Problem 11 Given that $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous mapping from the normed space $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ determine if the following sets of matrices are closed or open in the set of $n \times n$ matrices:

(a.) $GL(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$ (general linear group)

(b.) $SL(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$ (special linear group)

Problem 12 Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $F(A) = A^TA - I$ where I is the $n \times n$ identity matrix. Notice that F is clearly continuous since it has a formula which is formed from polynomials in the entries of A. Let $O(n) = F^{-1}\{0\}$.

(a.) Explain why $O(n)$ is closed.

(b.) Show that if $A, B \in O(n)$ then $AB, A^{-1} \in O(n)$ (this makes $O(n)$ a group since it clearly contains the identity matrix and matrix multiplication is associative)

(c.) Is $O(n)$ connected? Hint: consider the determinant function on $O(n)$, recall $\det(AB) = \det(A)\det(B)$.

Problem 13 Let $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by $F(A) = A^2$. Find the Frechet differential $dF_A(H)$ and prove your proposed linear transformation in H satisfies the required limit. My advice is as follows:

(a.) calculate $F(A+H) - F(A)$ and select what appears linear in H and claim it is $dF_A(H)$

(b.) verify the claim by plugging your guess into the definition of differentiable.

Problem 14 Suppose V, W are normed vector spaces. Show that if $F : V \to W$ is differentiable at x_o in a normed vector space V then F is continuous at x_o.

Problem 15 Show that if $F : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and $G : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ are differentiable at A then the matrix product function FG is likewise differentiable at A.
Hint: intuition suggests $F(A+H) \approx F(A) + dF_A(H)$ for H small, the same for G, multiply and take an educated guess

Problem 16 Find the Jacobian matrix for $F(x, y) = (x^2 - y^2, 2xy)$. Use the Jacobian matrix to help construct a linearization of F centered at $(1, 1)$.

Problem 17 Find the Jacobian matrix for $G(x, y, z) = x + y^2 + z^3$. Use the Jacobian matrix to help construct a linearization of G centered at $(1, \sqrt{2}, \sqrt[3]{3})$.

Problem 18 Find the Jacobian matrix for $H(r, t, s) = (r\cos(t)\sinh(s), r\cos(t)\sinh(s), r\cosh(s))$.

Problem 19 Calculate the differential for the determinant mapping $S : \mathbb{R}^{n\times n} \to \mathbb{R}$ at A where $S(A) = \det(A)$. Note a basis for $n \times n$ matrices is given by the matrix units $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ hence it suffices to calculate $dS_A(E_{ij})$ for arbitrary i, j. Significant partial credit will be awarded for working out the $n = 2$ case.

Problem 20 Calculate the differential for the trace mapping $T : \mathbb{R}^{n\times n} \to \mathbb{R}$ at A where $T(A) = \text{trace}(A) = \sum_{i=1}^{n} A_{ii}$. Once again, it suffices to calculate $dT_A(E_{ij})$ for arbitrary i, j. Significant partial credit will be awarded for working out the $n = 2$ case.

PROBLEM 11 We are given $\det : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is continuous. Determine if $GL(n) = \det^{-1}[(-\infty,0)\cup(0,\infty)]$ and $SL(n) = \det^{-1}\{1\}$ are open or closed (neither is logically possible fwiw.)

(a.) $(-\infty,0)\cup(0,\infty)$ is open in $\mathbb{R}$.
The inverse image of open sets is open under a continuous mapping. Thus $GL(n) = \det^{-1}[(-\infty,0)\cup(0,\infty)]$ is <u>open</u> as det is continuous.

(b.) $\{1\}$ is closed in $\mathbb{R}$ as $(-\infty,1)\cup(1,\infty) = \mathbb{R}-\{1\}$ is open. Thus $SL(n) = \det^{-1}\{1\}$ is closed as the continuous mapping det takes the inverse image of closed sets to closed sets.

