then describe $F^{-1}\{0\}$ as a point-set.

(b.) if $G : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $G(x,y,z) = (x^2 + y^2 + z^2, z)$ describe $G^{-1}(R^2, k)$ geometrically. What condition do we need for $G^{-1}(R^2, k) \neq \emptyset$? (assume $R, k \in \mathbb{R}$)

(c.) let $X : \mathbb{R}^2 \to \mathbb{R}^n$ be defined by $X(s,t) = p + sv + tw$ where v, w are linearly independent n-vectors and $p \in \mathbb{R}^n$. Describe $X([0,1]^2)$.

Problem 5 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ the **standard matrix** of T is denoted $[T] \in \mathbb{R}^{m\times n}$ and is defined by $[T]_{ij} = [T(e_j)]_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. In matrix notation, this definition is nicely written as $[T] = [T(e_1)|T(e_2)|\cdots|T(e_n)]$. Use the given definition to find the standard matrix for the linear transformations given below:

(a.) $T_1(x, y) = (x + 2y, 3x + 4y)$

(b.) $T_2(x, y) = (x + y, 2x + 2y, 3x + 3y)$

(c.) $T_3(x, y, z) = x + 2y + 3z$

(d.) $T_4(x) = (x, 2x, 3x)$

Problem 6 Calculate the composition $T_3 \circ T_2 \circ T_1$ in two ways:

(a.) from the definition of function composition $(T_3 \circ T_2 \circ T_1)(x, y) = T_3(T_2(T_1(x, y)))$,

(b.) via matrix multiplication $(T_3 \circ T_2 \circ T_1)(x, y) = [T_3 \circ T_2 \circ T_1](x, y) = [T_3][T_2][T_1][x, y]^T$.

remark: this is why the product of matrices is defined as it is.

Problem 7 Let $A_3 = \{A \in \mathbb{R}^{3\times3} \mid A^T = -A\}$. Find a basis $\{f_1, f_2, f_3\}$ for A_3 by writing $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and studying the condition $A^T = -A$. You should find $\dim(A_3) = 3$.

Bonus: *study the isomorphism $\Phi : A_3 \to \mathbb{R}^3$ defined by linearly extending $\Phi(f_i) = e_i$, if we think of antisymmetric 3×3 matrices as vectors then what is the geometric meaning of matrix multiplication for A_3?*

Problem 8* Suppose P is a paralellogram in $\mathbb{R}^3$ in the octant with positive coordinates. Furthermore, define $P = \{\vec{r}_o + u\vec{A} + v\vec{B} \mid (u, v) \in [0, 1]^2\}$.

(a.) find $Area(P)$.

(b.) define $L_{ij}(\vec{v}) = (\vec{v} \bullet \widehat{x}_i)\,\widehat{x}_i + (\vec{v} \bullet \widehat{x}_j)\,\widehat{x}_j$ and prove using properties of dot-products that L_{ij} is a linear transformation.

(c.) assume that linear transformations map parallelograms to lines, parallelograms or points and use this presupposition to establish the following equation:

$$Area(P)^2 = Area(L_{12}(P))^2 + Area(L_{31}(P))^2 + Area(L_{23}(P))^2$$

Problem 9 Let $c \in \mathbb{R}$, $A, X \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ we define $AB \in \mathbb{R}^{m\times p}$, $A + X, cA \in \mathbb{R}^{m\times n}$ by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}, \qquad (A + X)_{ij} = A_{ij} + X_{ij}, \qquad (cA)_{ij} = cA_{ij}.$$

Using the index notation given above, show (for appropriately sized matrices)

(a.) $A(B + C) = AB + AC$

(b.) if $(A^T)_{ij} = A_{ji}$ for all i, j then $(AB)^T = B^T A^T$ (socks-shoes identity)

(c.) If $I \in \mathbb{R}^{n\times n}$ such that $I_{ij} = \delta_{ij}$ and $X \in \mathbb{R}^{n\times p}$ then $IX = X$.

