Problem 61 Let $T[f] = \int_0^1 f(t)\,dt$ for all $f \in V = P_2 = \text{span}\{1, x, x^2\}$. Show that $T \in V^*$.

Problem 62 Suppose V has basis $\beta = \{f_i\}_{i=1}^n$ and V^* has dual basis $\beta^* = \{f^j\}_{j=1}^n$ for which we assume $f^j(f_i) = \delta_{ij}$. We learn how to select components via the appropriate basis-evaluation:

(a.) let $v \in V$. Show that if $v = \sum_{i=1}^n c^i f_i$ then $c^i = f^i(v)$.

(b.) let $\alpha \in V^*$. Show that if $\alpha = \sum_{i=1}^n c_i f^i$ then $c_i = \alpha(f_i)$.

Problem 63 Suppose we have two bases for V: $\beta = \{f_i\}_{i=1}^n$ and $\bar{\beta} = \{\bar{f}_i\}_{i=1}^n$. A nice notation when dealing with two coordinate systems is to use bars to denote quantities attached to the bar-basis. In particular, we write:

$$v = \sum_{i=1}^n v^i f_i \qquad \text{verses} \qquad v = \sum_{i=1}^n \bar{v}^j \bar{f}_j$$

Since $f_i \in \text{span}\bar{\beta}$ there exists a matrix $P \in \mathbb{R}^{n\times n}$ for which $f_i = \sum_{j=1}^n P_i{}^j \bar{f}_j$. Moreover, because the inverse expansion must also exist as β is also a basis we can expect $P \in \text{Gl}(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$. In particular, $Q \in \mathbb{R}^{n\times n}$ exists for which $\bar{f}_j = \sum_{k=1}^n Q_j{}^k f_k$. Observe that substituting the Q-expansion into the P-expansion yields:

$$f_i = \sum_{k,j=1}^n P_i{}^j Q_j{}^k f_k \qquad \Rightarrow \qquad \sum_{j=1}^n P_i{}^j Q_j{}^k = \delta_{ik}.$$

some people write $\delta_i{}^k$ in place of δ_{ik} to emphasize the pattern. Do note that $P^{-1} = Q$, we introduce Q to avoid writing P^{-1}. I think I've given enough background at this point to ask you the following:

(a.) let $x \in V$. Show that $\bar{x}^j = \sum_{i=1}^n P_i{}^j x^i$

(b.) likewise, let $\alpha \in V^*$ and suppose $\alpha = \sum_{i=1}^n \alpha_i f^i$ and $\alpha = \sum_{i=1}^n \bar{\alpha}_i \bar{f}^i$ show $\bar{\alpha}_j = \sum_{i=1}^n Q^i{}_j \alpha_i$.

(c.) differentiate part (a.) with respect to x^i to obtain $P_i{}^j = \frac{\partial \bar{x}^j}{\partial x^i} = \partial_i \bar{x}^j$.

Problem 64 Suppose V, W are vector spaces and V^*, W^* are the corresponding dual spaces. Moreover, suppose V has basis $\beta = \{f_i\}_{i=1}^n$ and V^* has dual basis $\beta^* = \{f^j\}_{j=1}^n$ for which we assume $f^j(f_i) = \delta_{ij}$. Likewise, suppose W has basis $\gamma = \{g_i\}_{i=1}^m$ and W^* has dual basis $\gamma^* = \{g^j\}_{j=1}^m$ for which we assume $g^j(g_i) = \delta_{ij}$. Furthermore, we suppose there exist barred-versions of all the bases given above; $\bar{\beta}, \bar{\beta}^*, \bar{\gamma}, \bar{\gamma}^*$. Using partial-derivatives to capture the coordinate change:

$$f_i = \sum_{j=1}^n \frac{\partial \bar{x}^j}{\partial x^i} \bar{f}_j, \qquad f^i = \sum_{j=1}^n \frac{\partial x^i}{\partial \bar{x}^j} \bar{f}^j, \qquad g_i = \sum_{j=1}^m \frac{\partial \bar{y}^j}{\partial y^i} \bar{g}_j, \qquad g^i = \sum_{j=1}^m \frac{\partial y^i}{\partial \bar{y}^j} \bar{g}^j.$$