PROBLEM 12 Let $F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ be defined by $F(A) = A^TA - I$ where $I_{ij} = \delta_{ij}$. ($n\times n$ identity matrix) We observe F is continuous. Let $O(n) = F^{-1}\{0\}$

(a.) again, $\{0\}$ is closed in $\mathbb{R}^{n\times n}$, F continuous $\therefore$ $F^{-1}\{0\}$ is closed in $\mathbb{R}^{n\times n}$.

+2 pts to anyone who did this ☺

Ok, I wonder, why did no one ask me how to <u>show</u> $\{0\}$ is closed in $\mathbb{R}^{n\times n}$. I'm guessing you all have assumed this. Well, to show it's closed we should show $\mathbb{R}^{n\times n} - \{0\}$ is open. Let $A \in \mathbb{R}^{n\times n} - \{0\}$ subject the norm $\|A\|^2 = tr(AA^T)$. Since $A \neq 0$ it follows $\|A\| \neq 0$. Intuitively, $B_\delta(A) \subset \mathbb{R}^{n\times n} - \{0\}$ if we set $\delta = \|A\|/2$ then $0 \notin B_\delta(A)$. Let $X \in B_\delta(A)$ with $\delta = \|A\|/2$, observe $\|X - A\| < \frac{\|A\|}{2}$ hence $X \neq 0$ as $\|0 - A\| = \|A\| \not< \frac{\|A\|}{2}$. Thus, A is an interior point of $\mathbb{R}^{n\times n} - \{0\}$. But, A is arbitrary $\therefore$ $\mathbb{R}^n - \{0\}$ is open so, by definition, $\{0\}$ is closed. //

PROBLEM 12

(b.) Let $A, B \in O(n) = \{A \mid A^TA - I = 0\}$.

We have $A^TA = I \Rightarrow (A^TA)^{-1} = I^{-1}$

$\Rightarrow A^{-1}(A^T)^{-1} = I$

$\Rightarrow A^{-1}(A^{-1})^T = I$

$\Rightarrow A^{-1} \in O(n)$.

Also $B^TB = I$ so consider,

$(AB)^TAB = B^TA^TAB = B^TIB = B^TB = I$

thus $AB, A^{-1} \in O(n)$ as claimed.

(c.) Is $O(n)$ connected.?

No. Consider $SO(n) = \{A \in O(n) \mid \det(A) = 1\}$
and $-SO(n) = \{A \in O(n) \mid \det(A) = -1\}$
Notice $A \in O(n) \Rightarrow A^TA = I \Rightarrow (\det(A))^2 = 1$
But, $x^2 = 1$ has $x = \pm 1$ (if $x \in \mathbb{R}$ is context).
Here $\det(A) \in \mathbb{R}$ thus $\det(A) = \pm 1$. It
follows $O(n) = SO(n) \cup [-SO(n)]$ where
$SO(n) \cap [-SO(n)] = \emptyset$ as $\det(A) = 1 \neq -1$.
Hence $O(n)$ has a separation and so fails to
be connected.

Remark: So I did not need to show you
f (connected) = connected for f continuous. I
hope you enjoyed it anyway. Maybe we'll
need it later...

$\boxed{\text{PROBLEM 13}}$ Let $F: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ with $F(A) = A^2$.

(a.)
$$\begin{aligned} \Delta F &= F(A+H) - F(A) \\ &= (A+H)(A+H) - A^2 \\ &= A^2 + \underbrace{AH + HA}_{dF_A(H)} + H^2 - A^2 \end{aligned}$$

$dF_A(H) \leftarrow AH\ HA!$ (linear ? ☆)

(b.)
$$\lim_{H\to 0}\left[\frac{F(A+H) - F(A) - dF_A(H)}{\|H\|}\right] =$$
$$= \lim_{H\to 0}\left[\frac{A^2 + AH + HA + H^2 - A^2 - (AH + HA)}{\|H\|}\right]$$
$$= \lim_{H\to 0}\left[\frac{H^2}{\|H\|}\right]$$

