Problem 81 Suppose α is a p-form on $\mathbb{R}^3$. Let $*$ denote the standard euclidean Hodge dual. Is it possible that $*d(\alpha) = d(*\alpha)$?

Problem 82 The Hodge dual operation on $\mathbb{R}^3$ allows us to intoduce another way to differentiate a p-form α:

$$\delta\alpha = (-1)^{np+n+1} * d * \alpha = (-1)^{3p} * d * \alpha$$

where I have set $n = 3$ since I intend to use this **coderivative** δ on $\mathbb{R}^3$. Let $\alpha = a dy \wedge dz + b dz \wedge dx + c dx \wedge dy$ where a, b, c are smooth functions on $\mathbb{R}^3$. Calculate the formula for $\delta\alpha$.

Problem 83 De Rahm, Hodge and others developed a theory to analyze closed vs. exact differential forms. See my notes for an example of how the shape of the domain can come into play. One interesting theorem Hodge proved was that if ω was any p-form on a Riemannian manifold then there exists a $(p-1)$-form α and a $(p+1)$-form β and a *harmonic form* γ such that

$$\omega = d\alpha + \delta\beta + \gamma.$$

In the special case $M = \mathbb{R}^3$ it is the case $\gamma = 0$. **Use the theorem due to Hodge to prove that any vector field can be written in terms of the gradient of a scalar function and the curl of some vector field; that is, for any vector field $\vec{F}$ there exists another vector field $\vec{G}$ and a function g such that $\vec{F} = \nabla g + \nabla \times \vec{G}$.** I think if you examine the case $\omega = \omega_{\vec{F}}$ then it ought to be about a line or two once you unravel the notation. I let Hodge do the really hard part. (you need to use the preceding problem to understand the coderivative part)

Problem 84 Consider $\omega = (x+y)dx + (y+z)dy + (z+x)dz$ on $\mathbb{R}^3$. Verify Hodge's Theorem (see preceding problem) by finding α and β such that $\omega = d\alpha + \delta\beta$. Begin your quest by understanding what the degrees of α and β must be in your context.

Warning: the notations *, $*$ and $_*$ mean different things. I do not use $_*$, but other authors use $df(v) = f_*(v)$; that is f_* is the push-forward by f. I use f^* as the pull-back by f and finally $*\gamma$ is the Hodge dual of γ which is only defined with respect to a metric (and for us, either $\mathbb{R}^3$ euclidean or $\mathbb{R}^4$ Minkowski for the application to Electrodynamics in a later chapter). Hodge duality is far less basic than the other two stars of this discussion. If you wish to read on Hodge duality in some generality then you might look at David Bleecker's text *Gauge Theory and Variational Principles* which is inexpensive in Dover format.

Problem 85 Consider $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(r, \theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$. View r, θ, ϕ as Cartesian coordinates on the domain of F and view x, y, z as coordinates for the codomain of F. In particular, this indicates $F^1 = x$ and $F^2 = y$ and $F^3 = z$. Consider the differential form $\gamma = dx \wedge dy \wedge dz$. Calculate $F^*(\gamma)$.

Problem 86 Continuing the previous problem, find $F^*(\beta)$ where $\beta = \frac{1}{x^2+y^2}[-ydx + xdy]$.

Problem 87 Let $T : V \to V$ be a linear transformation on a finite-dimensional vector space V. Let $\Lambda^k T$ be the function from $\underbrace{V \times V \times \cdots \times V}_{k-\text{copies of } V}$ to $\Lambda^k(V)$ defined as follows:

$$\Lambda^k T(v_1, v_2, \ldots, v_k) = T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_k)$$

Briefly explain why $\Lambda^k T$ is a multilinear completely antisymmetric mapping. Then, if $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V then show:

$$\Lambda^n T = M v^1 \wedge v^2 \wedge \cdots \wedge v^n$$

where $\beta^* = \{v^1, v^2, \dots, v^n\}$ is the dual basis to β. Also, explain what is M as it relates to T.