Here I introduce coordinate systems y and $\bar{y}$ for W which correspond the the γ and $\bar{\gamma}$ bases in the natural manner ($y = \sum_{i=1}^m y^i g_i = \sum_{i=1}^m \bar{y}^i \bar{g}_i$ etc...). Let $T : V \to W$ be a linear transformation. Define, for the sake of tradition, (later, it is the custom to use $T^i{}_j$ instead of $A^i{}_j$)

$$A^i{}_j = g^i(T(f_j)) \qquad \text{and, in the same way,} \qquad \bar{A}^i{}_j = \bar{g}^i(T(\bar{f}_j))$$

(a.) show that:

$$T(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} A^i{}_j x^j g_i.$$

(b.) find the relation between $A^i{}_j$ and $\bar{A}^i{}_j$.

Problem 65 Let $b : V \times V \to \mathbb{R}$ be a bilinear mapping on a vector space V with bases $\beta, \bar{\beta}$ (continuing to use the notation of the previous problem) then **show** $b(x,y) = \sum_{i,j} b_{ij} x^i y^j$ where $b_{ij} = b(f_i, f_j)$. Moreover, if $\bar{b}_{ij}$ are the components of b with respect to $\bar{\beta}$-basis then **show**

$$\bar{b}_{ij} = \sum_{k,l=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} b_{kl}.$$

Problem 66 Consider the tensor $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $g = \sum_{i=1}^{n} e^i \otimes e^i$ on $\mathbb{R}^n$ where $e^i(e_j) = \delta_{ij}$ and e_j denotes the usual j-th element of the standard basis for $\mathbb{R}^n$. Identify this tensor.

Problem 67 Define $\omega_{e_i} = e^i$ and show that linearly extending ω to $\mathbb{R}^3$ provides an isomorphism to $\Lambda^1(\mathbb{R}^3) = \mathbb{R}^{3*}$. Also Define $\Phi_{e_i} = \sum_{j,k=1}^{3} \epsilon_{ijk} e^j \otimes e^k$. Show that extending Φ linearly gives an isomorphism $\Phi : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^3)$. Verify that $\omega_{\vec{F}} \wedge \omega_{\vec{G}} = \Phi_{\vec{F}\times\vec{G}}$.

Problem 68 Let $V = \mathbb{R}^4$ and denote $V^* = \text{span}\{dt, dx, dy, dz\}$. Let $\alpha = 3dt + 6dx$ and $\beta = dx + dy$. Calculate $\alpha \wedge \alpha$, $\beta \wedge \beta$ and $\alpha \wedge \beta$.

Problem 69 Suppose $g : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ is given by $g(x,y) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$. Suppose we change coordinates by a matrix Λ where $\Lambda^T \eta \Lambda = \eta$ where we define $\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the formula for g in the $\bar{x}$-coordinate system. In particular, $\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$.

Problem 70 Finally, a physical application, the four-momentum is P^μ for $\mu = 0, 1, 2, 3$ where $P^0 = E$ (total energy) and $P^i = \gamma(v)v^i$ (relativistic momentum) and $\gamma(v) = \frac{1}{\sqrt{1-v^2}}$. Using the notation of the previous problem, the quantity $g(P, P)$ is the invariant interval of the 4-momentum. Calculate its value.

MISSION 7: SOLUTION

PROBLEM 61 Let $f \in V = P_2 = \text{span}\{1, x, x^2\}$. Show $T \in V^*$ meaning $T : V \to \mathbb{R}$ and T is linear, where $T[f] = \int_0^1 f(x)\,dx$

$$T[ax^2+bx+c] = \int_0^1 (ax^2+bx+c)\,dx$$
$$= \left(\tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx\right)\Big|_0^1$$
$$= \tfrac{1}{3}a + \tfrac{1}{2}b + c \in \mathbb{R} \qquad \text{hence } T : V \to \mathbb{R}$$

Furthermore,

$$T[f + cg] = \int_0^1 (f(x) + cg(x))\,dx$$
$$= \int_0^1 f(x)\,dx + c\int_0^1 g(x)\,dx$$
$$= T[f] + c\,T[g]. \implies T \in \mathcal{L}(V, \mathbb{R}) \therefore \underline{T \in V^*}.$$

PROBLEM 62 Suppose $V = \text{span}\{f_1, \dots, f_n\}$ where $\beta = \{f_1, \dots, f_n\}$ is basis for V. and $\beta^* = \{f^1, f^2, \dots, f^n\}$ is dual-basis to β meaning $f^i(f_j) = \delta_{ij}$ and we extend $f^i : V \to \mathbb{R}$ linearly from those values; $f^i(\sum_{j=1}^n x^j f_j) = \sum_{j=1}^n x^j f^i(f_j)$.