Note, $\lim_{H\to 0} G(H) = 0$ iff $\lim_{H\to 0}\|G(H)\| = 0$

So, observe $\left\|\frac{H^2}{\|H\|}\right\| \leq \frac{\|H\|^2}{\|H\|} = \|H\|$

thus $0 < \left\|\frac{H^2}{\|H\|}\right\| \leq \|H\| \Rightarrow \lim_{H\to 0}\left\|\frac{H^2}{\|H\|}\right\| = 0$

by squeeze Th$^{\underline{m}}$. Hence $\lim_{H\to 0}\left[\frac{H^2}{\|H\|}\right] = 0$

and we've shown $\boxed{dF_A(H) = AH + HA}$

☆: we should check linearity,

$$\begin{aligned} dF_A(cH+K) &= A(cH+K) + (cH+K)A \\ &= cAH + AK + cHA + KA \\ &= c(AH + HA) + AK + KA \\ &= c\,dF_A(H) + dF_A(K) \end{aligned}$$

$\therefore dF_A$ is linear.

PROBLEM 14 (following Edwards excellent hint on pg. 75)

Let $R(h) = \dfrac{F(a+h) - F(a) - dF_a(h)}{\|h\|}$ for $h \neq 0$

Observe, $F(a+h) = F(a) + dF_a(h) + R(h)\|h\|$

Furthermore, if F is differentiable at $x = a$ then by def$^{\underline{n}}$ $\lim_{h \to 0} R(h) = 0$ as $R(h)$ is precisely the Frechet quotient! Consider then,

$$\lim_{h \to 0} \big(F(a+h)\big) = \lim_{h \to 0} \Big(F(a) + dF_a(h) + R(h)\|h\| \Big)$$

$$= \lim_{h \to 0} \big(F(a)\big) + \lim_{h \to 0} \big(dF_a(h)\big) + \lim_{h \to 0} R(h)\|h\|$$

$$= F(a) + 0 + \underbrace{\lim_{h \to 0} R(h)}_{0} \underbrace{\lim_{h \to 0} \|h\|}_{0}$$

$$= F(a).$$

Woot. We have shown $F(a+h) \longrightarrow F(a)$ as $h \longrightarrow 0$ and so F is continuous at a.

(let $a = x_o$ sorry ☺)

PROBLEM 15 $F : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ and $G : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ both differentiable at A. Show FG is diff at A.

Let $$R_F(H) = \frac{F(A+H) - F(A) - dF_A(H)}{\|H\|}$$

and $$R_G(H) = \frac{G(A+H) - G(A) - dG_A(H)}{\|H\|}$$ for $H \neq 0$.

Observe $\lim_{H\to 0} R_F(H) = \lim_{H\to 0} R_G(H) = 0$ by differentiability of F and G at A. Furthermore,

$$F(A+H) = F(A) + dF_A(H) + R_F(H)\,\|H\|$$
$$G(A+H) = G(A) + dG_A(H) + R_G(H)\,\|H\|$$

again, for $H \neq 0$. Consider then,

$$F(A+H)\,G(A+H) = \big(F(A) + dF_A(H) + R_F(H)\,\|H\|\big) \cdot \big(G(A) + dG_A(H) + R_G(H)\,\|H\|\big)$$

$$\begin{aligned} = {} & F(A)G(A) + dF_A(H)\,G(A) + F(A)\,dG_A(H) \\ & + dF_A(H)\,dG_A(H) + dF_A(H)R_G(H)\,\|H\| \\ & + R_F(H)\,\|H\|\,G(A) + R_F(H)\|H\|\,dG_A(H) \\ & + R_F(H)\,\|H\|\,R_G(H)\,\|H\| + F(A)R_G(H)\,\|H\|. \end{aligned}$$

$$= F(A)G(A) + dF_A(H)G(A) + F(A)dG_A(H) + \cdots$$

clearly the $+\cdots$ terms will vanish in $H \to 0$ limit even after multiplication by $\frac{1}{\|H\|}$ as they contain at least two copies of a manifest H-dependence.