Problem 88 Prove $\det(AB) = \det(A)\det(B)$ via wedge products. You might want to use the $\epsilon_{i_1, i_2, \dots, i_n}$ formula for the determinant of $A, B \in \mathbb{R}^{n \times n}$

Problem 89 Suppose $F(r, s, t) = (rst, r^2+s^2, s^2+t^2, 3)$ defines a mapping from $\mathbb{R}^3$ to $\mathbb{R}^4$. Furthermore, suppose $\mathbb{R}^4$ has Cartesian coordinates (x, y, z, w). Consider the one-forms on $\mathbb{R}^4$ as follows:

$$\alpha = (x^2 + z)dw \qquad \& \qquad \beta = (z^2 + w^2)(dx + dy)$$

Verify the identities:

(i.) $d[F^*(\alpha)] = F^*[d\alpha]$
(ii.) $d[F^*(\beta)] = F^*[d\beta]$
(iii.) $F^*[\alpha \wedge \beta] = F^*[\alpha] \wedge F^*[\beta]$

Problem 90 Maxwell's equations are written in differential form on $\mathbb{R}^4$ in my notes. Essentially, ignoring a factor of c, the coordinates on spacetime are (t, x, y, z). Pull-back Maxwell's equations to volume of constant time $t = t_o$. What are the new equations which hold on the slice of spacetime where time is constant? Are these equations familar from Physics 232 (if you've had or ICED that course)

PROBLEM 81

Can $*(d\alpha) = d(*\alpha)$

No. Why? Counting. If α is p-form in $\mathbb{R}^3$ then $*\alpha$ is a $(3-p)$-form hence $d(*\alpha)$ is a $(3-p)+1 = (4-p)$-form.

On the other hand, $*(d\alpha)$ is a $3-(p+1) = 2-p$ form. Clearly $4-p \neq 2-p$ as $2 \neq 4$ etc...

PROBLEM 82 Given $\delta\alpha = (-1)^{3p} * d * \alpha$, for what follows $p = 2$

$$\begin{aligned}
\delta \Phi_{\langle a,b,c\rangle} &= * d * \Phi_{\langle a,b,c\rangle} \\
&= * d\omega_{\langle a,b,c\rangle} \\
&= * \Phi_{\nabla\times\langle a,b,c\rangle} \\
&= \boxed{\omega_{\nabla\times\langle a,b,c\rangle}} \\
&= \underline{(\partial_y c - \partial_z b)dx + (\partial_z a - \partial_x c)dy + (\partial_x b - \partial_y a)dz}.
\end{aligned}$$

Remark: I guess I might have set you up for a more complicated calculation ... I hope you got something from it.

PROBLEM 83 Hodge's Th$^{\underline{m}}$ for $\mathbb{R}^3 \Rightarrow$ given ω, $\exists \alpha, \beta$ s.t. $\underline{\omega = d\alpha + \delta\beta}$.

Let $\vec{F}$ be vector field then $\omega = \omega_{\vec{F}}$ is one-form hence by Hodge, $\exists\alpha$ a zero-form and β a two form for which $\omega_{\vec{F}} = d\alpha + \delta\beta$ but $d\alpha = \omega_{\nabla\alpha}$ and $\delta\beta = \delta\Phi_{\vec{G}} = \omega_{\nabla\times\vec{G}}$

hence $\omega_{\vec{F}} = \omega_{\nabla\alpha} + \omega_{\nabla\times\vec{G}} = \omega_{\nabla\alpha + \nabla\times\vec{G}} \Rightarrow \underline{\vec{F} = \nabla\alpha + \nabla\times\vec{G}}$.

as claimed

$\omega_{\vec{v}}$ is isomorphism.