(a.) Show $v \in V \implies v = \sum_{i=1}^n c^i f_i$ where $c^i = f^i(v)$.

(b.) Show $\alpha \in V^* \implies \alpha = \sum_{i=1}^n c_i f^i$ where $c_i = \alpha(f_i)$

(a.) Let $v \in V$ and suppose $v = \sum_{i=1}^n c^i f_i$. Consider,

$$f^j(v) = f^j\left(\sum_{i=1}^n c^i f_i\right) = \sum_{i=1}^n c^i \underbrace{f^j(f_i)}_{\delta_{ij}} = c^j.$$

(b.) Let $\alpha \in V^*$ and suppose $\alpha = \sum_{i=1}^n c_i f^i$. Consider,

$$\alpha(f_j) = \left(\sum_{i=1}^n c_i f^i\right)(f_j) = \sum_{i=1}^n c_i f^i(f_j) = \sum_{i=1}^n c_i \delta_{ij} = c_j.$$

<u>Remark</u>: $\hat{f}_j \in V^{**}$ is defined by $\hat{f}_j(\alpha) = \alpha(f_j)$ so $c_j = \hat{f}_j(\alpha)$.

PROBLEM 63

(a.)
$$\begin{aligned}
\bar{x}^j &= \bar{f}^j(x) &&: \text{by PROBLEM 62a.}\\
&= \bar{f}^j\Big(\sum_{i=1}^n x^i f_i\Big) &&: \text{def}^{\underline{n}} \text{ of } x^i\\
&= \sum_{i=1}^n x^i \bar{f}^j(f_i) &&: \text{linearity of } \bar{f}^j\\
&= \sum_{i=1}^n x^i \bar{f}^j\Big(\sum_{k=1}^n P_i{}^k \bar{f}_k\Big) &&: \text{given, def}^{\underline{n}} \text{ of } P \text{ here.}\\
&= \sum_{i,k=1}^n P_i{}^k x^i \bar{f}^j(\bar{f}_k) &&: \text{linearity of } \bar{f}^j \text{ once more.}\\
&= \sum_{i,k=1}^n P_i{}^k x^i \delta_{jk} &&: \bar{f}^j(\bar{f}_k) = \delta_{jk} \text{ essentially defines } \bar{f}^j\\
&= \underline{\sum_{i=1}^n P_i{}^j x^i} \; //
\end{aligned}$$

(b.)
$$\begin{aligned}
\bar{\alpha}_j &= \alpha(\bar{f}_j) &&: \text{by PROBLEM 62b}\\
&= \alpha\Big(\sum_{i=1}^n Q_j{}^i f_i\Big) &&: \text{this defined } Q \text{, it was given in problem statement.}\\
&= \sum_{i=1}^n Q_j{}^i \alpha(f_i) &&: \text{this is by linearity of } \alpha.\\
&= \underline{\sum_{i=1}^n Q_j{}^i \alpha_i} \; // &&: \text{by PROBLEM 62b once more.}
\end{aligned}$$

(c.) Differentiation of x^i w.r.t. x^j enjoys the beautiful f-la $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$ - Here I've <u>defined</u> $\frac{\partial g}{\partial x^i}(p) = \frac{d}{dt}\big(g(p + t f_i)\big)\Big|_{t=0}$ as usual, except here x^i is coordinate map on abstract vector space so it's a bit more general than the usual $\partial/\partial x^i$

$$\frac{\partial \bar{x}^j}{\partial x^i} = \frac{\partial}{\partial x^i}\Big(\sum_{k=1}^n P_k{}^j x^k\Big) = \sum_{k=1}^n P_k{}^j \frac{\partial x^k}{\partial x^i} = \sum_{k=1}^n P_k{}^j \delta_{ki} = \underline{\underline{P_i{}^j}}.$$

