Remember, I can only give you hints if you work on this before the day before it's due...

Problem 51 Given that $\mathbb{R}$ is complete, show that $\mathbb{R}^{n\times n}$ is complete. (I proved $\mathbb{R}^n$ is complete in lecture, think a bit, you can easily modify my argument)

Problem 52 Let A be a square matrix and P and invertible matrix. Show that $P^{-1}e^AP = e^{P^{-1}AP}$.

Problem 53 Show that if $AB = BA$ then $e^Ae^B = e^{A+B}$.

Problem 54 Show that $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$ where $[A, B] = AB - BA$ and the omit terms involve three or more terms in A and B. This is the Baker-Campbell-Hausdorff formula, the higher terms I have not written are all expressed as nestings of multiple commutators. For example, the next higher terms involve $[A, [A, B]$ and $[B, [B, A]]$.

Problem 55 Show that $\det(e^A) = e^{\text{tr}(A)}$. You can assume A is diagonalizable or, if you know what it means, assume A is in Jordan form. (this identity is general)

Problem 56 Suppose $SL(n) = \{A \in \mathbb{R}^{n\times n} \mid det(A) = 1\}$ defines the group of **special linear** matrices. Find a condition on B such that $\gamma(t) = e^{tB}$ forms a smooth curve in $SL(n)$ near the identity matrix. For future reference, the set of all matrices such as B forms $sl(n)$.

Problem 57 Suppose $O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^TA = I\}$ defines the group of **orthogonal** matrices. Find a condition on B such that $\gamma(t) = e^{tB}$ forms a smooth curve in $O(n)$. For future reference, the set of all matrices such as B forms $o(n)$.

Problem 58 Assume f_n, f are real-valued functions of a real variable and $n \in \mathbb{N}$. Suppose $f_n(x) \to f(x)$ for all $x \in [a, b]$ as $n \to \infty$. In such a case, we say that $f_n \to f$ pointwise on $[a, b]$. We also say, $\{f_n\}$ converges pointwise to f on $[a, b]$. We know continuous functions are Riemann integrable. If each f_n is continuous on $[a, b]$ then we can calculate $\int_a^b f_n(x)\,dx$ for $n = 1, 2, \ldots$. On the other hand, we may be able to calculate $\int_a^b f(x)\,dx$. Is it true that:

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx \ ?$$

The answer is, in general, **no.** Show how the following provides a **counter-example**

$$f_n(x) = \begin{cases} 4n^2x & 0 \le x \le 1/2n \\ 4n - 4n^2x & 1/n \le x \le 1/n \\ 0 & 1/n \le x \le 1 \end{cases}.$$

this exercise shows that it may not be possible to interchange the order of a limit and an integral. Clearly pointwise convergence does not allow the interchange in general. If for all $\epsilon > 0$ there exists N (independent of x) such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$ then we say $\{f_n\}$ converges to f uniformly on $[a, b]$. Uniform convergence often allows for the interchange of limits. See Rosenlicht for a nice introduction to these ideas, I got this example from page 138 of his text Introduction to Analysis.

Mission 6, MATH 332, FUN WITH MATRICES & ANALSIS

PROBLEM 51

Let $A : \mathbb{N} \longrightarrow \mathbb{R}^{n\times n}$ be a Cauchy sequence of matrices ; $\{A^{(n)}_{ij}\}$, where $A_{ij} : \mathbb{N} \longrightarrow \mathbb{R}$ $\forall i,j \in \mathbb{N}_n$. We wish to argue each $\{A_{ij}(n)\}_{n=1}^{\infty}$ is Cauchy sequence in $\mathbb{R}$. Observe, for $m,n \in \mathbb{N}$, $m>n$,

$$\underbrace{|A_{ij}(m) - A_{ij}(n)|}_{\text{absolute value}} \leq \underbrace{\|A(m) - A(n)\|}_{\|A\| = \sqrt{Tr(AA^T)}}$$