PROBLEM 84 Consider $\omega = (x+y)\,dx + (y+z)\,dy + (z+x)\,dz$

Find α, β for which $\omega = d\alpha + \delta\beta$.

$$\omega = x\,dx + y\,dy + z\,dz + y\,dx + z\,dy + x\,dz$$

$$= d\Big(\underbrace{\tfrac{1}{2}[x^2+y^2+z^2]}_{\alpha}\Big) + \underbrace{y\,dx + z\,dy + x\,dz}$$

$$\delta\beta = *\,d\,*\,(a\,dy\wedge dz + b\,dz\wedge dx + c\,dx\wedge dy)$$

$$= *\,(y\,dy\wedge dz + z\,dz\wedge dx + x\,dx\wedge dy)$$

$$= *\,d\left(\tfrac{1}{2}y^2\,dz + \tfrac{1}{2}z^2\,dx + \tfrac{1}{2}x^2\,dy\right)$$

$$= *\,d\,*\,\Phi_{\langle \frac{1}{2}z^2,\ \frac{1}{2}x^2,\ \frac{1}{2}y^2\rangle}$$

Thus, $\omega = df + \delta\Phi_{\vec{G}}$ for $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2)$

$\vec{G}(x,y,z) = \langle z^2/2,\ x^2/2,\ y^2/2\rangle$

Just to check,

$$\delta\Phi_G = *\,d\,*\,\Phi_{\vec{G}}$$

$$= *\,d\,\omega_{\vec{G}}$$

$$= *\,d\left(\frac{z^2}{2}\,dx + \frac{x^2}{2}\,dy + \frac{y^2}{2}\,dz\right)$$

$$= *\,(z\,dz\wedge dx + x\,dx\wedge dy + y\,dy\wedge dz)$$

$$= z\,dy + x\,dz + y\,dx \quad \text{(what we wanted.)}$$

PROBLEM 85 $F(r,\theta,\phi) = (r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi) = (x,y,z)$

Let $\gamma = dx\wedge dy\wedge dz$ calculate $F^*(\gamma)$. Omit pt. dependence,

$$F^*\gamma = F^*(dx\wedge dy\wedge dz)$$

$$= d(x\circ F)\wedge d(y\circ F)\wedge d(z\circ F)$$

$$= d(r\cos\theta\sin\phi)\wedge d(r\sin\theta\sin\phi)\wedge d(r\cos\phi)$$

$$= d(r\cos\theta\sin\phi)\wedge d(r\sin\theta\sin\phi)\wedge[dr\cos\phi - r\sin\phi\,d\phi]$$

$$= (-r\sin\theta\sin\phi\,d\theta + r\cos\theta\cos\phi\,d\phi)\wedge(r\cos\theta\sin\phi\,d\theta + r\sin\theta\cos\phi\,d\phi)\wedge dr\cos\phi$$

$$+ (\cos\theta\sin\phi\,dr - r\sin\theta\sin\phi\,d\theta)\wedge(\sin\theta\sin\phi\,dr + r\cos\theta\sin\phi\,d\theta)\wedge(-r\sin\phi\,d\phi)$$

$$= d\theta\wedge d\phi\,[-r^2\sin^2\theta\sin\phi\cos\phi - r^2\cos^2\theta\cos\phi\sin\phi]\wedge(\cos\phi)\,dr$$

$$+ dr\wedge d\theta\,[r\cos^2\theta\sin^2\phi + r\sin^2\theta\sin^2\phi]\wedge(-r\sin\phi\,d\phi)$$

$$= dr\wedge d\phi\wedge d\theta\,[r^2\cos^2\phi\sin\phi + r^2\sin^2\phi\sin\phi] = \boxed{r^2\sin\phi\,dr\wedge d\phi\wedge d\theta}$$

