Problem 21 Let $f(x,y) = x^y$ and calculate the differential df. Let $\gamma(t) = (t,t)$ and use the chain rule to calculate $(f \circ \gamma)'(t)$ (*note the chain rule is accomplished via multiplication of Jacobian matrices* $[df]$ *and* $[\gamma]$ *in this context*). Contrast this calculation to the calculation you used in calculus I to find $\frac{d}{dx}(x^x)$.

Problem 22 Suppose $X : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $X(s,t) = (x(s,t), y(s,t), z(s,t))$ and $\bar{X} : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $\bar{X}(\bar{s},\bar{t}) = (\bar{x}(\bar{s},\bar{t}), \bar{y}(\bar{s},\bar{t}), \bar{z}(\bar{s},\bar{t}))$. Suppose further that there exists some notation changing map $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $X = \bar{X} \circ T$ meaning:

$$\bar{X}(T(s,t)) = X(s,t)$$

where the notation $T(s,t) = (\bar{s}(s,t), \bar{t}(s,t))$ yields

$$\bar{X}((\bar{s}(s,t), \bar{t}(s,t))) = X(s,t)$$

Find $\frac{\partial X}{\partial s}$ and $\frac{\partial X}{\partial t}$ in terms of $\frac{\partial \bar{X}}{\partial \bar{s}}$, $\frac{\partial \bar{X}}{\partial \bar{t}}$ and $\frac{\partial \bar{s}}{\partial s}, \frac{\partial \bar{s}}{\partial t}$ and $\frac{\partial \bar{t}}{\partial s}, \frac{\partial \bar{t}}{\partial t}$.

Problem 23 (this is partly a continuation of the previous problem) Recall the surface integral of a vector field $\vec{F}$ on a surface S parameterized by $X : D \to S$ was defined by $\int_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(X(s,t)) \cdot \left(\frac{\partial X}{\partial s} \times \frac{\partial X}{\partial t}\right) ds\, dt$. Show that this definition is independent of the choice of parametrization. In particular, show that if you replace the expressions in terms of X and s,t in terms of $\bar{X}$ then you obtain the surface integral written in terms of the barred-parametrization. However, this is only true if we impose a certain condition on T. What condition is that?

Problem 24 Show that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is invertible on $U \subseteq \mathbb{R}^n$ then dF_{x_o} is invertible for each $x_o \in U$.

Problem 25 Edwards #3.6 on page 88.

Problem 26 Edwards #3.11 on page 89.

Problem 27 Suppose $\sin w = \exp(xyz)$ and $z^3 = x^2 + y^2 + \ln w$. Find $\left.\frac{\partial w}{\partial x}\right|_y, \left.\frac{\partial w}{\partial x}\right|_z$ at such points as the implicit function theorem allows. Comment in each case on your choice of dependent and independent variables.

?

Problem 28 Edwards #3.5 on page 194. (if it is possible, follow our intuitive proof to obtain approximate the solution)

Problem 29 Edwards #3.6 on page 194.(if it is possible, follow our intuitive proof to obtain approximate the solution)

Problem 30 Edwards #3.7 on page 194.(if it is possible, follow our intuitive proof to obtain approximate the solution)

PROBLEM 22

$$\frac{\partial \mathbf{X}}{\partial s} = \frac{\partial}{\partial s}\left(\bar{\mathbf{X}}\left(\bar{s}(s,t), \bar{t}(s,t)\right)\right)$$

$$\underline{\frac{\partial \mathbf{X}}{\partial s} = \frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial s} + \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial s}}. \qquad \text{Likewise} \quad \underline{\frac{\partial \mathbf{X}}{\partial t} = \frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial t} + \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t}}.$$

PROBLEM 23

$$\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} = \left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial s} + \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial s}\right) \times \left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}}\frac{\partial \bar{s}}{\partial t} + \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t}\right)$$

$$= \cancelto{0}{\left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}} \times \frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}}\right)}\frac{\partial \bar{s}}{\partial s}\frac{\partial \bar{s}}{\partial t} + \left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}} \times \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\right)\frac{\partial \bar{s}}{\partial s}\frac{\partial \bar{t}}{\partial t} +$$

$$+ \left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}} \times \frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}}\right)\frac{\partial \bar{t}}{\partial s}\frac{\partial \bar{s}}{\partial t} + \cancelto{0}{\left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}} \times \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\right)}\frac{\partial \bar{t}}{\partial s}\frac{\partial \bar{t}}{\partial t}$$

$$= \left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}} \times \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\right)\left(\frac{\partial \bar{s}}{\partial s}\frac{\partial \bar{t}}{\partial t} - \frac{\partial \bar{t}}{\partial s}\frac{\partial \bar{s}}{\partial t}\right)$$