We're <u>given</u> $\{A(m)\}_{m=1}^{\infty}$ is Cauchy. Consider, for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t for $m,n \in \mathbb{N}$ with $m > n > N \Rightarrow \|A(m) - A(n)\| < \epsilon$. Thus, consider,

$$|A_{ij}(m) - A_{ij}(n)| \leq \|A(m) - A(n)\| < \epsilon$$

hence $\{A_{ij}(m)\}_{m=1}^{\infty}$ is Cauchy for each choice of $i,j \in \mathbb{N}_n$. But, $\mathbb{R}$ is complete hence $\exists A^*_{ij} \in \mathbb{R}$ for which $\lim_{n\to\infty} A_{ij}(n) = A^*_{ij}$. We claim

$$\lim_{n\to\infty} A(n) = A^* \quad \text{where } (A^*)_{ij} = A^*_{ij}.$$

this claim follows from the vector of limits $th^{\underline{m}}$

$\lim_{x\to a} \vec{F}(x) = \vec{b} \iff \lim_{x\to a} F_i(x) = b_i \quad \forall i$

which also holds for sequential limits.

(we could prove it, but, I go on...)

PROBLEM S2 Let $A \in \mathbb{R}^{n\times n}$ and $P \in \underbrace{GL(n)}_{\text{invertible } n\times n \text{ matrices.}}$

$$\begin{aligned}
P^{-1}e^AP &= P^{-1}(I + A + \tfrac{1}{2}A^2 + \tfrac{1}{3!}A^3 + \cdots)P \\
&= P^{-1}IP + P^{-1}AP + \tfrac{1}{2}P^{-1}AAP + \tfrac{1}{3!}P^{-1}AAAP + \cdots \\
&= P^{-1}IP + \underline{P^{-1}AP} + \tfrac{1}{2}\underline{P^{-1}AP}\,\underline{P^{-1}AP} + \tfrac{1}{3!}\underline{P^{-1}AP}\,\underline{P^{-1}AP}\,\underline{P^{-1}AP} + \cdots \\
&= I + P^{-1}AP + \tfrac{1}{2}(P^{-1}AP)^2 + \tfrac{1}{3!}(P^{-1}AP)^3 + \cdots \\
&= e^{P^{-1}AP}.
\end{aligned}$$

A mathematically proper argument would use some inductive step. I'd prove that $P^{-1}A^kP = (P^{-1}AP)^k$ for $k \geq 1$. Assume true for $k > 1$ and consider

$$\begin{aligned}
P^{-1}A^{k+1}P &= P^{-1}A^kAP \\
&= P^{-1}A^kPP^{-1}AP \\
&= (P^{-1}AP)^kP^{-1}AP \quad \text{by ind. hypothesis} \\
&= (P^{-1}AP)^{k+1}
\end{aligned}$$

hence by PMI, we find $P^{-1}A^kP = (P^{-1}AP)^k \ \forall k \in \mathbb{N}$.

thus $P^{-1}\left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)P = \sum_{k=0}^{\infty} \frac{P^{-1}A^kP}{k!} = \sum_{k=0}^{\infty} \frac{(P^{-1}AP)^k}{k!} = e^{P^{-1}AP}$.

analysis happened here! we're using the rearrangement result or something ... gap in my argument technically.

PROBLEM 53

Suppose $AB = BA$ then

$$e^A e^B = \left(I + A + \frac{1}{2}A^2 + \cdots\right)\left(I + B + \frac{1}{2}B^2 + \cdots\right)$$

$$= I + A + B + AB + \frac{1}{2}IB^2 + \frac{1}{2}IA^2 + \cdots$$

$$= I + (A+B) + \frac{1}{2}\left(A^2 + AB + BA + B^2\right) + \cdots$$

$$= I + (A+B) + \frac{1}{2}(A+B)^2 + \cdots$$

$$= e^{A+B}$$

Alternatively,

$$(A+B)\left(A^2 + AB + BA + B^2\right)$$

$$e^{A+B} = I + (A+B) + \frac{1}{2}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots$$

$$= I + A + B + \frac{1}{2}\left(A^2 + AB + BA + B^2\right) + \frac{1}{3!}\left(A^3 + A^2B + ABA + BA^2 + BAB + B^2A + B^3\right)$$

$$= I + A + B + \frac{1}{2}A^2 + \frac{1}{2}(2AB) + \frac{1}{2}B^2 + \frac{1}{3!}\left(A^3 + 3A^2B + 3AB^2 + B^3\right) + \cdots$$

then multiply $e^A e^B$ to see the same terms.