Problem 10 It is convenient to introduce some notation: the *Kronecker delta* is defined by $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. Note the standard basis for $\mathbb{R}^n$ is nicely described by $(e_i)_j = \delta_{ij}$.

It is also often convenient to introduce the completely antisymmetric symbol in n-dimensions (the Levi-Civita symbol);

$$\epsilon_{i_1 i_2 \ldots i_n} = \det[e_{i_1}|e_{i_2}|\cdots|e_{i_n}].$$

The formula above means that $\epsilon_{12\ldots n} = det[I] = 1$ and all other nontrivial values can be obtained by swapping indices. Each swap changes the sign. For example, $\epsilon_{123\ldots n} = -\epsilon_{213\ldots n} = -1$. If any index is repeated then the antisymmetric symbol is zero.

A popular convention in physics and some math is that when an index is repeated a summation is implied. For example, $A_iB_i = \sum_i A_iB_i$ where the $\sum_i$ ranges over whatever is the accepted range of the index i. Let A_i, B_i denote three-dimensional vectors. Verify that

(a.) $A \bullet B = A_iB_i$

(b.) $A \times B = A_iB_j\epsilon_{ijk}e_k$

MISSION 1 : SOLUTION

PROBLEM 1 Suppose a line $\mathcal{L}$ in $\mathbb{R}^3$ contains $(1,1,1)$ and $(2,4,6)$.

(a.) Parametrically: $\varphi(\lambda) = (1,1,1) + \lambda[(2,4,6)-(1,1,1)]$

$$\varphi(\lambda) = (1,1,1) + \lambda(1,3,5)$$

$$\boxed{\varphi(\lambda) = (1+\lambda,\ 1+3\lambda,\ 1+5\lambda)}, \quad \underline{\mathcal{L} = \varphi(\mathbb{R})}.$$

(b.)
$$\left.\begin{array}{l} x = 1+\lambda \\ y = 1+3\lambda \\ z = 1+5\lambda \end{array}\right\} \longrightarrow \lambda = x-1 = \frac{y-1}{3} = \frac{z-1}{5}$$

$$\boxed{\mathcal{L} = \left\{ (x,y,z) \ \middle|\ x-1 = \frac{y-1}{3} \text{ and } \frac{y-1}{3} = \frac{z-1}{5} \right\}}$$

PROBLEM 2 $\mathcal{P}$ a plane with $(1,1,0,\dots,0) = e_1+e_2 = P$ and $(1,2,0,\dots,0) = \underbrace{e_1+2e_2}_{Q}$ and $(2,1,0,\dots,0) = \underbrace{2e_1+e_2}_{R}$ in $\mathbb{R}^n$.

(a.) describe $\mathcal{P}$ parametrically with parameters α, β.

$$\varphi(\alpha,\beta) = P + \alpha(Q-P) + \beta(R-P)$$

just thought of this today. This gives $\varphi(0,0) = P$, $\varphi(1,0) = Q$, $\varphi(0,1) = R$. It will parametrize plane in $\mathbb{R}^n$ with the triple P, Q, R of points.

$$\varphi(\alpha,\beta) = e_1+e_2 + \alpha(\cancel{e_1}+2e_2-\cancel{e_1}-e_2) + \beta(2e_1+\cancel{e_2}-e_1-\cancel{e_2})$$

$$\varphi(\alpha,\beta) = e_1+e_2+\alpha e_2+\beta e_1$$

$$\boxed{\varphi(\alpha,\beta) = (1+\beta,\ 1+\alpha,\ 0,\dots,0)}, \quad \underline{\mathcal{P} = \varphi(\mathbb{R}^2)}.$$

(b.) Clearly $\mathcal{P}$ is set of $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ such that $\boxed{x_3 = x_4 = \cdots = x_n = 0.}$ No condition is found for x_1, x_2 as $x_1 = 1+\beta$, $x_2 = 1+\alpha$ indicate <u>all</u> values are attained as α, β vary over $\mathbb{R}$.