Consider,

$$\iint_D \vec{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}\right) ds\, dt =$$

$$= \iint_D \vec{F}(\bar{\mathbf{X}}(\bar{s},\bar{t})) \cdot \left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}} \times \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\right)\underbrace{\left(\frac{\partial \bar{s}}{\partial s}\frac{\partial \bar{t}}{\partial t} - \frac{\partial \bar{t}}{\partial s}\frac{\partial \bar{s}}{\partial t}\right)} ds\, dt$$

need this is positive so we can replace it with $\left|\frac{\partial \bar{s}}{\partial s}\frac{\partial \bar{t}}{\partial t} - \frac{\partial \bar{t}}{\partial s}\frac{\partial \bar{s}}{\partial t}\right|$ which appears in the multivariate change of variables for integrals theorem.

$$= \iint_{\bar{D}} \vec{F}(\bar{\mathbf{X}}(\bar{s},\bar{t})) \cdot \left(\frac{\partial \bar{\mathbf{X}}}{\partial \bar{s}} \times \frac{\partial \bar{\mathbf{X}}}{\partial \bar{t}}\right) d\bar{s}\, d\bar{t}$$

To obtain the equality here we need $\det(T') > 0$ as

$$T' = \begin{bmatrix} \partial \bar{s}/\partial s & \partial \bar{s}/\partial t \\ \partial \bar{t}/\partial s & \partial \bar{t}/\partial t \end{bmatrix}$$

this makes $\mathbf{X}$ and $\bar{\mathbf{X}}$ share the same orientation. Or, they're compatible patches for oriented S.

PROBLEM 24 Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is invertible on $U \subseteq \mathbb{R}^n$ then $F \circ F^{-1} = id_{F(U)}$ and $F^{-1} \circ F = id_U$. By the chain-rule and the fact $d(id) = id$ as id is linear transformation we find

$$dF \circ dF^{-1} = id \ \ \& \ \ dF^{-1} \circ dF = id$$

Hence dF is invertible at each point.

Btw, $d(F^{-1} \circ F)_{x_0} = (dF^{-1})_{F(x_0)} \circ dF_{x_0} = Id$

I'm not sure including the point-dependence adds much here...

PROBLEM 25 Edwards #3.6 on page 88

Suppose $\Phi = 5 \dfrac{\partial^2 u}{\partial x^2} + 2 \dfrac{\partial^2 u}{\partial x \partial y} + 2 \dfrac{\partial^2 u}{\partial y^2} \Rightarrow \overline{\Phi} = \dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}$ given $x = 2s + t$, $y = s - t$, Begin by stating what this means in terms of functions.

$$\overline{\Phi} \doteq \Phi(x(s,t), y(s,t))$$

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial s}\left[u(x(s,t), y(s,t))\right] = \frac{\partial u}{\partial x}\overset{2}{\cancel{\frac{\partial x}{\partial s}}} + \frac{\partial u}{\partial y}\overset{1}{\cancel{\frac{\partial y}{\partial s}}} = 2u_x + u_y$$

$$\frac{\partial^2 u}{\partial s^2} = \left[2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right](2u_x + u_y) = 4u_{xx} + 2u_{yx} + 2u_{xy} + u_{yy}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\overset{1}{\cancel{\frac{\partial x}{\partial t}}} + \frac{\partial u}{\partial y}\overset{-1}{\cancel{\frac{\partial y}{\partial t}}} = u_x - u_y$$

$$\frac{\partial^2 u}{\partial t^2} = \left[\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right][u_x - u_y] = u_{xx} - u_{yx} - u_{xy} + u_{yy}$$

Thus, $\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2} = 5\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} + 2\dfrac{\partial^2 u}{\partial y^2}$.

PROBLEM 26 # 3.11 pg 89 Edwards

(a.) If $f(\vec{x}) = g(r)$ where $r = \|\vec{x}\|$ and $n \geq 3$ show

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = \frac{n-1}{r} g'(r) + g''(r) \quad \text{for } \vec{x} \neq 0.$$

(b.) If $\nabla^2 f = 0 \Rightarrow f(\vec{x}) = \frac{a}{r^{n-2}} + b$ for $x \neq 0$, $a, b \in \mathbb{R}$

(a.) Observe $\frac{\partial r}{\partial x_j}$ is easily calculated from $\vec{x} \cdot \vec{x} = r^2$

$$\vec{x} \cdot \vec{x} = \sum_{i=1}^{n} x_i^2 = r^2. \text{ Note, } \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} x_i^2\right) = \sum_{i=1}^{n} 2x_i \delta_{ij} = 2x_j$$

thus $2r \frac{\partial r}{\partial x_j} = 2x_j \Rightarrow \frac{\partial r}{\partial x_j} = \frac{x_j}{r}$ for $j = 1, 2, \ldots, n$.

