Problem 31 A complex number $a+ib$ can be represented by a 2×2 matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Let $\Psi(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. It is clear this mapping is a vector space isomorphism. In addition,

(a.) show that $\Psi(zw) = \Psi(z)\Psi(w)$ where zw denotes complex number multiplication and $\Psi(z)\Psi(w)$ denotes matrix multiplication,

(b.) if $||A||^2 = \text{trace}(A^TA)$ for matrices and $|a+ib|^2 = a^2+b^2$ for complex numbers then show $||\Psi(a+ib)||^2 = 2|a+ib|^2$.

Problem 32 We derived that $f=(u,v):\mathbb{R}^2\to\mathbb{R}^2$ is **complex differentiable** on $U\subseteq\mathbb{R}^2$ iff $f' = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$ on U. Complex differentiabiity is a special structure, take Math 331 if you'd like to know a lot more about this. Calculate the Jacobians of the following maps and determine if these real mappings denote complex-differentiable functions on $\mathbb{R}^2$:

(a.) $f(x,y) = (x^2-y^2, 2xy)$

(b.) $f(x,y) = (y,x)$

(c.) $f(x,y) = (e^x\cos(y), e^x\sin(y))$

Incidentally, the examples above in complex notation $z=x+iy$ are $f(z)=z^2$, $i\bar{z}$ and e^z.

Problem 33 Suppose the solution set of $x_1+x_2+2x_4 = 11$ and $2x_1 - x_3 = -1$ describes a plane S in $\mathbb{R}^4$. Find the equations of the plane at $(1,2,3,4)$ which is normal to S.

Problem 34 Let $\gamma(t) = (t,t^2,t^3,t^4)$ parametrize the curve C and let $\gamma(1)=p$. Find T_pC and N_pC (your choice, you can either express the tangent or normal space impicitly as a level set or expicitly via a parametrization)

Problem 35 Let $G(x,y,z,w) = (x^2+y^2+w,\ y^2+z^2-w)$. Define the affine space $S = G^{-1}(-1,10)$. Consider the point $p=(1,2,0,-6)\in S$. Calculate T_pS and N_pS and express each as follows:

(a.) $T_pS = p+\text{span}\{v_1,v_2\}$

(b.) $N_pS = p+\text{span}\{w_1,w_2\}$

Problem 36 Continuing the previous problem, Calculate T_pS and N_pS as follows:

(a.) $T_pS = F^{-1}\{(0,0)\}$

(b.) $N_pS = H^{-1}\{(0,0)\}$

where F and H are mappings from $\mathbb{R}^4$ to $\mathbb{R}^2$ whose level sets give the tangent and normal planes in $\mathbb{R}^4$ as indicated.

Problem 37 Edwards #5.4 on page 116. (Lagrange Multipliers) (see P25 of · IIs10 sol'n)

Problem 38 Edwards #5.10 on page 116. (Lagrange Multipliers) (see P26 of IIs10 sol'n

Problem 39 Edwards #5.11 on page 116. (Lagrange Multipliers) ✓ P26 of S10 again.

Problem 40 Edwards #5.12 on page 116. (Lagrange Multipliers) ✓

Mission 4 Solution

Problem 31 Let $\Psi(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. It is clear this is a vector space isomorphism $\Psi : \mathbb{C} \longrightarrow M \subset \mathbb{R}^{2\times 2}$.

(a.) Let $z = x+iy$ and $w = a+ib$ then

$$zw = (x+iy)(a+ib) = xa - yb + i(ya + xb)$$

Thus

$$\Psi(zw) = \begin{bmatrix} xa - yb & -(ya+xb) \\ ya + xb & xa - yb \end{bmatrix}.$$

Also, consider

$$\Psi(z)\,\Psi(w) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} xa - yb & -xb - ay \\ ya + xb & -yb + xa \end{bmatrix}$$

$$= \Psi(zw).$$

We see Ψ preserves <u>multiplication</u>.

(b.) If $\|A\|^2 = \text{trace}(A^TA)$ and $|a+ib|^2 = a^2+b^2$ then show $\|\Psi(a+ib)\|^2 = 2|a+ib|^2$

$$\|\Psi(a+ib)\|^2 = \text{trace}\left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^T \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right)$$

$$= \text{trace}\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right)$$

$$= \text{trace}\left(\begin{bmatrix} a^2+b^2 & -ab+ba \\ -ba+ab & b^2+a^2 \end{bmatrix}\right)$$

$$= 2(a^2+b^2)$$

$$= 2|a+ib|^2. \quad //$$

PROBLEM 32 We derive $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ is complex differentiable on $U \subseteq \mathbb{R}^2$ iff $f' = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$. Calculate f' for the following maps and determine if they are complex-differentiable on $\mathbb{R}^2$.