PROBLEM 15 continued

Define, or claim $d(FG)_A(H) = dF_A(H)G(A) + F(A)dG_A(H)$.

Consider then, for $H \neq 0$ we define:

$$R_{FG}(H) = \frac{F(A+H)\,G(A+H) - F(A)\,G(A) - d(FG)_A(H)}{\|H\|}$$

We can simplify the formula for $R_{FG}(H)$ in view of the algebra given on previous page! Only the $+\cdots$ terms are uncancelled.

$$R_{FG}(H) = \frac{1}{\|H\|}\Big(dF_A(H)\left[dG_A(H) + R_G(H)\|H\|\right] + F(A)R_G(H)\|H\| + R_F(H)\|H\|\left[G(A) + dG_A(H) + R_G(H)\|H\|\right]\Big)$$

Counting, we had 9 total terms, 3 cancelled and 6 remain

Let me simplify the expression above, for $H \neq 0$,

$$R_{FG}(H) = \frac{1}{\|H\|} dF_A(H)dG_A(H) + dF_A(H)R_G(H) + F(A)R_G(H) + R_F(H)\left[G(A) + dG_A(H) + R_G(H)\|H\|\right]$$

$$= \frac{1}{\|H\|} dF_A(H)\,dG_A(H) + \left[dF_A(H) + F(A)\right]R_G(H) + R_F(H)\left[G(A) + dG_A(H) + R_G(H)\|H\|\right]$$

Observe, by continuity of dG_A, dF_A at $H=0$ we have $dF_A(H), dG_A(H) \longrightarrow 0$ as $H \longrightarrow 0$. By diff. of F and G we also have $R_F(H), R_G(H) \longrightarrow 0$ as $H \to 0$. Thus, by limit laws we find all terms vanish except the term $\frac{1}{\|H\|} dF_A(H)\,dG_A(H)$ which requires further analysis

PROBLEM 15 continued

$$\lim_{H\to 0}\left(\frac{1}{\|H\|}dF_A(H)\,dG_A(H)\right) = 0 \qquad (\underline{\text{why ?}})$$

Observe $\frac{1}{\|H\|}dF_A(H) = dF_A\left(\frac{H}{\|H\|}\right)$.

Thus $\left\|\frac{H}{\|H\|}\right\| = 1$. It should follow $\left\|dF_A\left(\frac{H}{\|H\|}\right)\right\| < \infty$.

Observe
$$dF_A(V) = dF_A\left(\sum_{i,j} V_{ij}E_{ij}\right) = \sum_{i,j} V_{ij}\,dF_A(E_{ij})$$

Clearly $\|dF_A(V)\| \leq \underbrace{n^2\max\{|V_{ij}|, \|dF_A(E_{ij})\|\}}_{M}$

Then, as $V = \frac{H}{\|H\|}$ for $H \neq 0$ certainly has $|V_{ij}| \leq 1$ we have $\left\|dF_A\left(\frac{H}{\|H\|}\right)\right\| \leq M$ __independent__ of H. Oh, so, let's squeeze this to oblivion,

$$0 \leq \left\|dF_A\left(\frac{H}{\|H\|}\right)\right\| \leq M$$

uses Banach Algebra prop. of $\mathbb{R}^{n\times n}$!

$$(\star)\quad 0 \leq \left\|\frac{1}{\|H\|}dF_A(H)\,dG_A(H)\right\| \leq \left\|dF_A\left(\frac{H}{\|H\|}\right)\right\|\|dG_A(H)\| \leq M\|dG_A(H)\|$$

Therefore, $\left\|\frac{1}{\|H\|}dF_A(H)\,dG_A(H)\right\| \longrightarrow 0$ as $H \to 0$ since $dG_A(H) \to dG_A(0) = 0$ and the squeeze Th$^{\underline{m}}$ applies to the above $\star$

PROBLEM 15 continued, again

Collecting our work thus far we have shown that

$$\lim_{H \to 0} \left(R_{FG}(H) \right) = 0$$

But, this implies FG is differentiable at A.