PROBLEM S3: Given $AB = BA$ for $A, B \in \mathbb{R}^{n \times n}$

Claim: $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k$

Proof: same as the usual binomial Th$^{\underline{m}}$ with a few extra steps to collect powers of A, B after using $AB = BA$. //

Note: $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ "n choose k"

Very well, let's apply this to the matrix exponential of $A+B$,

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} A^{n-k} B^k$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{A^{n-k}}{(n-k)!} \frac{B^k}{k!} \right)$$

Cauchy Product.

$$= \left(\sum_{n=0}^{\infty} \frac{A^n}{n!} \right) \left(\sum_{k=0}^{\infty} \frac{B^k}{k!} \right)$$

$$= e^A e^B.$$

$\boxed{\text{PROBLEM 54}}$

$$e^A e^B = \left(I + A + \tfrac{1}{2}A^2\right)\left(I + B + \tfrac{1}{2}B^2\right) + \cdots$$
$$= \underbrace{I + A + B + \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 + AB}_{*} + \cdots$$

$$\exp\left(A + B + \tfrac{1}{2}[A,B]\right) =$$
$$= I + A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{2}\left(A + B + \tfrac{1}{2}[A,B]\right)^2 + \cdots$$
$$= I + A + B + \tfrac{1}{2}(AB - BA) + \tfrac{1}{2}\overbrace{(A+B)(A+B)}^{\text{keeping only quad. terms.}} + \cdots$$
$$= I + A + B + \tfrac{1}{2}(AB - \cancel{BA}) + \tfrac{1}{2}(A^2 + AB + \cancel{BA} + B^2)$$
$$= \underbrace{I + A + B + \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 + AB}_{*} + \cdots$$

Therefore, at least two second order, we find
$$\exp(A)\exp(B) = \exp\left(A + B + \tfrac{1}{2}[A,B] + \cdots\right)$$

<u>Remark</u>: If we know $[A,B]$ then this tells us <u>how</u> to multiply e^A & e^B. In this sense the Lie Algebra <u>generates</u> the Lie Group.

$\boxed{\text{PROBLEM 55}}$ the non-diagonalizable case is nontrivial. I believe my 321 notes contain an argument for that case. Here I assume $\exists P \in GL(n)$ such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \Rightarrow \underline{A = PDP^{-1}}$$

$$\text{trace}(P^{-1}AP) = \text{trace}(APP^{-1}) = tr(A)$$

$$\Rightarrow \text{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{j=1}^{n} \lambda_j.$$

On the other hand,

$$\det(A) = \det(PDP^{-1}) = \cancel{\det(P)\det(D)\det(P^{-1})}$$

$$\Rightarrow \det(A) = \lambda_1 \lambda_2 \cdots \lambda_n = \prod_{j=1}^{n} \lambda_j$$

Thus, if e^A has e-values $\alpha_1, \alpha_2, \dots, \alpha_n$ then

$$\det(e^A) = \alpha_1 \alpha_2 \cdots \alpha_n$$

Notice $Av = \lambda v \Rightarrow A^2 v = \lambda A v = \lambda^2 v \cdots \Rightarrow A^k v = \lambda^k v$
hence $(e^A)v = (I + A + \cdots)v = (I + \lambda + \frac{1}{2}\lambda^2 + \cdots)v = e^{\lambda} I v$
thus v has e-value e^{λ} for e^A and we ~~identify~~ label.
$\alpha_1 = e^{\lambda_1}, \alpha_2 = e^{\lambda_2}, \dots, \alpha_n = e^{\lambda_n}$.

$$\det(e^A) = e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n}$$
$$= e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n}$$
$$= e^{\text{trace}(A)}$$

//

$\boxed{\text{PROBLEM 56}}$

$\gamma(t) \in SL(n) \Rightarrow \det(\gamma(t)) = 1$ for all $t \in \mathrm{dom}(\gamma)$.