(a.) $f(x,y) = (x^2 - y^2,\ 2xy)$

$$f' = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$ } correct pattern $\therefore$ YES. f is complex diff. on $\mathbb{R}^2$

[Remark $f(z) = z^2$ gives $(x+iy)(x+iy) = x^2 - y^2 + i(2xy)$ this example is just the square of a complex variable. moreover, $f'(z) = 2z$ and you can easily see $f' \longrightarrow 2(x+iy)$ under the Ψ^{-1} mapping.

(b.) $f(x,y) = (y, x)$

$$f' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ } wrong pattern, need the $(1,2)$ and $(2,1)$ components of f' to differ by sign. Hence, f not complex differentiable.

[Remark: $z = x + iy$ and $\bar{z} = x - iy$ can be solved for x, y as $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$ it follows $f(x,y) = (y,x) = y + ix = \frac{1}{2i}(z - \bar{z}) + \frac{i}{2}(z + \bar{z})$ simplifies to $f(z) = i\bar{z} = i(x - iy)$. Complex diff. fncts. are functions of z (holomorphic). This is a fnct of $\bar{z}$ (antiholomorphic)

(c.) $f(x,y) = (e^x \cos y,\ e^x \sin y) \Rightarrow f' = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$. thus f is complex diff. on $\mathbb{R}^2$

[Remark: $f(z) = e^z = e^{x+iy} = e^x \cos y + ie^x \sin y$. this is the complex exponential in real notation.

PROBLEM 33

$$S = G^{-1}\{(11,-1)\} \text{ where } G(x_1,x_2,x_3,x_4) = \begin{bmatrix} x_1 + x_2 + 2x_4 \\ 2x_1 - x_3 \end{bmatrix}$$

Find the eq^{n}'s (cartesian) of the normal plane to S at $(1,2,3,4)$

Observe $\nabla G_1, \nabla G_2$ attached to p are ~~tangent~~ vectors. We need tangent vectors which are $\perp$ to normal plane to write the cartesian eq^{n}'s for N_pS viewed as a point set. Technically, $N_pS = \{(p,v) \mid v \cdot w = 0 \ \forall (p,w) \in T_pS\}$ but we're looking for eq^{n}'s for $\{p + v \mid (p,v) \in N_pS\}$

$\nabla G_1 = \langle 1,1,0,2\rangle$ want $\nabla G_1 \cdot v = 0$ & $\nabla G_2 \cdot v = 0$

$\nabla G_2 = \langle 2,0,-1,0\rangle$ $\Rightarrow v \in \mathrm{Null}\left(\begin{smallmatrix}\nabla G_1 \\ \nabla G_2\end{smallmatrix}\right)$.

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & -1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 2 & 0 & 4 \\ 0 & -2 & -1 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & -2 & -1 & -4 \end{bmatrix}$$

Hence $v \in \mathrm{Null}(G') \Rightarrow 2v_1 - v_3 = 0$ & $-2v_2 - v_3 - 4v_4 = 0$

So $v = (v_1,v_2,v_3,v_4) = \left(\tfrac{1}{2}v_3, \tfrac{1}{2}(-v_3 - 4v_4), v_3, v_4\right)$

$= v_3\left(\tfrac{1}{2}, \tfrac{-1}{2}, 1, 0\right) + v_4(0,-2,0,1)$

It follows that the eq^{n}'s for N_pS as a point-set are,

$$\boxed{\begin{aligned} &\tfrac{1}{2}(x_1 - 1) - \tfrac{1}{2}(x_2 - 2) + x_3 - 3 = 0 \\ &-2(x_2 - 2) + (x_4 - 4) = 0 \end{aligned}}$$

Remark: sadly this was my 3rd attempt at calculating Null G'. I've checked, my alleged tangent vectors $\langle 1/2, -1/2, 1, 0\rangle$ and $\langle 0,-2,0,1\rangle$ are indeed $\perp$ to ∇G_1 & ∇G_2. At last!

PROBLEM 34

$\gamma(t) = \langle t, t^2, t^3, t^4 \rangle$ parametrizes C, $\gamma(1) = P$.

