Problem 41 Let $Q(x, y) = x^2 - 2xy$. Find the matrix of Q, what are its eigenvalues? Write the simple formula for Q in terms of the eigencoordinates $\bar{x}, \bar{y}$.

$$[Q] = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \quad \text{gives} \quad Q(v) = v^T \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} v \quad \text{for } v = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\det \begin{bmatrix} 1-\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda(\lambda - 1) - 1$$
$$= \lambda^2 - \lambda - 1$$
$$= \left(\lambda - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - \tfrac{4}{4}$$
$$= \left(\lambda - \tfrac{1}{2}\right)^2 - \tfrac{5}{4} \implies \lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$Q(\bar{x}, \bar{y}) = \lambda_1 \bar{x}^2 + \lambda_2 \bar{y}^2$$

$$\boxed{Q(\bar{x}, \bar{y}) = \left(\frac{1 - \sqrt{5}}{2}\right)\bar{x}^2 + \left(\frac{1 + \sqrt{5}}{2}\right)\bar{y}^2}$$

(technically, If $\Phi(x, y) = (\bar{x}, \bar{y})$ then $Q(\bar{x}, \bar{y}) = (Q \circ \Phi)(x, y)$ so, it's not really "Q", but this above is common.)

Problem 42 Let $Q(x, y) = 2xy$. Find the matrix of Q, what are its eigenvalues? Write the simple formula for Q in terms of the eigencoordinates $\bar{x}, \bar{y}$.

$$[Q] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftarrow \text{matrix of } Q$$

$$\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1) = 0$$

$$\therefore \underline{\lambda_1 = -1, \ \lambda_2 = 1}.$$

eigenvalues of Q

$$Q(\bar{x}, \bar{y}) = -\bar{x}^2 + \bar{y}^2$$

(I didn't ask you all to do ↓↓↓ but, I thought it might be fun to see.

Note, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \pm \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{bmatrix} v \\ u \end{bmatrix} = \pm \begin{bmatrix} u \\ v \end{bmatrix}$

Thus we easily derive $\vec{u}_{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and $\Phi_\beta(x, y) = [\beta]^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$ for $\beta = \{\vec{u}_{-1}, \vec{u}_1\}$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} x - y \\ x + y \end{bmatrix} \Rightarrow \begin{array}{l} \bar{x} = \frac{1}{\sqrt{2}}(x - y) \\ \bar{y} = \frac{1}{\sqrt{2}}(x + y) \end{array}$$

Inversely, $(\Phi_\beta^{-1})(\bar{x}, \bar{y}) = [\beta]\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \bar{x} + \bar{y} \\ -\bar{x} + \bar{y} \end{bmatrix}$

thus $x = \frac{1}{\sqrt{2}}(\bar{x} + \bar{y})$ and $y = \frac{1}{\sqrt{2}}(-\bar{x} + \bar{y})$. Calculate,

$$\begin{aligned} Q(x, y) = 2xy &= 2\frac{1}{\sqrt{2}}(\bar{x} + \bar{y})\frac{1}{\sqrt{2}}(-\bar{x} + \bar{y}) \\ &= (\bar{x} + \bar{y})(-\bar{x} + \bar{y}) \\ &= \underline{-\bar{x}^2 + \bar{y}^2}. \end{aligned}$$

Problem 43 Let $Q(x, y, z) = 2(xy + yz)$. Find the matrix of Q, what are its eigenvalues? Write the simple formula for Q in terms of the eigencoordinates $\bar{x}, \bar{y}, \bar{z}$.

$$[Q] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Q(x,y,z) = [x, y, z]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x, y, z]\begin{bmatrix} y+z \\ x \\ x \end{bmatrix}$$

$$= x(y+z) + xy + xz$$

$$= 2(xy + yz)$$

Believe It!