PROBLEM 86 Once more $F(r,\theta,\phi)$ as in 85,

$$F^*(\beta) = F^*\left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right)$$

$$= \frac{1}{(r\cos\theta\sin\phi)^2 + (r\sin\theta\sin\phi)^2}\left[-r\sin\theta\sin\phi\, d(r\cos\theta\sin\phi) + r\cos\theta\sin\phi\, d(r\sin\theta\sin\phi)\right]$$

$$= \frac{1}{r^2\sin^2\phi}\Big[-r\sin\theta\sin\phi\left[\cos\theta\sin\phi\, dr - r\sin\theta\sin\phi\, d\theta + r\cos\theta\cos\phi\, d\phi\right]$$
$$+ r\cos\theta\sin\phi\left[\sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi\right]\Big]$$

$$= \frac{1}{r^2\sin^2\phi}\Big\{dr\left(-r\sin\theta\cos\theta\sin^2\phi + r\sin\theta\cos\theta\sin^2\phi\right)$$
$$+ d\theta\left(r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\sin^2\phi\right)$$
$$+ d\phi\left(-r^2\sin\theta\cos\theta\sin\phi\cos\phi + r^2\sin\theta\cos\theta\cos\phi\sin\phi\right)\Big\}$$

$$= \frac{1}{r^2\sin^2\phi}\, d\theta\,(r^2\sin^2\phi)$$

$$= \boxed{d\theta}$$

In other words, $d\theta$ written in Cartesian coordinates is simply $\beta = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$.

PROBLEM 87 Let V be finite dim'l with $T: V \to V$ linear trans. define $\Lambda^k T : V \times V \times \cdots \times V \longrightarrow \Lambda^k V^*$ ← k-vectors, wedge product of k-vectors.

$$\Lambda^k T(v_1,\dots,v_k) = T(v_1)\wedge\cdots\wedge T(v_k)$$

Observe that

$$\Lambda^k T(v_1,\dots, v_j + cw_j, \dots, v_k) = T(v_1)\wedge\cdots\wedge T(v_j + cw_j)\wedge\cdots\wedge T(v_k)$$
$$= T(v_1)\wedge\cdots\wedge(T(v_j) + cT(w_j))\wedge\cdots\wedge T(v_k) \quad \{\text{linear of } T\}$$
$$= T(v_1)\wedge\cdots\wedge T(v_j)\wedge\cdots\wedge T(v_k) + c\,T(v_1)\wedge\cdots\wedge T(w_j)\wedge\cdots\wedge T(v_k) \quad (\text{dist of } \wedge \text{ product})$$
$$= \Lambda^k T(v_1,\dots, v_j,\dots, v_k) + c\,T(v_1,\dots, w_j,\dots, v_k)$$

Thus, $\Lambda^k T$ is linear in the j^{th} slot ⇒ multilinear as j was arbitrary. It's antisymmetric because the wedge product is also.

$\boxed{\text{PROBLEM 87}}$ continued, Let $\{v_1, v_2, \dots, v_n\}$ be basis for V and calculate $\Lambda^n T$

$$\begin{aligned}
\Lambda^n T(v_1, v_2, \dots, v_n) &= T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_n) \\
&= \left(\sum_{i_1=1}^{n} M_{i_1 1} v_{i_1}\right) \wedge \left(\sum_{i_2=1}^{n} M_{i_2 2} v_{i_2}\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^{n} M_{i_n n} v_{i_n}\right) \\
&= \sum_{i_1 \dots i_n = 1}^{n} M_{i_1 1} M_{i_2 2} \cdots M_{i_n n} \, v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_n} \\
&= \underline{\sum_{i_1, \dots, i_n = 1}^{n} \epsilon_{i_1 i_2 \dots i_n} M_{i_1 1} M_{i_2 2} \cdots M_{i_n n}} \, v_1 \wedge v_2 \wedge \cdots \wedge v_n \\
&= \underline{\det(M)} \, v_1 \wedge v_2 \wedge \cdots \wedge v_n
\end{aligned}$$

this is my $\underline{\text{definition}}$ for $\det M$.

Here $M = [T]_{\beta\beta}$ in my usual linear algebra notation.

In particular, $[T]_{\beta,\beta} = \big[\, [T(v_1)]_\beta \,|\, [T(v_2)]_\beta \,|\, \cdots \,|\, [T(v_n)]_\beta \,\big]$

thus $\mathrm{Col}_j([T]_{\beta,\beta}) = [T(v_j)]_\beta$

Here $\Phi_\beta(x_1 v_1 + \cdots + x_n v_n) = (x_1, \dots, x_n)$, $\Phi_\beta : V \to \mathbb{R}^n$ is the β-coordinate chart.