$\gamma'(t) = \langle 1, 2t, 3t^2, 4t^3 \rangle$

Thus $\gamma'(1) = \langle 1, 2, 3, 4 \rangle$ we find that

$$\boxed{T_pC = \{(P, v) \mid v = k\langle 1,2,3,4\rangle,\ k \in \mathbb{R}\}}$$

Or as a point set, $\{P + k\langle 1,2,3,4\rangle \mid k \in \mathbb{R}\}$.

To find N_pC we need three $\perp$ vectors if we wish to write it in the same fashion as T_pC. However, as a point-set, $P = \gamma(1) = \langle 1,1,1,1 \rangle$

$$\boxed{N_pC = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - 1 + 2(x_2 - 1) + 3(x_3 - 1) + 4(x_4 - 1) = 0\}}$$

optimally lazy answer. Oh, let's find the parametric version need v_1, v_2, v_3 for bases of $\text{Null}([1,2,3,4])$. But,

$$[1,2,3,4]v = 0 \implies v_1 + 2v_2 + 3v_3 + 4v_4 = 0$$

$$(v_1, v_2, v_3, v_4) = (-2v_2 - 3v_3 - 4v_4,\ v_2,\ v_3,\ v_4)$$

$$= v_2(-2,1,0,0) + v_3(-3,0,1,0) + v_4(-4,0,0,1)$$

well that wasn't too bad.

$$\boxed{N_pC = \{(P, v) \mid v \in \text{span}\{(-2,1,0,0), (-3,0,1,0), (-4,0,0,1)\}\}}$$

~~I refuse~~

I refuse to find cartesian eq^n's for T_pC. It's just wrong ☺.

PROBLEM 35 Let $G(x,y,z,w) = (x^2+y^2+w,\ y^2+z^2-w)$. Let $S = G^{-1}\{(-1,10)\}$. Consider $p = (1,2,0,-6) \in S$. Calculate T_pS and the N_pS and express each as a point-set written as $p + \text{span}\{v_1, v_2\} = \{p+v \mid v \in \text{span}\{v_1,v_2\}\}$.

$$\nabla G = \begin{bmatrix} 2x & 2y & 0 & 1 \\ 0 & 2y & 2z & -1 \end{bmatrix} \qquad \nabla G(1,2,0,-6) = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 0 & 4 & 0 & -1 \end{bmatrix}$$

To begin, $\nabla G_1, \nabla G_2$ span the normal directions at p hence,

$$\boxed{N_pS = (1,2,0,-6) + \text{span}\{\langle 2,4,0,1\rangle, \langle 0,4,0,-1\rangle\}}$$ ← answer to (b.)

To find T_pS we must find $v_1, v_2 \perp \langle 2,4,0,1\rangle, \langle 0,4,0,-1\rangle$ this means we should find basis for $\text{Null}\begin{bmatrix} 2 & 4 & 0 & 1 \\ 0 & 4 & 0 & -1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 4 & 0 & 1 \\ 0 & 4 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 4 & 0 & -1 \end{bmatrix} \therefore\ v \in \text{Null}\begin{bmatrix} 2 & 4 & 0 & 1 \\ 0 & 4 & 0 & -1 \end{bmatrix} \text{ has } 2v_1 + 2v_4 = 0 \ \& \ 4v_2 - v_4 = 0$$

$$v = (-v_4, \tfrac{1}{4}v_4, v_3, v_4)$$
$$= v_4(-1, 1/4, 0, 1) + v_3(0,0,1,0)$$

Thus, $\boxed{T_pS = (1,2,0,-6) + \text{span}\{\langle -1, 1/4, 0, 1\rangle, \langle 0,0,1,0\rangle\}}$

PROBLEM 36 Express T_pS and N_pS of 35 as inverse images.

(a.) Let $F(x,y,z,w) = (2(x-1) + 4(y-2),\ 4(y-2) - (w+6))$

$F^{-1}\{(0,0)\} = T_pS$.

(b.) Let $H(x,y,z,w) = (-(x-1) + \frac{1}{4}(y-2) + (w+6),\ z)$

$H^{-1}\{(0,0)\} = N_pS$.

the vectors which span T_pS give coeff. for cart. eq^{n}s of N_pS.
the vectors which span N_pS give coeff for cartesian eq^{n}s of T_pS.

- also use generalization of Calculus III eq^{n} for plane formula: $n = \langle a,b,c\rangle$ and pt. (x_0,y_0,z_0) give eq^{n}
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$
I use same idea here repeatedly.