Find e-values from characteristic eq$^{\underline{n}}$ as usual, by Laplace Exp. by minors

$$\det\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = -\lambda \det\begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} - 1\det\begin{bmatrix} 1 & 0 \\ 1 & -\lambda \end{bmatrix} + 1\det\begin{bmatrix} 1 & -\lambda \\ 1 & 0 \end{bmatrix}$$

$$= -\lambda(\lambda^2) - (-\lambda) + (\lambda)$$

$$= -\lambda^3 + 2\lambda$$

$$= \lambda(2 - \lambda^2)$$

$$= -\lambda(\lambda + \sqrt{2})(\lambda - \sqrt{2}) = 0$$

$$\lambda_1 = -\sqrt{2}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{2}$$

$$\boxed{Q(\bar{x}, \bar{y}, \bar{z}) = -\sqrt{2}\,\bar{x}^2 + \sqrt{2}\,\bar{z}^2}$$

($0 \cdot \bar{y}^2$ didn't seem worth including)

Problem 44 Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ where f has a single local minimum at $x = x_o$ and g has only one local maximum at $y = y_o$. If we define $h(x, y) = [f(x)g(y)]^2$ then does h have any local extrema? Find the taylor expansion of h near any critical point(s) in terms of the values of f and g and their derivatives. Break into cases if necessary.

non zero!

Critical Pts. for $h(x,y)$ arise from $\nabla h = 0$. Assume f & g are differentiable on $\mathbb{R}$. Calculate,

$$\nabla h = \left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right\rangle$$
$$= \left\langle \frac{\partial}{\partial x}[f(x)g(y)]^2, \frac{\partial}{\partial y}[f(x)g(y)]^2 \right\rangle$$
$$= \left\langle 2[f(x)g(y)]\frac{\partial}{\partial x}[f(x)g(y)], 2[f(x)g(y)]\frac{\partial}{\partial y}[f(x)g(y)] \right\rangle$$
$$= 2f(x)g(y)\left\langle g(y)\frac{df}{dx}, f(x)\frac{dg}{dy} \right\rangle$$

We find critical points for h arise from several conditions for $f(x)$, $g(y)$: $\nabla h(a,b) = \langle 0, 0\rangle$ given any of the following:

1.) If $f(a) = 0$ or $g(b) = 0$. (makes $f(x)g(y) = 0$)

2.) $a = x_o$ and $b = y_o$ (makes $\frac{df}{dx} = 0$ and $\frac{dg}{dy} = 0$)

Begin with case 2.

$$f(x) = f(x_o) + \tfrac{1}{2}f''(x_o)(x - x_o)^2 + \cdots \qquad (\text{min} \Rightarrow f''(x_o) > 0)$$
$$g(y) = g(y_o) + \tfrac{1}{2}g''(y_o)(y - y_o)^2 + \cdots \qquad (\text{max} \Rightarrow g''(y_o) < 0)$$

$$f(x)g(y) = (f(x_o) + \alpha\Delta x^2)(g(y_o) - \beta\Delta y^2) \qquad \left[\begin{array}{l}\alpha = f''(x_o) \\ \beta = -g''(y_o)\end{array}\right.$$
$$= f(x_o)g(y_o) + \alpha g(y_o)(\Delta x)^2 - \beta f(x_o)(\Delta y)^2 + \cdots$$

$$h(x,y) = (f(x)g(y))^2 = [f(x_o)g(y_o) + \alpha g(y_o)(\Delta x)^2 - \beta f(x_o)(\Delta y)^2 + \cdots]^2$$
$$= (f(x_o)g(y_o))^2 + \alpha f(x_o)g(y_o)^2(\Delta x)^2 - \beta f(x_o)^2 g(y_o)(\Delta y)^2 + \cdots$$
$$= \boxed{h(x_o,y_o) + \tfrac{1}{2}f''(x_o)f(x_o)g^2(y_o)(x - x_o)^2 + \tfrac{1}{2}g''(y_o)f^2(x_o)g(y_o)(y - y_o)^2} + \cdots$$

analyze quadratic part at x_o, y_o to see min/max/saddle.

If $f(x_o) > 0$, $g(y_o) > 0$ then quadratic has $+, -$ $\Rightarrow$ saddle.
If $f(x_o) < 0$, $g(y_o) < 0$ then quad. has $-, +$ $\Rightarrow$ saddle.
If $f(x_o) < 0$, $g(y_o) > 0$ then quad has $-, -$ $\Rightarrow$ maximum.
If $f(x_o) > 0$, $g(y_o) < 0$ then quad has $+, +$ $\Rightarrow$ minimum.