PROBLEM 64

(a.) $T(x) = T\left(\sum_{i=1}^{n} x^i f_i\right)$: as $V = \text{span}\{f_1, \dots, f_n\}$ $\exists x^1, \dots, x^n$ such that $x = \sum_{i=1}^{n} x^i f_i$

$$= \sum_{i=1}^{n} x^i \, T(f_i) \quad : \text{linearity of } T$$

$$= \sum_{i=1}^{n} x^i \sum_{j=1}^{m} g^j(T(f_i))\, g_j \quad : \forall w \in W \text{ we have that } w = \sum_{j=1}^{m} g^j(w)\, g_j$$

$$= \underline{\sum_{i=1}^{n} \sum_{j=1}^{m} x^i A^j{}_i \, g_j} \quad : A^j{}_i = g^j(T(f_i)) \text{ defines } A.$$

(b.) Same argument as above shows $T(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{x}^i \bar{A}^j{}_i \, \bar{g}_j$ where $\bar{A}^j{}_i = \bar{g}^j(T(\bar{f}_i))$ by definition. I suspect the following is the easiest route:

$$\bar{A}^j{}_i = \bar{g}^j(T(\bar{f}_i))$$

$$= \bar{g}^j\left(T\left(\sum_{k=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i} f_k\right)\right)$$

$$= \sum_{k=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i} \, \bar{g}^j(T(f_k))$$

$$= \sum_{k=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i} \sum_{l=1}^{m} \frac{\partial \bar{y}^j}{\partial y^l} g^l(T(f_k))$$

$$= \underline{\sum_{k=1}^{n} \sum_{l=1}^{m} \frac{\partial \bar{y}^j}{\partial y^l} A^l{}_k \frac{\partial x^k}{\partial \bar{x}^i}}.$$

<u>Remark</u>: this formula explains the $A^j{}_i$ notation

notice how the up / down indices transform inversely

$$\bar{\partial}_i = \frac{\partial}{\partial \bar{x}^i} = \sum_{j=1}^{n} \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial}{\partial x^j} \quad \text{vs.} \quad d\bar{x}^i = \sum_{j=1}^{n} \frac{\partial \bar{x}^i}{\partial x^j} dx^j$$

in contrast see b of the next problem. Totally different coord. change props.

$\boxed{\text{PROBLEM 65}}$ $b : V \times V \longrightarrow \mathbb{R}$, define $b_{ij} = b(f_i, f_j)$ whereas $\bar{b}_{ij} = b(\bar{f}_i, \bar{f}_j)$. Let $x, y \in V$,

$$b(x,y) = b\left(\sum_{i=1}^{n} x^i f_i \,,\, \sum_{j=1}^{n} y^j f_j\right)$$

bilinearity

$$= \sum_{i,j=1}^{n} x^i y^j \, b(f_i, f_j)$$

def$^{\underline{n}}$ of b_{ij}

$$= \underline{\sum_{i,j=1}^{n} b_{ij} \, x^i y^j} \quad //$$

Next, consider,

$$\bar{b}_{ij} = b(\bar{f}_i, \bar{f}_j)$$

$$= b\left(\sum_{k=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i} f_k \,,\, \sum_{l=1}^{n} \frac{\partial x^l}{\partial \bar{x}^j} f_l\right)$$

$$= \sum_{k,l=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \, b(f_k, f_l)$$

$$= \underline{\sum_{k,l=1}^{n} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} \, b_{kl}} \quad //$$

$\boxed{\text{PROBLEM 66}}$ Let $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $g = \sum_{i=1}^{n} e^i \otimes e^i$

Let $x, y \in \mathbb{R}^n$ and w.r.t standard basis $e_1, e_2, \ldots, e_n$, $x = \sum_{i=1}^{n} x^i e_i$ and $y = \sum_{j=1}^{n} y^j e_j$,

$$g(x,y) = \left(\sum_{i=1}^{n} e^i \otimes e^i\right)(x,y)$$

$$= \sum_{i=1}^{n} e^i(x)\, e^i(y)$$

$$= \sum_{i=1}^{n} x^i y^i = x^1 y^1 + \cdots + x^n y^n = x \cdot y$$

g is the dot-product mapping a.k.a. the Euclidean metric on $\mathbb{R}^n$.

