Problem 71 Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is a linearly **dependent** set of dual-vectors over a finite dimensional vector space V. **Show that** $\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k = 0$. For your convenience, you may assume that $\alpha_1 = \sum_{j=2}^{k} c_j \alpha_j$ for some constants $c_2, \ldots, c_k \in \mathbb{R}$.

Problem 72 Suppose γ is a p-form in $\Omega(V)$ where V is a n-dimensional vector space over $\mathbb{R}$. Assume $p \leq n$. If $\gamma \wedge e^j = 0$ for $j = 1, 2, \ldots, n$ then does it follow $\gamma = 0$? Discuss.

Problem 73 Finish the proof of Theorem 10.2.6 part 3. In particular, show: for smooth functions f, g on a manifold $\mathcal{M}$ with coordinates (x^i):

$$\frac{\partial}{\partial x^i}(fg) = \frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i}$$

here I omitted the point-dependence for brevity.

Problem 74 Let $F(x, y, z) = (\cos(x), \sin(xy))$. Calculate the push-forward of $X = a\partial_x + b\partial_y + c\partial_z$.

Problem 75 Let $\mathrm{SL}(2) = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) = 1\}$. Find a coordinate chart which includes the identity matrix.

Problem 76 Suppose $B^T = -B \in \mathbb{R}^{2\times 2}$ let $\gamma(t) = e^{tB}$ define a curve in $\mathbb{R}^{2\times 2}$. Show that γ gives a path in $\mathrm{SL}(2)$ and express the tangent vector to the path at $t = 0$ in terms of coordinate vector fields derived from the coordinate system you derived in the previous problem. To find the tangent vector to the curve you should push-forward the derivation $\frac{d}{dt}\big|_0$ in the domain of the path to $d\gamma_o(\frac{d}{dt}\big|_0) \in T_I\mathrm{SL}(2)$.

Problem 77 Observe $\chi = (\theta, \phi)$ gives a coordinate chart on $S_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. The inverse of this chart is given by the patch $\chi^{-1}(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. Calculate the push-forward under χ^{-1} of ∂_θ and ∂_ϕ in terms of the standard coordinate derivations $\partial_x, \partial_y, \partial_z$ the Cartesian coordinate system (x, y, z) for $\mathbb{R}^3$.

Problem 78 Let $\alpha = ydx + zdy$ and $\beta = xydz \wedge dt + dx \wedge dy$. Assume that x, y, z, t are independent Cartesian coordinates on $\mathbb{R}^4$.

(a.) calculate $d\alpha$

(b.) calculate $d\beta$

(c.) calculate $\alpha \wedge \beta$

(d.) calculate $d(\alpha \wedge \beta)$ in two ways.

Problem 79 Use the table of basic Hodge duals in my notes extended linearly to prove $*\omega_{\vec{v}} = \Phi_{\vec{v}}$. Also, show that $*\Phi_{\vec{v}} = \omega_{\vec{v}}$.

Problem 80 Explain what vector-calculus identities follow from $d(d\alpha) = 0$ and the correspondances $df = \omega_f$, $d\omega_{\vec{F}} = \Phi_{\nabla\times\vec{F}}$ and $d\Phi_{\vec{G}} = (\nabla \cdot \vec{G})dx \wedge dy \wedge dz$. I did one of these in lecture.

MATH 332 MISSION 8 SOLUTION

PROBLEM 71 Suppose $\alpha_1 = \sum_{j=2}^{k} c_j \alpha_j$ for some constants $c_2, \dots, c_k \in \mathbb{R}$.

Consider,

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k = \left(\sum_{j=2}^{k} c_j \alpha_j\right) \wedge \alpha_2 \wedge \cdots \wedge \alpha_k$$

$$= c_2 \alpha_2 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k + c_3 \alpha_3 \wedge \alpha_2 \wedge \alpha_3 \wedge \cdots \wedge \alpha_k + \cdots + c_k \alpha_k \wedge \overbrace{\alpha_2 \wedge \cdots \wedge \alpha_{k-1}}^{k-2} \wedge \alpha_k$$

$$= c_2 (\alpha_2 \wedge \alpha_2) \wedge \cdots \wedge \alpha_k - c_3 \alpha_2 \wedge (\alpha_3 \wedge \alpha_3) \wedge \cdots \wedge \alpha_k + \cdots + (-1)^{k-2} c_k \alpha_2 \wedge \cdots \wedge \alpha_{k-1} \wedge (\alpha_k \wedge \alpha_k)$$

$$= 0$$

As $\alpha_2 \wedge \alpha_2 = \alpha_3 \wedge \alpha_3 = \cdots = \alpha_k \wedge \alpha_k = 0$, and $0 \wedge \alpha = 0$ in general.