Further, $\frac{\partial g}{\partial x_j} = g'(r) \frac{\partial r}{\partial x_j} = \frac{g'(r)}{r} x_j$

Thus

$$\frac{\partial^2 g}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left[g'(r) \frac{x_j}{r}\right] = g''(r) \frac{\partial r}{\partial x_j} \frac{x_j}{r} + g'(r)$$

$$= g''(r) \frac{\partial r}{\partial x_j} \frac{x_j}{r} + g'(r) \frac{\partial}{\partial x_j}\left(\frac{x_j}{r}\right)$$

$$= g''(r) \frac{x_j^2}{r^2} + g'(r)\left[\frac{r - x_j\left(\frac{x_j}{r}\right)}{r^2}\right]$$

$$= g''(r) \frac{x_j^2}{r^2} + g'(r)\left[\frac{r^2 - x_j^2}{r^3}\right]$$

Therefore,

$$\nabla^2 f = g''(r)\left[\frac{x_1^2 + \cdots + x_n^2}{r^2}\right] + \frac{g'(r)}{r^3}\left[nr^2 - x_1^2 - \cdots - x_n^2\right]$$

$$= g''(r) + \frac{g'(r)}{r^3}\left[nr^2 - r^2\right]$$

$$= \underline{g''(r) + \frac{n-1}{r} g'(r)}.$$

(b.) $g''(r) = \frac{1-n}{r} g'(r)$

$\frac{dv}{dr} = \frac{1-n}{r} v$

$(v = g'(r))$

$\frac{dv}{v} = \frac{(1-n)dr}{r} \Rightarrow \ln|v| = (1-n)\ln|r| + C_1$

$\Rightarrow \frac{dg}{dr} = \tilde{a} r^{1-n}$

$\Rightarrow g(r) = \frac{\tilde{a} r^{2-n}}{2-n} + b$

$\therefore \boxed{f(r) = a/r^{n-2} + b}$

PROBLEM 27

Given $\sin w = \exp(xyz)$ and $z^3 = x^2 + y^2 + \ln w$
Find $\frac{\partial w}{\partial x}\Big|_y$ and $\frac{\partial w}{\partial x}\Big|_z$ at such pts as the implicit fnct. Th$^{\underline{m}}$ allows

$$G(x,y,z,w) = (\sin w - \exp(xyz),\ z^3 - x^2 - y^2 - \ln w)$$

and we consider $G^{-1}\{(0,0)\}$. The implicit fnct Th$^{\underline{m}}$ allows us to solve for two of the variables in terms of the remaining two.

$$\frac{\partial w}{\partial x}\Big|_y \Rightarrow \begin{array}{l} w = w(x,y) \\ z = z(x,y) \end{array}$$

$$\frac{\partial w}{\partial x}\Big|_z \Rightarrow \begin{array}{l} w = w(x,z) \\ y = y(x,z) \end{array}$$

$$G' = \left[\begin{array}{c|c|c|c} -yze^{xyz} & -xze^{xyz} & -xye^{xyz} & \cos w \\ -2x & -2y & 3z^2 & -1/w \end{array}\right]$$

(I) For z, w - dependent need $\left[\frac{\partial G}{\partial z}\Big|\frac{\partial G}{\partial w}\right]$ invertible.
(II) For y, w - dependent need $\left[\frac{\partial G}{\partial y}\Big|\frac{\partial G}{\partial w}\right]$ invertible.

for (I) need $\det\left[\partial_z G | \partial_w G\right] = \frac{xye^{xyz}}{w} - 3z^2\cos w \neq 0$.

for (II) need $\det\left[\partial_y G | \partial_w G\right] = \frac{xze^{xyz}}{w} + 2y\cos w \neq 0$.