Problem 45 work Edwards problem 7.10 from page 141.

Find and classify critical pts. of $f(x,y) = (x^2+y^2)e^{x^2-y^2}$

$$\nabla f = \left\langle 2xe^{x^2-y^2} + (x^2+y^2)e^{x^2-y^2}(2x),\ 2ye^{x^2-y^2} + (x^2+y^2)e^{x^2-y^2}(-2y)\right\rangle$$

$$= e^{x^2-y^2}\left\langle 2x(1+x^2+y^2),\ 2y(1-x^2-y^2)\right\rangle$$

Observe $1+x^2+y^2 \neq 0$ and $e^{x^2-y^2} \neq 0$ thus $2x = 0$ is necessary for $\nabla f = \langle 0,0\rangle$. However, $2x=0$ is not sufficient, we <u>also</u> need $2y(1-x^2-y^2) = 0$

but, as $x=0 \Rightarrow 2y(1-y^2) = 0$

$\Rightarrow 2y(1+y)(1-y) = 0$

$\Rightarrow \underbrace{(0,0)}_{\text{I}},\ \underbrace{(0,-1)}_{\text{II}},\ \underbrace{(0,1)}_{\text{III}}$ critical points

I. $f(x,y) = (x^2+y^2)(1-x^2-y^2+\cdots) = \underline{x^2+y^2} + \cdots$

$\underline{f(0,0) = 0 \text{ local min}}$

<u>Continuing to II and III</u>

$$f(x,y) = (x^2+y^2)e^{x^2-y^2}$$

$$\partial_x f = (2x + (x^2+y^2)(2x))e^{x^2-y^2} = [2x+2x^3+2xy^2]e^{x^2-y^2}$$

$$\partial_y f = (2y + (x^2+y^2)(-2y))e^{x^2-y^2} = [2y-2y^3-2x^2y]e^{x^2-y^2}$$

$$\partial_{xx} f = [2+6x^2+2y^2 + 2x(2x+2x^3+2xy^2)]e^{x^2-y^2}$$

$$\partial_{yy} f = [2-6y^2-2x^2 - 2y(2y-2y^3-2x^2y)]e^{x^2-y^2}$$

$$\partial_{xy} f = [4xy - 2y(2x+2x^3+2xy^2)]e^{x^2-y^2}$$

Observe $\partial_x f, \partial_y f = 0$ for $(0,-1), (0,1)$ and $\underbrace{f_{xy}(0,\pm1) = 0}_{\text{thankfully!}}$ otherwise need eigenvalue analysis

II $f(x,y) = f(0,-1) + \frac{1}{2}\left[f_{xx}(0,-1)x^2 + 2f_{xy}(0,-1)x(y+1) + f_{yy}(0,-1)(y+1)^2\right] + \cdots$

$= e^{-1} + \frac{1}{2}\left[4e^{-1}x^2 - 4e^{-1}(y+1)^2\right] + \cdots$

$= e^{-1}(1 + 2x^2 - 2(y+1)^2 + \cdots) \Rightarrow \underline{f(0,-1) \text{ is not max/min}}$

III $f(x,y) = f(0,1) + \frac{1}{2}\left[f_{xx}(0,1)x^2 + 2f_{xy}(0,1)x(y-1) + f_{yy}(0,1)(y-1)^2\right] + \cdots$

$= e^{-1} + \frac{1}{2}\left[4e^{-1}x^2 - 4e^{-1}(y-1)^2\right] + \cdots$

$= e^{-1}(1 + 2x^2 - 2(y-1)^2 + \cdots) \Rightarrow \underline{f(0,1) \text{ not local max/min}}$

Problem 46 work Edwards problem 7.12 from page 141.