$\boxed{\text{PROBLEM 67}}$ Let $W_{e_i} = e^i$ and extend $W : \mathbb{R}^3 \to (\mathbb{R}^3)^*$ linearly,

$$W_v = \sum_{i=1}^{3} e^i(v)\, e^i$$

Note, $W_{e_j} = \sum_{i=1}^{3} e^i(e_j)\, e^i = \sum_{i=1}^{3} \delta_{ji}\, e^i = e^j$ hence W defined above is the desired linear extension. Linearity,

$$\begin{aligned} W_{v+cw} &= \sum_{i=1}^{3} e^i(v+cw)\, e^i \\ &= \sum_{i=1}^{3} e^i(v) e^i + c \sum_{i=1}^{3} e^i(w) e^i \\ &= W_v + c W_w \end{aligned}$$

Note, W maps $\{e_1, e_2, e_3\}$ to $\{e^1, e^2, e^3\}$ thus W is onto $(\mathbb{R}^3)^*$ and it follows by the standard dimension theorems of linear algebra W is isomorphism of $\mathbb{R}^3$ and $(\mathbb{R}^3)^*$. Likewise, sorting out $\Phi_{e_i} = \sum_{j,k=1}^{3} \epsilon_{ijk}\, e^j \otimes e^k$ yields

$$\begin{aligned} \Phi_{e_1} &= e^2 \otimes e^3 - e^3 \otimes e^2 &\Rightarrow\quad \Phi_{e_1} &= e^2 \wedge e^3 \\ \Phi_{e_2} &= e^3 \otimes e^1 - e^1 \otimes e^3 &\Rightarrow\quad \Phi_{e_2} &= e^3 \wedge e^1 \\ \Phi_{e_3} &= e^1 \otimes e^2 - e^2 \otimes e^1 &\Rightarrow\quad \Phi_{e_3} &= e^1 \wedge e^2 \end{aligned}$$

extending linearity gives isomorphism.

Thus Φ is a surjection onto the 3-dim'l $\Lambda^2(\mathbb{R}^3)$. Finally, the interesting part ☺,

$$\begin{aligned} W_{\vec{F}} \wedge W_{\vec{G}} &= (F_1 e^1 + F_2 e^2 + F_3 e^3) \wedge (G_1 e^1 + G_2 e^2 + G_3 e^3) \\ &= F_1 G_1 e^1 \wedge e^1 + F_1 G_2 e^1 \wedge e^2 + F_1 G_3 e^1 \wedge e^3 \\ &\quad + F_2 G_1 e^2 \wedge e^1 + F_2 G_2 e^2 \wedge e^2 + F_2 G_3 e^2 \wedge e^3 \\ &\quad + F_3 G_1 e^3 \wedge e^1 + F_3 G_2 e^3 \wedge e^2 + F_3 G_3 e^3 \wedge e^3 \\ &= (F_2 G_3 - F_3 G_2)\, e^2 \wedge e^3 + (F_3 G_1 - F_1 G_3)\, e^3 \wedge e^1 + (F_1 G_2 - F_2 G_1)\, e^1 \wedge e^2 \\ &= \underline{\Phi_{\vec{F} \times \vec{G}}}. \end{aligned}$$

these are components 1, 2, 3 of $\vec{F} \times \vec{G}$.

PROBLEM 68 $\alpha = 3dt + 6dx$, $\beta = dx + dy$, calculate $\alpha\wedge\alpha$, $\beta\wedge\beta$ and $\alpha\wedge\beta$

$$\alpha\wedge\alpha = (3dt + 6dx)\wedge(3dt+6dx)$$
$$= 18\, dt\wedge dx + 18\, dx\wedge dt$$, note $dt\wedge dt = dx\wedge dx = 0.$
$$= \boxed{0.}$$, $dx\wedge dt = -dt\wedge dx$

$$\beta\wedge\beta = (dx+dy)\wedge(dx+dy)$$: again $dx\wedge dx = dy\wedge dy = 0$
$$= dx\wedge dy + dy\wedge dx$$, $dx\wedge dy = -dy\wedge dx$
$$= \boxed{0}$$

$$\alpha\wedge\beta = (3dt+6dx)\wedge(dx+dy)$$
$$= 3dt\wedge dx + 3dt\wedge dy + \cancel{6dx\wedge dx}^{\,0} + 6dx\wedge dy$$
$$= \boxed{3dt\wedge dx + 3dt\wedge dy + 6dx\wedge dy}$$