Sol$^{\underline{n}}$:
$$\left.\begin{array}{l} \cos w\, dw = \exp(xyz)\left[yz\,dx + xz\,dy + xy\,dz\right] \\ \frac{1}{w}\, dw = 3z^2 dz - 2x\,dx - 2y\,dy \end{array}\right\} *$$

We wish to calculate $\frac{\partial w}{\partial x}\Big|_y \Rightarrow w = w(x,y)$ & $z = z(x,y)$

we need to solve $*$ for dw & dz

$$\cos(w)\,dw - xy\exp(xyz)\,dz = \exp(xyz)\left[yz\,dx + xz\,dy\right]$$
$$\frac{1}{w}\,dw - 3z^2\,dz = -2x\,dx - 2y\,dy$$

see case I needed

$$\begin{bmatrix} \cos w & -xye^{xyz} \\ 1/w & -3z^2 \end{bmatrix}\begin{bmatrix} dw \\ dz \end{bmatrix} = \begin{bmatrix} \exp(xyz)\left[yz\,dx + xz\,dy\right] \\ -2x\,dx - 2y\,dy \end{bmatrix}$$

PROBLEM 27 continued

Remark: I would not be surprised if I have made some error here...

$$\begin{bmatrix} dw \\ dz \end{bmatrix} = \frac{1}{-3z^2\cos w + \frac{xy}{w}e^{xyz}} \begin{bmatrix} -3z^2 & xye^{xyz} \\ -1/w & \cos w \end{bmatrix} \begin{bmatrix} e^{xyz}[yz\,dx + xz\,dy] \\ -2x\,dx - 2y\,dy \end{bmatrix}$$

Thus,

$$dw = \frac{1}{-3z^2\cos w + \frac{xye^{xyz}}{w}} \Big[-3z^2 e^{xyz}(yz\,dx + xz\,dy) + xye^{xyz}(-2x\,dx - 2y\,dy) \Big]$$

$$= \frac{we^{xyz}}{-3wz^2\cos w + xye^{xyz}} \Big[(-3yz^3 - 2x^2y)dx + (-3xz^2 - 2xy^2)dy \Big]$$

Hence,

$$\boxed{\left(\frac{\partial w}{\partial x}\right)\Big|_y = \frac{we^{xyz}(-3yz^3 - 2x^2y)}{-3wz^2\cos w + xye^{xyz}}}$$

I'll use the other technique for $\frac{\partial w}{\partial x}\big|_z$ for which $w = w(x,z)$ and $y = y(x,z)$ hence by assumption $\frac{\partial x}{\partial z} = \frac{\partial z}{\partial x} = 0$.

$$\frac{\partial w}{\partial x}\Big|_z = \frac{\partial}{\partial x}\Big|_z \Big(\exp(\underbrace{z^3 - x^2 - y^2}_{\alpha}) \Big)$$

(solved $z^3 = x^2 + y^2 + \ln w$ for w.)

$$= \exp\alpha \left[\frac{\partial}{\partial x}\Big|_z (z^3) - \frac{\partial}{\partial x}\Big|_z (x^2) - \frac{\partial}{\partial x}\Big|_z y^2 \right]$$

$$= \exp\alpha \left[-2x - 2y \frac{\partial y}{\partial x}\Big|_z \right]$$

Notice, $\sin w = \exp(xyz)$

$$\Rightarrow \cos(w)\frac{\partial w}{\partial x}\Big|_z = \exp(xyz)\left[yz + x\frac{\partial y}{\partial x}\Big|_z z \right]$$

Hence

$$\frac{\partial y}{\partial x}\Big|_z = \left(\cos w \exp(-xyz)\frac{\partial w}{\partial x}\Big|_z - yz \right)\frac{1}{xz}$$

$$\frac{\partial w}{\partial x}\Big|_z = \exp(\alpha)\left(-2x - 2y\left[\cos w\, e^{-xyz}\frac{\partial w}{\partial x}\Big|_z - yz \right]\frac{1}{xz} \right)$$

Now, solve for $\frac{\partial w}{\partial x}\big|_z$

$$\boxed{\frac{\partial w}{\partial x}\Big|_z = \frac{1}{e^{-z^3+x^2+y^2} + \frac{2y\cos w \exp(-xyz)}{xz}}\left[-2x + \frac{2y^2}{x} \right]}$$

PROBLEM 28 Edwards # 3.5 pg. 194 (also give intuition calculation to justify result)

Show that the eq^{n}'s

I.) $\sin(x+z) + \ln(yz^2) = 0$

II.) $\exp(x+z) + yz = 0$

implicitly define z near -1 as fnct of (x,y) near $(1,1)$.

$$G = \begin{bmatrix} \sin(x+z) + \ln(y) + 2\ln(z) \\ \exp(x+z) + yz \end{bmatrix}$$

$$G' = \left[\frac{\partial G}{\partial x} \middle| \frac{\partial G}{\partial y} \middle| \frac{\partial G}{\partial z} \right] = \left[\begin{array}{c|c|c} \cos(x+z) & 1/y & \cos(x+z) + 2/z \\ \exp(x+z) & z & \exp(x+z) + y \end{array} \right]$$

If $x=1, y=1$ and z is given such that I. & II. are solved then $\sin(1+z) = \ln(z^2)$ and $\exp(1+z) + z = 0$. To solve for $z = z(x,y)$ we need for I,

$$\frac{\partial G_I}{\partial z} = \cos(x+z) + \frac{2}{z} \neq 0$$ when $x=1, y=1$ and $G_I(1,1,z)=0$

Whereas for II,

$$\frac{\partial G_{II}}{\partial z} = \underbrace{\exp(x+z) + y}_{\exp(1+z)+1} \neq 0$$ when $x=1, y=1$ and $\underline{G_{II}(1,1,z)=0}$.