PROBLEM 3 $\beta = \{1, (x-1), (x-1)^2\}$ a basis for P_2.
Find $\Phi_\beta(f(x))$ for $f(x) = ax^2+bx+c$. Also, find $[T]_{\beta,\beta}$ for $T(f) = f''$.

There are many ways. I'll add zero.

$$\begin{aligned}
f(x) &= a(x-1+1)^2 + b(x-1+1) + c \\
&= a\left[(x-1)^2 + 2(x-1) + 1\right] + b(x-1) + b + c \\
&= a(x-1)^2 + [2a+b](x-1) + a+b+c
\end{aligned}$$

$$\Rightarrow \boxed{\Phi_\beta(f(x)) = (a+b+c,\ 2a+b,\ a)}$$

$$\begin{tikzcd}
P_2 \ni f(x),\ f(x) = ax^2+bx+c \arrow[r, "T"] \arrow[d] & f''(x),\ 2a = T(f(x)) \in P_2 \arrow[d] \\
\mathbb{R}^3 \ni \begin{bmatrix} a+b+c \\ 2a+b \\ a \end{bmatrix} \arrow[r, "{[T]_{\beta,\beta}}"] & \begin{bmatrix} 2a \\ 0 \\ 0 \end{bmatrix} = [T(f(x))]_\beta \in \mathbb{R}^3
\end{tikzcd}$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{[T]_{\beta,\beta}} \begin{bmatrix} a+b+c \\ 2a+b \\ a \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \\ 0 \end{bmatrix}$$

(I like the next page's approach a bit more)

PROBLEM 3 continued

$$\begin{tikzcd}
f(x) = a + b(x-1) + c(x-1)^2 \arrow[r] \arrow[d, "\Phi_\beta"'] & T(f(x)) = 2c \arrow[d, "\Phi_\beta"] \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} \arrow[r, "{[T]_{\beta,\beta}}"] & \begin{bmatrix} 2c \\ 0 \\ 0 \end{bmatrix}
\end{tikzcd}$$

$\mathbb{R}^3$ $\qquad\qquad$ $\mathbb{R}^3$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2c \\ 0 \\ 0 \end{bmatrix} \quad \therefore \quad \boxed{[T]_{\beta,\beta} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

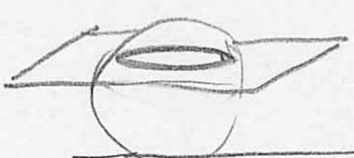
PROBLEM 4

(a.) $F(x,y) = y - x^2 - 1$

$$F^{-1}\{0\} = \{(x,y) \mid F(x,y) = 0\} = \boxed{\{(x,y) \mid y - x^2 - 1 = 0\}}$$

this is the graph $y = x^2 + 1$ (a parabola)

(b.) $$G^{-1}(R^2, k) = \{(x,y,z) \mid G(x,y,z) = (R^2, k)\}$$
$$= \{(x,y,z) \mid (x^2+y^2+z^2, z) = (R^2, k)\}$$
$$= \boxed{\{(x,y,z) \mid \underbrace{x^2+y^2+z^2 = R^2 \text{ \underline{and} } z = k}_{\text{intersection of sphere and horizontal plane.}}\}}$$

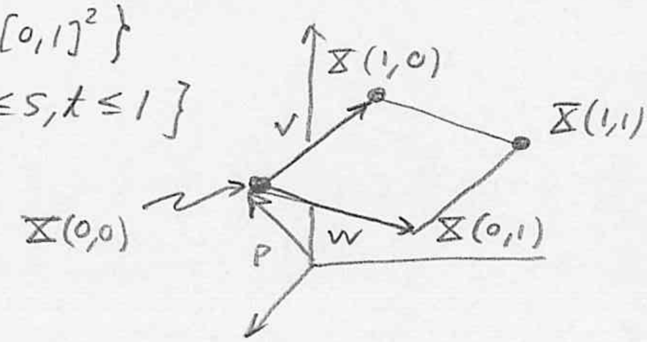


$\boxed{\text{We need } -R \le k \le R \text{ for } G^{-1}(R^2,k) \neq \phi}$

(c.)