But $\gamma(t) = e^{tB}$ by assumption hence:

$$\det(e^{tB}) = 1$$

$$\Rightarrow \exp(\mathrm{trace}(tB)) = 1 = \exp(0)$$

By 1-1 prop. of exponential on $\mathbb{R}$,

$$\mathrm{trace}(tB) = 0$$

but $\mathrm{trace}(tB) = t\,\mathrm{trace}(B)$ and this holds $\forall t \Rightarrow \boxed{\mathrm{trace}(B) = 0}$

$$\underline{sl(n) = \{B \in \mathbb{R}^{n\times n} \mid \mathrm{trace}(B) = 0\}}.$$

$\boxed{\text{PROBLEM 57}}$

$O(n) = \{A \in \mathbb{R}^{n\times n} \mid A^T A = I\}$

$\gamma(t) = e^{tB} \in O(n)$ for $t \in \mathbb{R}$, $\gamma(0) = e^0 = I$.

$$(\gamma(t))^T \gamma(t) = I$$

Differentiate,

$$\left(\frac{d\gamma}{dt}\right)^T \gamma(t) + (\gamma(t))^T \frac{d\gamma}{dt} = \frac{d}{dt}(I) = 0 \quad (\star)$$

evaluate at $t=0$, notice $\frac{d}{dt}(e^{tB}) = Be^{tB}$ hence $\gamma(0) = I$ but $\gamma'(0) = Be^0 = B$. Thus $\star$ tells us that $B^T I + I^T B = 0 \Rightarrow \underline{B^T = -B}$.

Therefore $\boxed{o(n) = \{B \in \mathbb{R}^{n\times n} \mid B^T = -B\}}$

PROBLEM 59

(a.) $\int_0^\infty \frac{\sin x}{x}\,dx = \underline{\int_0^\infty \frac{\sin(tY)}{Y}\,dY = \frac{\pi}{2}}$. $\left[\begin{array}{l} x = ty \\ dx = t\,dy \end{array}\right.$

(b.) $\frac{d}{dt}\int_0^\infty \frac{\sin(tY)}{Y}\,dY = \frac{d}{dt}\left(\frac{\pi}{2}\right)$

$\Rightarrow \int_0^\infty \frac{1}{Y}\frac{d}{dt}(\sin tY)\,dY = 0$

$\Rightarrow \int_0^\infty \frac{Y}{Y}\cos(tY)\,dY = 0 \quad \therefore \quad \underline{\int_0^\infty \cos(tY)\,dY = 0}$

(which is false!)

PROBLEM 60 (can do much more than I do here ☺)

$\int \cos(tx)\,dx = \frac{\sin(tx)}{t}$

$\Rightarrow \int \frac{d}{dt}[\cos(tx)]\,dx = \frac{d}{dt}\left(\frac{\sin tx}{t}\right)$

$\int -x\sin(tx)\,dx = \frac{t\cos tx - \sin tx}{t^2}$

$\int x\sin(tx)\,dx = \frac{\sin tx - \cancel{x}\cos tx}{t^2}$

$\Rightarrow \boxed{\int x\sin x\,dx = \sin x - x\cos(x) + C}$

$\int x^2\cos(tx)\,dx = \frac{d}{dt}\left[\frac{\sin tx - x\cos tx}{t^2}\right]$

$= \frac{[(x\cos tx)(t^2) + (x^2\sin tx)(t^2) - 2t(\text{—})]}{t^4}$

$= \frac{t^2(x\cos tx + x^2\sin tx) - 2t\sin tx + 2tx\cos tx}{t^4}$

$t=1)$ $\boxed{\int x^2\cos(\cancel{t}x) = x^2\sin(x) + 3x\cos x - 2\sin x + C}$