$z = -\exp(1+z)$

?

$\boxed{\text{PROBLEM 29}}$ Edwards #3.6 on p. 194

Can the surface whose eqⁿ is

$$G = xy - y\ln(z) + \sin(xz) = 0$$

be represented as a graph $z = f(x,y)$ near $(0,2,1)$?

Notice $\left.\frac{\partial G}{\partial z}\right|_{(0,2,1)} = \left.\left(\frac{-y}{z} + x\cos(xz)\right)\right|_{(0,2,1)} = -2+0 = -2 \neq 0$

therefore, as $G \in C'(0,2,1)$ the implicit fnct theorem applies and we find $\exists f$ for which $G(x,y,f(x,y)) = 0$ for all points near $(0,2,1)$.

$$dG = \left(y + z\cos(xz)\right)dx + \left(x - \ln(z)\right)dy + \left(\frac{-y}{z} + x\cos(xz)\right)dz$$

obtain the linearization by setting

$$dx = x-0, \quad dy = y-2, \quad dz = z-1$$

and evaluate dG at $(0,2,1)$. This gives,

$$(2+1)x + (1-0)(y-2) + (-2)(z-1) = 0$$

or,

$$3x + y - \cancel{2} - 2z + \cancel{2} = 0 \therefore \boxed{z \cong \tfrac{1}{2}(3x+y)}$$

this is the 1st approximation to the solⁿ to $G=0$ near $(0,2,1)$ for $z = f(x,y)$.

$\boxed{\text{PROBLEM 30}}$ Edwards #3.7 on pg. 194

PROBLEM 30 Edwards #3.7 pg. 194

$$G_1 = xu^2 + yzv + x^2 z = 3$$
$$G_2 \quad xyv^3 + 2zu - u^2 v^2 = 2$$

$$G' = \left[\begin{array}{c|c|c|c|c} u^2+2xz & zv & yv+x^2 & G_u & G_v \\ yv^3 & xv^3 & 2u & & \end{array}\right]$$

later

Here $(G_1, G_2) = G : \mathbb{R}^5_{xyzuv} \to \mathbb{R}^2$. Can we solve for (u,v) near $(1,1)$ as a function of (x,y,z) near $(1,1,1)$?

$$G' = \left[\frac{\partial G}{\partial x} \,\Big|\, \frac{\partial G}{\partial y} \,\Big|\, \frac{\partial G}{\partial z} \,\Big|\, \frac{\partial G}{\partial u} \,\Big|\, \frac{\partial G}{\partial v} \right]$$

$$\underbrace{[G_x | G_y | G_z]\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} + [G_u | G_v]\begin{bmatrix} u-1 \\ v-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

evaluating the derivatives at $(1,1,1,1,1)$ gives the linearization of $G = (3,2)$ at $(1,1,1,1,1)$. We can solve for u, v if $[G_u / G_v]$ is non-singular, observe

$$[G_u / G_v] = \begin{bmatrix} 2xu & yz \\ 2z - 2uv^2 & 3xyv^2 - 2u^2 v \end{bmatrix}$$

$$[G_u / G_v](1,1,1,1,1) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \det\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = 2 \neq 0$$

Therefore, as $G \in C'(1,1,1,1,1)$ and the necessary submatrix of G' is invertible it follows that we may solve for u, v as fncts of x, y, z as desired. Moreover, the 1^{st} approx is obtained from:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3(x-1) + y-1 + 2(z-1) \\ x-1 + y-1 + 2(z-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2(x-1) \\ 2(x-1) + 2(y-1) + 4(z-1) \end{bmatrix} = \begin{bmatrix} 1 - (x-1) \\ 1 - (x-1) - (y-1) - 2(z-1) \end{bmatrix}$$

PROBLEM 30 continued

We find, as 1^{st} approximation to $G=0$,

$$\boxed{\begin{aligned} u &= 2-x \\ v &= -x-y-2z+5 \end{aligned}}$$

Solves $G(x,y,z,u,v) = (3,2)$ near $(1,1,1,1,1)$.