Remark: I assumed here $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ and $0 \wedge \alpha = 0$ we're already shown elsewhere. Can you show these using only basic anticommuting of basis & distributivity?

PROBLEM 72 Suppose $\gamma = \sum_{i_1, i_2 \dots i_p} \frac{1}{p!} \gamma_{i_1 i_2 \dots i_p} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}$ where $(\mathbb{R}^n)^* = \text{span}\{e^1, \dots, e^n\}$ and $n \geq p$. Suppose $\gamma \wedge e^j = 0 \ \forall j \in \mathbb{N}_n$ does it follow $\gamma = 0$?

No. If $p = n$ then $\gamma = e^1 \wedge e^2 \wedge \cdots \wedge e^n$ has $\gamma \wedge e^j = 0 \ \forall j \in \mathbb{N}_n$ however, clearly, $\gamma \neq 0$. If $p < n$ then the question requires more thought. Consider $p = 1$,

$$\gamma = \sum_{i=1}^{n} \gamma_i e^i$$

$$\gamma \wedge e^j = \sum_{i=1}^{n} \gamma_i e^i \wedge e^j = 0$$

Notice, $e^i \wedge e^j \neq 0$ when $i \neq j$ (assuming $n \geq 2$). Thus we obtain $\gamma_i = 0 \ \forall i \neq j$ just from a fixed e^j. But, we're given all $j \in \mathbb{N}_n$ force $\gamma \wedge e^j = 0$ hence $\gamma \wedge e^{j+1} = 0 \Rightarrow \gamma_j = 0$ as well hence $\gamma_i = 0 \ \forall i \in \mathbb{N}_n$ and it follows $\gamma = 0$. A similar, and more annoying

$\mathcal{C}\ell_p$ = set of all multindex of increasing indices of length p from $\mathbb{N}_n$ here

may be given for higher p-forms. Let us argue with a multindex notation. Let $\gamma = \sum_{I \in \mathcal{C}\ell_p}{}' \gamma_I \, e^I$ and $\gamma \wedge e^j = 0 \quad \forall j = 1, 2, \dots, n$. Consider,

$$\gamma \wedge e^j = \sum_{I \in \mathcal{C}\ell_p}{}' \gamma_I \, e^I \wedge e^j = 0$$

Moreover, $e^I \wedge e^j \neq 0$ for all $I \in \mathcal{C}\ell_p$ for which $j \notin I$. This gives $\gamma_I = 0$ for all $I \in \mathcal{C}\ell_p$ with $j \in I$. Lets be concrete and set $j = 1$ to begin. We have that $\gamma_I = 0 \quad \forall \ I \in \mathcal{C}\ell_p$ for which $1 \notin I$.

Consider next $\gamma \wedge e^2 = 0 \Rightarrow \gamma_I = 0 \quad \forall I \in \mathcal{C}\ell_p$ s.t. $2 \in I$.

Then continue for $j = 2, 3, \dots, n$ and we obtain $\gamma_I = 0$ for all $I \in \mathcal{C}\ell_p$ not containing $2, 3, \dots, n$ respective. A moments reflection reveals this gives all $I \in \mathcal{C}\ell_p$ as we consider the totality of all the cases hence $\gamma_I = 0 \quad \forall \ I \in \mathcal{C}\ell_p \quad \therefore \quad \gamma = 0$.

Remark: If $\exists j \in \mathbb{N}_n$ for which $\gamma \wedge e^j = 0$ this is certainly not enough data to force $\gamma = 0$ as we obtain no information about γ_I with $j \in I$.