Moreover, we found

$$\boxed{d(FG)_A(H) = dF_A(H)\,G(A) + F(A)\,dG_A(H)}$$

Application ! $F(A) = A$, $G(A) = A$

$$d(FG)(H) = dF_A(H)\,G(A) + F(A)\,dG_A(H)$$
$$= HA + AH.$$

Could write, $dA^2(H) = HA + AH$.

Continuing for fun,

$$(dA^3)(H) = d(A^2A)(H)$$
$$= dA^2(H)\,A + A^2\,dA(H)$$
$$= (HA + AH)A + A^2H$$
$$= HA^2 + AHA + A^2H.$$

Continuing, could you derive $dA^n(H)$?

PROBLEM 16

$$F(x,y) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix} \Rightarrow F'(x,y) = \left[\partial_x F \middle| \partial_y F \right] = \boxed{\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}}$$

$$F(1+h, 1+k) \approx F(1,1) + F'(1,1)\begin{bmatrix} h \\ k \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2h - 2k \\ 2h + 2k \end{bmatrix}$$

$$= \boxed{(2h - 2k,\ 2 + 2h + 2k)}$$

PROBLEM 17

$$G(x,y,z) = x + y^2 + z^3 \Rightarrow \boxed{G'(x,y,z) = [1,\ 2y,\ 3z^2]}$$

$$G(1+a, \sqrt{2}+b, \sqrt[3]{3}+c) \approx G(1, \sqrt{2}, \sqrt[3]{3}) + G'(1, \sqrt{2}, \sqrt[3]{3})\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\approx 1 + 2 + 3 + [1,\ 2\sqrt{2},\ 3 \cdot 3^{2/3}]\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\approx \boxed{6 + a + 2\sqrt{2}\, b + 3^{5/3} c}$$

PROBLEM 18

$$H(r,t,s) = (r\cos(t)\sinh(s),\ r\cos(t)\sinh(s),\ r\cosh(s))$$

$$H'(r,t,s) = \left[\frac{\partial H}{\partial r} \middle| \frac{\partial H}{\partial t} \middle| \frac{\partial H}{\partial s} \right] = \boxed{\left[\begin{array}{c|c|c} \cos t \sinh s & -r\sin t \sinh s & r\cos t \cosh s \\ \cos t \sinh s & -r\sin t \sinh s & r\cos t \cosh s \\ \cosh s & 0 & r\sinh s \end{array}\right]}$$

PROBLEM 19 $\epsilon_{123\cdots n} = 1$ and all othervalues obtained by complete antisymmetry.

Definition: $\det(A) = \sum_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} A_{i_1 1} A_{i_2 2} \cdots A_{i_n n}$

Let us study the directional derivative along E_{ij} at A. We feed det the path $\varphi(t) = A + tE_{ij}$ for fixed, but arbitrary, $i, j \in \mathbb{N}_n$. Observe that:

$$\left(D_{E_{ij}} \det\right)(A) = \frac{d}{dt}\left[\det(A + tE_{ij})\right]\Big|_{t=0}$$

But, more to the point, we must study the difference quotient formed as follows:

$$\lim_{t \to 0} \left[\frac{\det(A + tE_{ij}) - \det(A)}{t}\right] = \ ?$$

Consider then, as $(E_{ij})_{kl} = \delta_{ik}\delta_{jl} \Rightarrow$ non zero entry in j^{th} column.

$$A + tE_{ij} = \left[A_{i,1} \,\middle|\, A_{i_2 2} \,\middle|\, \cdots \,\middle|\, A_j + te_i \,\middle|\, \cdots \,\middle|\, A_{in}\right]$$