PROBLEM 4c Desribe $\mathbf{X}([0,1]^2)$ for $\mathbf{X}(s,t) = P + sv + tw$ where $\{v, w\}$ is LI set of vectors in $\mathbb{R}^n$ ($\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^n$)

$$\mathbf{X}([0,1]^2) = \{ \mathbf{X}(s,t) \mid (s,t) \in [0,1]^2 \}$$
$$= \{ P + sv + tw \mid 0 \leq s,t \leq 1 \}$$



It gives a parallelogram sitting inside $\mathbb{R}^n$ with one corner at P.

PROBLEM 5 Find standard matrix of maps below.

(a.) $T_1(x,y) = \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow \boxed{[T_1] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$

(b.) $T_2(x,y) = \begin{bmatrix} x+y \\ 2x+2y \\ 3x+3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \boxed{[T_2] = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}}$

(c.) $T_3(x,y,z) = x + 2y + 3z = [1 \ 2 \ 3]\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \boxed{[T_3] = [1 \ 2 \ 3]}$

(d.) $T_4(x) = \begin{bmatrix} x \\ 2x \\ 3x \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \boxed{[T_4] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$

PROBLEM 6 Calculate composition $T_3 \circ T_2 \circ T_1$ in <u>two</u> ways:

(a.) $(T_3 \circ T_2 \circ T_1)(x,y) = T_3(T_2(x+2y, 3x+4y))$ $\quad u = x+2y$, $v = 3x+4y$

$= T_3(u+v, \ 2u+2v, \ 3u+3v)$

$= (u+v) + 2(2u+2v) + 3(3u+3v)$

$= 14u + 14v$

$= 14(x+2y) + 14(3x+4y) = \boxed{56x + 84y}$

(b.) $(T_3 \circ T_2 \circ T_1)(x,y) = [T_3][T_2][T_1]\begin{bmatrix} x \\ y \end{bmatrix}$

$= [1 \ 2 \ 3]\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [14, 14]\begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}$

(did you know this?) 

$= 14(x+2y) + 14(3x+4y)$

$= \boxed{56x + 84y}$

PROBLEM 7 Let $A_3 = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$. Find basis for A_3

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^T = -\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$$

Thus, $a = -a$, $e = -e$, $i = -i \Rightarrow a = e = i = 0$.
and $d = -b$, $g = -c$, $h = -f$ so we find

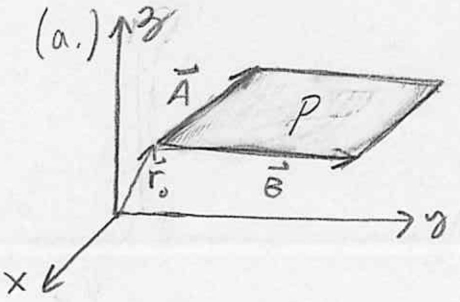
$$A = \begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Thus, as LI of the following set is pretty obvious,

$$\boxed{\beta = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.}$$

PROBLEM 8 Suppose P is parallelogram in $\mathbb{R}^3$ in the $x, y, z > 0$ octant. Furthermore, $P = \{\vec{r}_o + u\vec{A} + v\vec{B} \mid (u,v) \in [0,1]^2\}$

(a.)



$$\underline{\text{Area}(P) = \|\vec{A} \times \vec{B}\| = AB\sin\theta}.$$

(b.) $$L_{ij}(\vec{v} + c\vec{w}) = \left((\vec{v} + c\vec{w}) \cdot \hat{x}_i\right)\hat{x}_i + \left((\vec{v} + c\vec{w}) \cdot \hat{x}_j\right)\hat{x}_j$$
$$= \left(\vec{v} \cdot \hat{x}_i + c\vec{w} \cdot \hat{x}_i\right)\hat{x}_i + \left(\vec{v} \cdot \hat{x}_j + c\vec{w} \cdot \hat{x}_j\right)\hat{x}_j$$
$$= (\vec{v} \cdot \hat{x}_i)\hat{x}_i + (\vec{v} \cdot \hat{x}_j)\hat{x}_j + c\left((\vec{w} \cdot \hat{x}_i)\hat{x}_i + (\vec{w} \cdot \hat{x}_j)\hat{x}_j\right)$$
$$= L_{ij}(\vec{v}) + cL_{ij}(\vec{w})$$

Thus $L_{ij} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation.