PROBLEM 73 Show $\frac{\partial}{\partial x^i}(fg) = \frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i}$ for $\chi = (x^i)$ a coordinate chart on a manifold $\mathcal{M}$

Recall the notation near proof of Th^{m} 10.2.6. We use $u^1, u^2, \dots, u^n$ for Cartesian coordinates on $\mathbb{R}^n$ and $X = (x^i)$ then in this notation, for $f : M \longrightarrow \mathbb{R}$,

$$\frac{\partial f}{\partial x^i}(P) = \frac{\partial}{\partial u^i}\Big[(f \circ X^{-1})(u^1, \dots, u^n) \Big]\Big|_{X(P)} \quad \text{or} \quad \frac{\partial f}{\partial x^i} = \frac{\partial}{\partial u^i}\big[f \circ X^{-1} \big]\Big|_{X}$$

Notice $f \circ X^{-1} : \mathbb{R}^n \longrightarrow M \longrightarrow \mathbb{R}$ is function from $\mathbb{R}^n \longrightarrow \mathbb{R}$ hence ordinary partial differentiation makes sense. It's just directional differentiation in the i^{th} coordinate direction at P. With all this in mind, consider,

$$\begin{aligned}
\frac{\partial}{\partial x^i}\big[fg\big](P) &= \frac{\partial}{\partial u^i}\Big[(fg) \circ X^{-1}(u^1, \dots, u^n) \Big]\Big|_{X(P)} \\
&= \frac{\partial}{\partial u^i}\Big[(f \circ X^{-1})(u^1, \dots, u^n)\, (g \circ X^{-1})(u^1, \dots, u^n) \Big]\Big|_{X(P)} \\
&= \Big[\frac{\partial}{\partial u^i}\big[(f \circ X^{-1})(u^1, \dots, u^n)\big](g \circ X^{-1})(u^1, \dots, u^n) + (f \circ X^{-1})(\vec{u}) \frac{\partial}{\partial u^i}\big[(g \circ X^{-1})(\vec{u})\big] \Big]\Big|_{X(P)} \\
&= \Big[\frac{\partial}{\partial u^i}(f \circ X^{-1})(\vec{u}) \Big]\Big|_{X(P)} g(X^{-1}(X(P))) + f(X^{-1}(X(P))) \Big[\frac{\partial}{\partial u^i}(g \circ X^{-1})(\vec{u}) \Big]\Big|_{X(P)} \\
&= \frac{\partial f}{\partial x^i}(P)\, g(P) + f(P) \frac{\partial g}{\partial x^i}(P)
\end{aligned}$$

Key Step

Hence, as this holds $\forall P$,

$$\frac{\partial}{\partial x^i}(fg) = \frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i}$$

Here the key Step was, w/o surprise I hope, the ordinary product rule for partial derivatives.

PROBLEM 74 Let $F(x,y,z) = (\cos(x),\ \sin(xy)) = (u,v)$
Calculate the push-forward for $X = a\partial_x + b\partial_y + c\partial_z$
Here we assume a, b, c are constants.

$$dF_p(X) = a\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right) + b\left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right) + c\left(\frac{\partial u}{\partial z}\frac{\partial}{\partial u} + \frac{\partial v}{\partial z}\frac{\partial}{\partial v}\right)$$

$$= a\left(-\sin(x)\partial_u + y\cos(xy)\partial_v\right) + b\left(0\,\partial_u + x\cos(xy)\partial_v\right) + c\left(0\,\partial_u + 0\,\partial_v\right)$$

$$= \boxed{-a\sin(x)\partial_u + \left(ay\cos(xy) + bx\cos(xy)\right)\partial_v}$$

Here, x, y are evaluated at $p = (x,y,z)$ and technically ∂_u is $\partial_u\big|_{F(p)}$. We should understand $dF_p : T_p\mathbb{R}^3 \longrightarrow T_{F(p)}\mathbb{R}^2$. But, our main objective here is just the computation (which is boxed)

PROBLEM 75 Let $SL(2) = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) = 1.\}$
Find coordinate chart on $SL(2)$ whose domain includes $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$SL(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; ad - bc = 1 \right\}$$

We wish to have $b = c = 0$ in our chart domain, so we best not divide by b or c. I'll pick on a, well, d

$$a = \frac{1+bc}{d} \quad \text{for } d \neq 0.$$

Let $\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{1}{d}(1+bc), b, c\right)$. Clearly for $d \neq 0$ χ gives a smooth chart, well, with a single chart this is almost tautological. Anyway, notice χ is injective,

$$\chi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \Rightarrow \left(\frac{1}{\bar{d}}(1+\bar{b}\bar{c}), \bar{b}, \bar{c}\right) = \left(\frac{1}{d}(1+bc), b, c\right)$$

$$\Rightarrow b = \bar{b},\ c = \bar{c} \Rightarrow \frac{1}{d} = \frac{1}{\bar{d}} \therefore d = \bar{d}.$$

Furthermore, $a, \bar{a}$ are uniquely determined by b, c, d as we consider $SL(2)$.