By multilinearity of determinant. (It's linear in each column),

$$\det(A + tE_{ij}) = \det\left[A_{i,1} \,\middle|\, \cdots \,\middle|\, A_j \,\middle|\, \cdots \,\middle|\, A_{in}\right] + t\det\left[A_{i1} \,\middle|\, \cdots \,\middle|\, e_i \,\middle|\, \cdots \,\middle|\, A_{in}\right]$$

Thus,

$$\det(A + tE_{ij}) - \det(A) = t\det\left[A_{i1} \,\middle|\, A_{i2} \,\middle|\, \cdots \,\middle|\, e_i \,\middle|\, \cdots \,\middle|\, A_{in}\right]$$

thus, as t - cancels in

$$\left(D_{E_{ij}} \det\right)(A) = \underbrace{\det\left[col_1(A) \,\middle|\, \cdots \,\middle|\, col_{i-1}(A) \,\middle|\, e_i \,\middle|\, col_{i+1}(A) \,\middle|\, \cdots \,\middle|\, col_n(A)\right]}_{\det(M_{ij})}$$

Continuing

$\boxed{\text{PROBLEM 19}}$ continuing.

We can piece together $d_A(\det)(H)$ as a sum of directional derivatives. We don't have a Jacobian matrix here, but this is next best idea. I <u>propose</u>

$$d_A(\det)(H) = \sum_{i,j=1}^{n} H_{ij}(D_{E_{ij}} \det)(A)$$

$$= \boxed{\sum_{i,j=1}^{n} H_{ij} \det(M_{ij})}$$

Let me exhibit $n=2,3$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{aligned} M_{11} &= d \\ M_{22} &= a \\ M_{21} &= -b \\ M_{12} &= -c \end{aligned}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \begin{aligned} M_{11} &= \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} \\ M_{12} &= -\det \begin{bmatrix} d & f \\ g & i \end{bmatrix} \\ M_{13} &= \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} \\ M_{21} &= -\det \begin{bmatrix} b & c \\ h & i \end{bmatrix} \\ M_{22} &= \det \begin{bmatrix} a & c \\ g & i \end{bmatrix} \end{aligned} \text{ etc...}$$

<u>$n=2$</u>

$$\boxed{d_A(\det)\begin{pmatrix} H_{11} & H_{12} \\ H_{22} & H_{22} \end{pmatrix} = H_{11} d - H_{12} c - H_{21} b + H_{22} a}$$

<u>Remark</u>: the reader will forgive me if I fail to prove the Frechet quotient $\longrightarrow 0$ for the proposed differential ☺

PROBLEM 20

Let $T : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ where $T(A) = \sum_{i=1}^{n} A_{ii} \equiv tr(A)$.
Calculate dT_A and show T diff at A.

Observe, $$T(A+H) = tr(A+H) = \sum_{i=1}^{n} (A+H)_{ii}$$
$$= \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} H_{ii}$$
$$= tr(A) + tr(H).$$

Consequently, $T(A+H) - T(A) = tr(H)$

If we define $dT_A(H) = tr(H)$ then clearly dT_A is linear and

$$\lim_{H\to 0} \left(\frac{T(A+H) - T(A) - dT_A(H)}{\|H\|} \right) = \lim_{H\to 0} \left(\frac{0}{\|H\|} \right) = 0.$$

Thus T is diff at A and $dT_A(H) = tr(H)$.

Just for fun, to exhibit constructing dT from DT,

$$\left(D_{E_{ij}} T\right)(H) = d_H T(E_{ij}) = tr(E_{ij}) = \delta_{ij}$$

Thus
$$d_A T(H) = \sum_{i,j=1}^{n} H_{ij}\delta_{ij} = \sum_{i=1}^{n} H_{ii} = tr(H)$$

(to make this less circular comment, one could calculate $(D_{E_{ij}} \;)(H)$ directly; $(D_{E_{ij}} T)(H) = \frac{d}{dt}\left(Tr(H + tE_{ij})\right)\Big|_{t=0}$
$= \frac{d}{dt}\left(H_{11} + \cdots + H_{nn} + t\delta_{ij}\right)$
$= \delta_{ij}$.)