PROBLEM 8 continued

(c.) Show $\text{Area}(P)^2 = \text{Area}(L_{12}(P))^2 + \text{Area}(L_{31}(P))^2 + \text{Area}(L_{23}(P))^2$ ☆

Short of a clever argument... brute force.

$$\vec{A} = \langle A_1, A_2, A_3 \rangle$$
$$\vec{B} = \langle B_1, B_2, B_3 \rangle$$
$$\vec{A} \times \vec{B} = \langle A_2 B_3 - A_3 B_2,\ A_3 B_1 - A_1 B_3,\ A_1 B_2 - A_2 B_1 \rangle$$

$$L_{12}(\vec{A}) = \langle A_1, A_2, 0 \rangle \qquad L_{12}(\vec{B}) = \langle B_1, B_2, 0 \rangle$$
$$L_{31}(\vec{A}) = \langle A_1, 0, A_3 \rangle \qquad L_{31}(\vec{B}) = \langle B_1, 0, B_3 \rangle$$
$$L_{23}(\vec{A}) = \langle 0, A_2, A_3 \rangle \qquad L_{23}(\vec{B}) = \langle 0, B_2, B_3 \rangle$$

The sides of $L_{12}(P)$ are given by $L_{12}(\vec{A})$ and $L_{12}(\vec{B})$ thus,

$$\begin{aligned} \text{Area}(L_{12}(P))^2 &= \| L_{12}(\vec{A}) \times L_{12}(\vec{B}) \|^2 \\ &= \| \langle A_1, A_2, 0 \rangle \times \langle B_1, B_2, 0 \rangle \|^2 \\ &= \| \langle 0, 0, A_1 B_2 - A_2 B_1 \rangle \|^2 \\ &= (A_1 B_2 - A_2 B_1)^2. \end{aligned}$$

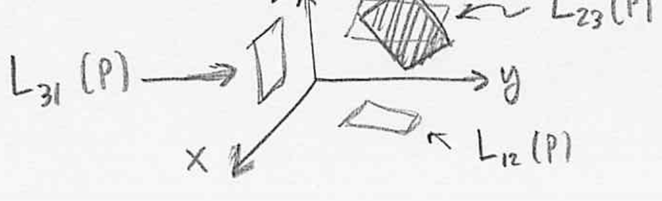
Likewise, $L_{31}(P)$ has edges $L_{31}(\vec{A}), L_{31}(\vec{B})$ hence,

$$\begin{aligned} \text{Area}(L_{31}(P))^2 &= \| \langle A_1, 0, A_3 \rangle \times \langle B_1, 0, B_3 \rangle \|^2 \\ &= \| \langle 0, A_3 B_1 - A_1 B_3, 0 \rangle \|^2 \\ &= (A_3 B_1 - A_1 B_3)^2. \end{aligned}$$

And of course, $L_{23}(P)$ has edges $L_{23}(\vec{A}), L_{23}(\vec{B})$ hence,

$$\begin{aligned} \text{Area}(L_{23}(P))^2 &= \| \langle 0, A_2, A_3 \rangle \times \langle 0, B_2, B_3 \rangle \|^2 \\ &= \| \langle A_2 B_3 - A_3 B_2, 0, 0 \rangle \|^2 \\ &= (A_2 B_3 - A_3 B_2)^2. \end{aligned}$$

Thus, as $\|\vec{A} \times \vec{B}\|^2 = (A_2 B_3 - A_3 B_2)^2 + (A_3 B_1 - A_1 B_3)^2 + (A_1 B_2 - A_2 B_1)^2$ we find that the claim ☆ is true.



Remark: this is also true in two dimensions ☺ or n (!)