PROBLEM 75 continued

You could also have used $\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(b, c, \frac{1+bc}{a}\right)$ (solve $ad - bc = 1$ for $d = \frac{1+bc}{a}$). Notice, to see injectivity we could also exhibit an inverse,

$$\Phi^{-1}(x, y, z) = \left[\ ?\ \right]$$

a moments reflection reveals $x \longmapsto (1,2)$ spot and $y \longmapsto (2,1)$
Now, z requires some thought.

$$\left(b, c, \frac{1+bc}{a}\right) = (x, y, z) \quad \begin{array}{l} \to x = b \\ \to y = c \\ \to z = \dfrac{1+bc}{a} \end{array}$$

then $az = 1 + bc = 1 + xy \longrightarrow a = \frac{1+xy}{z}$

finally, $d = \frac{1+bc}{a} = \frac{1+xy}{(1+xy)/z} = z$. So,

$$\Phi^{-1}(x, y, z) = \left[\begin{array}{c|c} \frac{1}{z}(1+xy) & x \\ \hline y & z \end{array}\right]$$

Let's check it, for $z \neq 0$,

$$\begin{aligned} \Phi\left(\Phi^{-1}(x, y, z)\right) &= \Phi\left(\begin{array}{c|c} \frac{1}{z}(1+xy) & x \\ \hline y & z \end{array}\right) \\ &= \left(x, y, \frac{1+xy}{\frac{1}{z}(1+xy)}\right) \\ &= (x, y, z). \end{aligned}$$

Also, $\det\left(\Phi^{-1}(x, y, z)\right) = \det\left[\begin{array}{c|c} \frac{1}{z}(1+xy) & x \\ \hline y & z \end{array}\right] = 1 + xy - xy \stackrel{\checkmark}{=} 1.$

PROBLEM 75 continued

Another thought $\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (b, c, d)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$ for which $d \neq 0$. This is clearly injective for $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in SL(2)$ with $d_1, d_2 \neq 0$

$$\psi\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \psi\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \Rightarrow (b_1, c_1, d_1) = (b_2, c_2, d_2)$$
$$\Rightarrow b_1 = b_2,\ c_1 = c_2,\ d_1 = d_2$$

What about a_1, a_2? Remember,

$$a_1 d_1 - b_1 c_1 = 1 \quad \text{and} \quad a_2 d_2 - b_2 c_2 = 1$$

Hence, $a_1 = \dfrac{1 + b_1 c_1}{d_1} = \dfrac{1 + b_2 c_2}{d_2} = a_2$ hence ψ injective.

Moreover,

$$\psi^{-1}(b, c, d) = \left[\begin{array}{c|c} \frac{1+bc}{d} & b \\ \hline c & d \end{array}\right].$$

There are doubtless other choices. I think I like ψ best for the purposes of 76. Notice

$$\psi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 0, 1).$$

PROBLEM 76 Assume $B^T = -B$ for $B \in \mathbb{R}^{2\times 2}$ and define $\gamma(t) = \exp(tB)$. Consider, $\det(\exp(tB)) = \exp(\text{trace}(tB)) = \exp(0) = 1$. Therefore, $\gamma(t) \in SL(2)$ $\forall t \in \mathbb{R}$. Also, $\gamma(0) = \exp(0) = I_{2\times 2}$.

Observe $B^T = -B$ in $\mathbb{R}^{2\times 2}$ is rather boring,

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^T = -\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \Rightarrow \begin{array}{l} B_{11} = B_{22} = 0 \\ B_{12} = -B_{21} \end{array}$$

Hence, set $B = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$ without loss of generality.

PROBLEM 76 continued

$$\gamma(t) = \exp\left(t\begin{bmatrix} 0 & \beta \\ -\beta & 0\end{bmatrix}\right)$$

$$= \exp(t\beta J) \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$$

$$= I + t\beta J + \tfrac{1}{2}t^2\beta^2J^2 + \tfrac{1}{3!}t^3\beta^3J^3 + \tfrac{1}{4!}t^4\beta^4J^4 + \cdots$$

$$= I\left(1 - \tfrac{1}{2}(t\beta)^2 + \tfrac{1}{4!}(t\beta)^4 + \cdots\right) + J\left(t\beta - \tfrac{1}{3!}(t\beta)^3 + \cdots\right)$$

You check it?