<u>Remark</u>: sorry about the Problem Statement on this one, I copied it wrong and/or "fixed" it as I copied. I don't think I needed to discuss V^* here. The only perhaps weird thing is that $\Lambda^k T$ is multi-vector valued. (we've mostly worked with forms, but multivectors share most of the same algebra, well, not the exterior derivative, but otherwise ...

PROBLEM 88

$$\text{Vol}_n \det(A) = Ae_1 \wedge Ae_2 \wedge \cdots \wedge Ae_n \Rightarrow \underline{\det A \, e_1 \wedge \cdots \wedge e_n = \text{Col}_1(A) \wedge \cdots \wedge \text{Col}_n(A)}$$

Likewise

$$\begin{aligned}
\det(AB) e_1 \wedge \cdots \wedge e_n &= \text{Col}_1(AB) \wedge \text{col}_2(AB) \wedge \cdots \wedge \text{Col}_n(AB) \\
&= A\,\text{col}_1(B) \wedge A\,\text{col}_2(B) \wedge \cdots \wedge A\,\text{col}_n(B) \\
&= \left(\sum_{k_1=1}^{n} B_{k_1 1} \text{col}_{k_1}(A)\right) \wedge \left(\sum_{k_2=1}^{n} B_{k_2 2} \text{col}_{k_2}(A)\right) \wedge \cdots \wedge \left(\sum_{k_n=1}^{n} B_{k_n n} \text{col}_{k_n}(A)\right) \\
&= \sum_{k_1, \dots, k_n = 1}^{n} B_{k_1 1} B_{k_2 2} \cdots B_{k_n n} \, \text{Col}_{k_1}(A) \wedge \text{col}_{k_2}(A) \wedge \cdots \wedge \text{Col}_{k_n}(A) \quad (*) \\
&= \underbrace{\sum_{k_1 \cdots k_n = 1}^{n} \epsilon_{k_1 k_2 \cdots k_n} B_{k_1 1} B_{k_2 2} \cdots B_{k_n n}} \; \text{Col}_1(A) \wedge \text{col}_2(A) \wedge \cdots \wedge \text{col}_n(A) \\
&= \det(B) \det(A) \, e_1 \wedge e_2 \wedge \cdots \wedge e_n
\end{aligned}$$

Thus $\underline{\det(AB) = \det A \det B}$.

(*) notice the signs needed to reorder $k_1 k_2 \cdots k_n \longrightarrow 1 2 3 \cdots n$ is precisely quantified by the completely antisymmetric symbol $\epsilon_{k_1 k_2 \cdots k_n}$.

Remark: in Math 321 my proof of $\det(AB) = \det A \det B$ requires case-by-case analysis of elementary matrix calculations. This is much cleaner (believe it!)

PROBLEM 89 continued

$$F^*(d\beta) = F^*\big((2z\,dz + 2w\,dw)\wedge(dx+dy)\big)$$
$$= F^*\big(2z\,dz\wedge(dx+dy) + 2w\,\underset{0}{dw}\wedge(dx+dy)\big)$$
$$= 2(s^2+t^2)\,d(s^2+t^2)\wedge[d(rst) + d(r^2+s^2)] \quad *$$

Compare with,

$$d(F^*(\beta)) = d\Big([(s^2+t^2)^2 + 9][d(rst) + d(r^2+s^2)]\Big)$$
$$= d(s^2+t^2)^2 \wedge [d(rst) + d(r^2+s^2)]$$
$$= 2(s^2+t^2)\,d(s^2+t^2)\wedge[d(rst) + d(r^2+s^2)]_*$$

Thus, $\underline{F^*(d\beta) = d(F^*(\beta))}$.