PROBLEM 9

(a.) Show $A(B+C) = AB + AC$ for $A \in \mathbb{R}^{m \times n}$, $B, C \in \mathbb{R}^{n \times p}$

$$(A(B+C))_{ij} = \sum_{k=1}^{n} A_{ik} (B+C)_{kj} \quad : \text{def}^{\underline{n}} \text{ of mat. mult.}$$

$$= \sum_{k=1}^{n} A_{ik} (B_{kj} + C_{kj}) \quad : \text{def}^{\underline{n}} \text{ of mat. } (+)$$

$$= \sum_{k=1}^{n} A_{ik} B_{kj} + \sum_{k=1}^{n} A_{ik} C_{kj} \quad : \text{prop. of finite sums.}$$

$$= (AB)_{ij} + (AC)_{ij} \quad : \text{def}^{\underline{n}} \text{ of mat. mult.}$$

$$= (AB + AC)_{ij} \quad : \text{def}^{\underline{n}} \text{ of mat. } (+)$$

This holds $\forall i,j$ hence $A(B+C) = AB + AC$ as desired. //

(b.) Show $(AB)^T = B^T A^T$ Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

$$[(AB)^T]_{ij} = (AB)_{ji} \quad : \text{def}^{\underline{n}} \text{ of transpose.}$$

$$= \sum_{k=1}^{n} A_{jk} B_{ki} \quad : \text{def}^{\underline{n}} \text{ of mat. mult.}$$

$$= \sum_{k=1}^{n} B_{ki} A_{jk} \quad : \text{real \# mult. commutes.}$$

$$= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} \quad : \text{def}^{\underline{n}} \text{ of transpose}$$

$$= (B^T A^T)_{ij} \quad \forall i,j \; \therefore \; \underline{(AB)^T = B^T A^T} \; //$$

(C.) If $I \in \mathbb{R}^{n \times n}$ s.t. $I_{ij} = \delta_{ij}$ and $X \in \mathbb{R}^{n \times p}$ then $IX = X$.

$$(IX)_{ij} = \sum_{k=1}^{n} I_{ik} X_{kj} = \sum_{k=1}^{n} \delta_{ik} X_{kj} = X_{ij}$$

get it?

Thus $IX = X$. //

$$\delta_{ik} = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases}$$

$\boxed{\text{PROBLEM 10}}$

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \quad , (e_i)_j = \delta_{ij} \quad \text{slick def}^{\underline{t}} \text{ for standard basis.}$$

$$\epsilon_{i_1 i_2 \cdots i_n} = \det[e_{i_1} | e_{i_2} | \cdots | e_{i_n}]$$

(a.) $A \cdot B = A_1 B_1 + A_2 B_2 + A_3 B_3 = \underbrace{A_i B_i}$

Einstein Σ notation.

(b.) $A \times B = A_i B_j \epsilon_{ijk} e_k$

$$= A_1 B_2 \epsilon_{12k} e_k + A_1 B_3 \epsilon_{13k} e_k + A_2 B_3 \epsilon_{23k} e_k + A_2 B_1 \epsilon_{21k} e_k + A_3 B_1 \epsilon_{31k} e_k + A_3 B_2 \epsilon_{32k} e_k$$

$$= A_1 B_2 \epsilon_{123} e_3 + A_1 B_3 \epsilon_{132} e_2 + A_2 B_3 \epsilon_{231} e_1 + A_2 B_1 \epsilon_{213} e_3 + A_3 B_1 \epsilon_{312} e_2 + A_3 B_2 \epsilon_{321} e_1$$

$$= (A_1 B_2 - A_2 B_1) e_3 + (A_3 B_1 - A_1 B_3) e_2 + (A_2 B_3 - A_3 B_2) e_1$$

$$= \langle A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1 \rangle$$

Here I used

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$$

these all follow from prop. of determinants. For example, flipping pair of columns generates minus

$$\epsilon_{321} = \det[e_3 | e_2 | e_1] = -\det[\underbrace{e_1 | e_2 | e_3}_{I}] = -1.$$