$J^2 = -I$
$J^3 = -J$
$J^4 = I$
etc...

$$= I\cos t\beta + J\sin\beta t$$

$$= \begin{bmatrix} \cos t\beta & \sin\beta t \\ -\sin\beta t & \cos\beta t\end{bmatrix}$$

I'll use $\psi\begin{pmatrix} a & x \\ y & z\end{pmatrix} = (x, y, z)$ for our $SL(2)$ chart where $z \neq 0$. We have

$$\gamma(t) = \begin{bmatrix} \cos(t\beta) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t)\end{bmatrix}$$

$(x\circ\gamma)(t) = \sin\beta t \qquad (x\circ\gamma)'(t) = \beta\cos\beta t$

$(y\circ\gamma)(t) = -\sin\beta t \qquad (y\circ\gamma)'(t) = -\beta\cos\beta t$

$(z\circ\gamma)(t) = \cos\beta t \qquad (z\circ\gamma)'(t) = \beta\sin\beta t$

Thus,

$$d\gamma_0\left(\frac{d}{dt}\Big|_0\right) = \frac{d(x\circ\gamma)(t)}{dt}\Big|_{t=0}\frac{\partial}{\partial x}\Big|_I + \frac{d(y\circ\gamma)(t)}{dt}\Big|_{t=0}\frac{\partial}{\partial y}\Big|_I + \frac{d(z\circ\gamma)(t)}{dt}\Big|_{t=0}\frac{\partial}{\partial z}\Big|_I$$

$$= \boxed{\beta\frac{\partial}{\partial x}\Big|_I - \beta\frac{\partial}{\partial y}\Big|_I}$$

If we identify $\frac{\partial}{\partial x}\Big|_I = \begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}$ and $\frac{\partial}{\partial y}\Big|_I = \begin{bmatrix} 0 & 0 \\ +1 & 0\end{bmatrix}$ then $d\gamma_0\left(\frac{d}{dt}\Big|_0\right) \longrightarrow \begin{bmatrix} 0 & \beta \\ -\beta & 0\end{bmatrix}$ which is not at all surprising;

$$\frac{d}{dt}\left[\exp\left(t\begin{bmatrix} 0 & \beta \\ -\beta & 0\end{bmatrix}\right)\right] = \begin{bmatrix} 0 & \beta \\ -\beta & 0\end{bmatrix}\underbrace{\exp\left(t\begin{bmatrix} 0 & \beta \\ -\beta & 0\end{bmatrix}\right)}_{I \text{ when } t=0}$$

$\boxed{\text{PROBLEM 77}}$

$\chi = (\theta, \phi)$ gives chart for $S_2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$
Notice $\chi^{-1}(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Calculate push-forward of $\partial/\partial\theta$ and $\partial/\partial\phi$ under χ^{-1}

$$d\chi^{-1}\left(\frac{\partial}{\partial\theta}\right) = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\theta}\frac{\partial}{\partial z}$$

$$= \boxed{-\sin\theta\sin\phi\frac{\partial}{\partial x} + \cos\theta\sin\phi\frac{\partial}{\partial y}}$$

$$d\chi^{-1}\left(\frac{\partial}{\partial\phi}\right) = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial}{\partial z}$$

$$= \boxed{\cos\theta\cos\phi\frac{\partial}{\partial x} + \cos\phi\sin\theta\frac{\partial}{\partial y} - \sin\phi\frac{\partial}{\partial z}}$$

Oops, not, quite, need to convert $\sin\theta$, $\sin\phi$ etc... to Cartesian coordinates x, y, z ... unfun, no, fun! For the sphere $\rho = 1$ hence

$$x = \cos\theta\sin\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\phi$$

$$x^2+y^2 = \sin^2\phi \hookrightarrow \sin\phi = \pm\sqrt{x^2+y^2} = \sqrt{x^2+y^2}$$

as $0 \leq \phi \leq \pi$ traditionally.