In Exercises 7.5 and 7.6 we learned,

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots$$

$$\sin^2(x) = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots$$

With those given, classify critical pt. $(0,0,0)$ of:

$$f(x,y,z) = x^2+y^2+e^{xy} - y\tan^{-1}(x) + \sin^2 z$$

$$= x^2+y^2+1+xy - y\left(x - \frac{1}{3}x^3\right) + z^2 + \cdots$$

$$= \underline{1+x^2+y^2+z^2+\cdots}$$

kept only to quadratic order.

Therefore, $f(0,0,0)=1$ is a local minimum of f.

<u>Remark</u>: I'm quite happy xy cancelled with $-xy$ since otherwise the presence of crossterms neccessitates an eigenvalue of the Hessian analysis.

Problem 47 Consider the function $f(x,y,z) = e^{-z^2}\sin(x^2+y^2)$. Find points at which this function takes on local extreme values.

$$\nabla f = \left\langle e^{-z^2}\cos(x^2+y^2)\,2x,\ e^{-z^2}\cos(x^2+y^2)\,2y,\ -2ze^{-z^2}\sin(x^2+y^2)\right\rangle$$

Observe $(0,0,0)$ is critical. Also, if $z=0$ and $\cos(x^2+y^2)=0$. And $x=y=0$ with $\sin(x^2+y^2)=0$, but, that's just $(0,0,0)$. We find critical pts:

1.) $(0,0,0)$

2.) $(x,y,0)$ with $\cos(x^2+y^2)=0$.

$$\Rightarrow \underbrace{x^2+y^2 = \frac{\pi}{2}(2k-1)}_{\substack{\text{family of cylinders}\\ \text{of radius } R_k = \sqrt{\frac{\pi}{2}(2k-1)} = \sqrt{\frac{\pi}{2}},\ \sqrt{\frac{3\pi}{2}},\ \dots}} \text{ for some } k \in \mathbb{N}.$$

These are the possible locations of extreme values. To check if these are max/min / saddle / other further analysis is required.

1.) $f(x,y,z) = (1-z^2)(x^2+y^2+\cdots) = x^2+y^2+\cdots$

observe the absence of z^2-term hence $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 0$, non-definite, can't tell without higher-order analysis. Observe $f(0,0,z) = 0$; f is constant along z-axis. $\boxed{\text{(not max/min at } (0,0,0))}$

2.) $z=0$, $x^2+y^2 = \frac{\pi}{2}(2k-1)$ for $k \in \mathbb{N}$

Convenient to substitute $r = \sqrt{x^2+y^2}$ and expand $f(r,z)$ about $r_k = \sqrt{\frac{\pi}{2}(2k-1)}$ for $k \in \mathbb{N}$

$$\frac{\partial f}{\partial r} = e^{-z^2}\cos(r^2)(2r) = 2re^{-z^2}\cos(r^2)$$

$$\frac{\partial^2 f}{\partial r^2} = e^{-z^2}\left[2\cos(r^2) - 4r^2\sin(r^2)\right]$$

Observe $\left.\dfrac{\partial^2 f}{\partial r^2}\right|_{(r_k,0)} = -4r_k^2\sin\left(\frac{\pi}{2}(2k-1)\right)$ $\to$ $(-)\ \frac{\pi}{2}, \frac{5\pi}{2}, \dots$; $(+)\ \frac{3\pi}{2}, \frac{7\pi}{2}, \dots$

Thus,

$$\boxed{\begin{array}{l} (x,y,0) \text{ with } x^2+y^2 = \frac{\pi}{2}(4k-3) \Rightarrow \text{local max.} \\ (x,y,0) \text{ with } x^2+y^2 = \frac{\pi}{2}(4k-1) \Rightarrow \text{local min.} \end{array}}$$

Problem 48 Suppose we define $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. Show that $e^x e^y = e^{x+y}$ by multiplying the series via the Cauchy product. Partial credit can be obtained for working this out to third order.