Remark: $\alpha\wedge\beta = (-1)^{pq}\beta\wedge\alpha$ so, if $p=q=1$ naturally $\alpha\wedge\beta = -\beta\wedge\alpha$ and if $\alpha=\beta$ then $\alpha\wedge\alpha = -\alpha\wedge\alpha$ hence $\alpha\wedge\alpha = 0$ for a 1-form α. Thus $\alpha\wedge\alpha = \beta\wedge\beta = 0$ is not at all surprising.

PROBLEM 69 Let $g(x,y) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$

hence $g(x,y) = x^T \underbrace{\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{\eta} y$. In problem 65

we found $b = \sum_{i,j=1}^{n} b_{ij}\, e^i \otimes e^j$ (noting $x^i y^j = (e^i \otimes e^j)(x,y)$ and peeling off the (x,y) from $b(x,y) = \left(\sum b_{ij}\, e^i \otimes e^j\right)(x,y)$.

where $b_{ij} = b(e_i, e_j)$. Observe

we're given $g_{\mu\nu} = g(e_\mu, e_\nu) = \eta_{\mu\nu} = \begin{cases} -1 & \mu = \nu = 0 \\ 1 & \mu = \nu = 1,2,3 \\ 0 & \mu \neq \nu \end{cases}$

Now consider $\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$

where $\Lambda^T \eta \Lambda = \eta$ is a condition on the coord. change matrix Λ. Study $\bar{g}_{\mu\nu}$,

$$\bar{g}_{\mu\nu} = g(\bar{e}_\mu, \bar{e}_\nu)$$
$$= \sum_{\alpha,\beta} (\Lambda^{-1})_\mu{}^\alpha (\Lambda^{-1})_\nu{}^\beta \underbrace{g(e_\alpha, e_\beta)}_{\eta_{\alpha\beta}}$$

Not too enlightening, if you work on it.... eventually you'll see what we find in an easier notation below,

$$g(x,y) = \bar{x}^T \bar{g}\, \bar{y}$$
$$= (\Lambda x)^T \bar{g} \Lambda y$$
$$= x^T \Lambda^T \bar{g} \Lambda y = x^T \eta y \quad \forall x, y$$

$$\Rightarrow \Lambda^T \bar{g} \Lambda = \eta \quad \text{but, } \eta = \Lambda^T \eta \Lambda$$

$$\Rightarrow \Lambda^T \bar{g} \Lambda = \Lambda^T \eta \Lambda$$, note Λ, Λ^T invertible!

$$\Rightarrow \boxed{\bar{g} = \eta}$$

Thus, $g(x,y) = \underbrace{-\bar{x}^0\bar{y}^0 + \bar{x}^1\bar{y}^1 + \bar{x}^2\bar{y}^2 + \bar{x}^3\bar{y}^3}_{\text{invariant w.r.t Lorentzian coord. change}}$.

PROBLEM 70

$$p^\mu = (E, \gamma(v) v^i) \quad i=1,2,3, \quad \gamma(v) \equiv \frac{1}{\sqrt{1-v^2}}$$

$$g(p,p) = -E^2 + \gamma^2 v^1 v^1 + \gamma^2 v^2 v^2 + \gamma^2 v^3 v^3$$

If $\vec{v} = 0$ then $\boxed{g(p,p) = -E^2}$. that is, the invariant interval of the 4-momentum is the negative of the so-called rest-energy. (With m_0, c explicit we'd find $g(p,p) = -\frac{(m_0 c^2)^2}{c^2} = -m_0^2 c^2$ since $p^0 = E/c$ and $E = m_0 c^2$ for $v=0$ thus $\frac{E}{c} = m_0 c$.) This is a often used calculational technique for relativistic collision problems. To study a collision, look at the invariant interval in a coordinate system where the problem is easily understood, rest frame or center of momentum frame etc...