Also, $\frac{y}{x} = \tan\theta$

Oh, $x = \cos\theta\sin\phi = \cos\theta\sqrt{x^2+y^2} \hookrightarrow \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$

also, $y = \sin\theta\sin\phi = \sin\theta\sqrt{x^2+y^2} \hookrightarrow \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

Hence, (better answer 2)

$$\boxed{\begin{aligned} d\chi^{-1}\left(\frac{\partial}{\partial\theta}\right) &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\ d\chi^{-1}\left(\frac{\partial}{\partial\phi}\right) &= \frac{xz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{yz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} - \sqrt{x^2+y^2}\frac{\partial}{\partial z}\end{aligned}}$$

PROBLEM 78 Let $\alpha = y\,dx + z\,dy$
$\beta = xy\,dz\wedge dt + dx\wedge dy$

(a.) $d\alpha = \underline{dy\wedge dx + dz\wedge dy}$.

(b.) $d\beta = (y\,dx + x\,dy)\wedge dz\wedge dt + \underbrace{\cancel{d(dx\wedge dy)}}_{0}$
$= \underline{y\,dx\wedge dz\wedge dt + x\,dy\wedge dz\wedge dt}$.

(c.) $\alpha\wedge\beta = (y\,dx + z\,dy)\wedge(xy\,dz\wedge dt + dx\wedge dy)$
$= \underline{xy^2\,dx\wedge dz\wedge dt + xyz\,dy\wedge dz\wedge dt} + 0 + 0$.

(d.) $d(\alpha\wedge\beta) = d(xy^2)\wedge dx\wedge dz\wedge dt + d(xyz)\wedge dy\wedge dz\wedge dt$
$= \underline{2xy\,dy\wedge dx\wedge dz\wedge dt + yz\,dx\wedge dy\wedge dz\wedge dt}$.
$= \boxed{(yz - 2xy)\,dx\wedge dy\wedge dz\wedge dt}$ ← I like this answer.

Remark: I simplify my life by not writing $d(xyz) = yz\,dx + xz\,dy + xy\,dz$ since I know $dy\wedge dz\wedge dt$ automatically kills the dy & dz term. Hence, just write $yz\,dx$ and go on...

The other method, by Leibniz (graded)

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge d\beta$$
$$= (dy\wedge dx + dz\wedge dy)\wedge(xy\,dz\wedge dt + dx\wedge dy)$$
$$- (y\,dx + z\,dy)\wedge(y\,dx\wedge dz\wedge dt + x\,dy\wedge dz\wedge dt)$$
$$= xy\,dy\wedge dx\wedge dz\wedge dt - yx\,dx\wedge dy\wedge dz\wedge dt - zy\,dy\wedge dx\wedge dz\wedge dt$$
$$= -2xy\,dx\wedge dy\wedge dz\wedge dt + yz\,dx\wedge dy\wedge dz\wedge dt$$
$$= \boxed{(yz - 2xy)\,dx\wedge dy\wedge dz\wedge dt}$$

PROBLEM 79

$$
\begin{aligned}
*\omega_{\vec{v}} &= *(a\,dx + b\,dy + c\,dz) \\
&= a(*dx) + b(*dy) + c(*dz) \\
&= a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy \\
&= \Phi_{\vec{v}} \qquad \text{where } \vec{v} = \langle a, b, c\rangle.
\end{aligned}
$$

$$
\begin{aligned}
*\Phi_{\vec{v}} &= *(a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy) \\
&= a(*[dy\wedge dz]) + b[*(dz\wedge dx)] + c[*(dx\wedge dy)] \\
&= a\,dx + b\,dy + c\,dz \\
&= \omega_{\vec{v}}.
\end{aligned}
$$

PROBLEM 80 We showed in lecture that

$$df = \omega_{\nabla f} \quad \text{and} \quad d\omega_{\vec{F}} = \Phi_{\nabla\times\vec{F}} \quad \text{and} \quad d\Phi_{\vec{F}} = (\nabla\cdot\vec{F})\,dx\wedge dy\wedge dz$$

Consider then,

$$0 = d(df) = d\omega_{\nabla f} = \Phi_{\nabla\times\nabla f} \implies \boxed{\nabla\times\nabla f = 0}$$

$$0 = d(d\omega_{\vec{F}}) = d\Phi_{\nabla\times\vec{F}} = \nabla\cdot(\nabla\times\vec{F})\,dx\wedge dy\wedge dz$$

$$\implies \boxed{\nabla\cdot(\nabla\times\vec{F}) = 0}$$