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$$

Where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ defines Cauchy Product.

$$e^x e^y = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} c_n$$

Cauchy Product

$$\Rightarrow \quad c_n = \sum_{k=0}^{n} \frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!}$$

$$c_0 = 1$$
$$c_1 = y + x$$
$$c_2 = \tfrac{1}{2}y^2 + xy + \tfrac{1}{2}x^2 = \tfrac{1}{2}(x+y)^2$$
$$c_3 = \tfrac{1}{3!}y^3 + \tfrac{1}{2}xy^2 + \tfrac{1}{2}x^2y + \tfrac{1}{3!}x^3 = \tfrac{1}{3!}\left(y^3 + 3xy^2 + 3x^2y + x^3\right) = \tfrac{1}{3!}(x+y)^3$$
$$\vdots$$
$$c_n = \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}x^k y^{n-k} = \frac{1}{n!}(x+y)^n$$

Binomial Th$^{\underline{m}}$!

Therefore,

$$e^x e^y = \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n = e^{x+y}.$$

Remark: If we define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ then the calculation here proves the law of exponents as an implication of the Binomial Th$^{\underline{m}}$. Perhaps one of the most complete, logical and intuitive methods to construct sine, cosine, exp. etc.. is by the carefully developed theory of power series

not in my notes, but Apostol Math. Analysis is good

Problem 49 The inertia tensor $[I_{ij}]$ for a rigid object B provides nice formulas to describe the possible rotational motions of the rigid body. In particular, we define,

$$I_{ij} = \iiint_B \rho\left[(x_1^2 + x_2^2 + x_3^2)\delta_{ij} - x_i x_j\right] dV$$

It can be shown that if the body B rotates with angular velocity $\vec{\omega}$ then the total kinetic energy of B is given by

$$T = \sum_{i,j=1}^{3} I_{ij}\omega_i\omega_j = \vec{\omega}^T I \vec{\omega}.$$

The total angular momentum of B is given by

$$\vec{L} = \sum_{i,j=1}^{3} I_{ij}\omega_i e_j = I\vec{\omega}.$$

Finally, the net torque on B governs the change in the angular momentum:

$$\vec{\tau} = \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt}$$

Show that if $\vec{\tau} = 0$ then the kinetic energy is conserved. In other words, show that if $\vec{\tau} = 0$ then $\frac{dT}{dt} = 0$. warning: if your solution is longer than about a line then you're not thinking about this in the best way.

$I^T = I$

$$\vec{\tau} = 0 \Rightarrow I\frac{d\vec{\omega}}{dt} = 0 \Rightarrow \frac{d\vec{\omega}^T}{dt} I^T = 0 \Rightarrow \frac{d\vec{\omega}^T}{dt} I = 0$$

Consider, $T = \vec{\omega}^T I \vec{\omega}$ hence, product rule yields: (notice I is constant, the body is rigid)

$$\frac{dT}{dt} = \underbrace{\frac{d\vec{\omega}^T}{dt} I}_{0}\vec{\omega} + \vec{\omega}^T \underbrace{I\frac{d\vec{\omega}}{dt}}_{0} = 0$$

Thus $T = T_o$ ← constant, the kinetic energy does not change with time.

Problem 50 Consider the intertia tensor below:

$$[I_{ij}] = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

This is the inertia tensor for a cube of side length a with one corner at the origin. The total mass M of the cube is distributed evenly to give $\rho = M/a^3$. Find the eigenvalues and vectors and consider their physical significance.

$$\det\begin{bmatrix} 8-\lambda & -3 & -3 \\ -3 & 8-\lambda & -3 \\ -3 & -3 & 8-\lambda \end{bmatrix} = -\lambda^3 + 24\lambda^2 - 165\lambda + 242$$
$$= -(\lambda-2)(\lambda-11)^2$$

clearly.

We can calculate,

$\lambda_1 = 2$ $\quad (I - 2\mathbb{1})\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

$\lambda_2 = 11$ $\quad (I - 11\mathbb{1})\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\text{rref}\begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

well, duh.

$$u + v + w = 0 \Rightarrow u = -v - w,$$

$$\vec{u}_2 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = v\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the principle axes of the cube are along $\vec{u}_1 = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$ and any direction in plane spanned by $\vec{u}_2 = \frac{1}{\sqrt{2}}\langle -1,1,0\rangle$ and $\vec{u}_3 = \frac{1}{\sqrt{2}}\langle -1,0,1\rangle$.