(iii.) Verify $\underline{F^*[\alpha\wedge\beta] = F^*[\alpha]\wedge F^*[\beta]}$

$$F^*[\alpha\wedge\beta] = F^*[(x^2+z)(z^2+w^2)\,dw\wedge(dx+dy)]$$
$$= [(rst)^2 + s^2+t^2][(s^2+t^2)^2+9]\,d(3)\wedge(d(rst)+d(r^2+s^2))$$
$$= 0.$$

$F^*(\alpha)\wedge F^*(\beta) = 0$ also as $F^*(\alpha) = 0$

as $\alpha = (x^2+z)\,dw$ pulls back to zero

under $w = 3$.

PROBLEM 89

$$F(r,s,t) = (rst,\ r^2+s^2,\ s^2+t^2,\ 3) = (x, y, z, w)$$

abuse of notation, I β technically $x \circ F$, $y \circ F$, $z \circ F$, $w \circ F$ would be more accurate.

Let $\alpha = (x^2 + z)\,dw$

$\beta = (z^2 + w^2)(dx + dy)$

(i.) verify $d(F^*(\alpha)) = F^*(d\alpha)$

$$F^*(\alpha) = F^*((x^2+z)\,dw)$$
$$= (x^2+z)\,d(3) \quad \text{(with } x = rst,\ z = s^2+t^2\text{)}$$
$$= 0 \quad \therefore \quad dF^*(\alpha) = 0.$$

$$F^*(d\alpha) = F^*(2x\,dx \wedge dw + dz \wedge dw)$$
$$= F^*(2x\,dx + dz) \wedge F^*(dw) \qquad w = 3 \Rightarrow dw = 0.$$
$$= F^*(2x\,dx + dz) \wedge 0$$
$$= 0 \quad \therefore \quad F^*(d\alpha) = 0.$$

Thus, $d(F^*(\alpha)) = F^*(d\alpha)$.

(ii.) verify $d(F^*(\beta)) = F^*(d\beta)$

$$F^*(\beta) = (z^2+w^2)\left[d(rst) + d(r^2+s^2)\right] \quad \text{where } z = s^2+t^2,\ w = 3,$$
$$= \left((s^2+t^2)^2 + 9\right)\left[st\,dr + rt\,ds + rs\,dt + 2r\,dr + 2s\,ds\right]$$
$$= \left(9 + (s^2+t^2)^2\right)\left[(st + 2r)\,dr + (rt + 2s)\,ds + rs\,dt\right]$$

$$d(F^*(\beta)) = dr \wedge \Big[\big(2(s^2+t^2)2s\,ds + 2(s^2+t^2)2t\,dt\big)(st + 2r) + (9 + (s^2+t^2)^2)(t\,ds + s\,dt)\Big]$$
$$+ \cdots$$

— Sorry about this. —

Somewhat fixed $\rightarrow$ key is not to actually compute $d(rst)$ etc... pattern easier to match w/o those computations explicit.

$\boxed{\text{PROBLEM 90}}$ $F = \omega_{\vec{E}} \wedge dt + \Phi_{\vec{B}}$ ← Faraday tensor

$*F = \Phi_{\vec{E}} - \omega_{\vec{B}} \wedge dt$ ← Hodge Dual.

Maxwell's Eq='s in spacetime are given by:

$$dF = 0 \quad \text{and} \quad d(*F) = \mu_0 * J$$

where $*J = \rho\, dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge dt$ $\left[\begin{array}{l} \rho = \frac{dQ}{dV} \text{ charge density} \\ \vec{J} = \frac{d\vec{I}}{dA} \text{ current density.} \end{array}\right.$

Consider $\Phi(x, y, z) = (t_0, x, y, z)$. this makes

$\Phi : \mathbb{R}_{xyz} \longrightarrow \mathbb{R}_{txyz}$ has $\Phi^*(dt) = 0$

$$\left.\begin{array}{l} \Phi^*(dx) = dx \\ \Phi^*(dy) = dy \\ \Phi^*(dz) = dz \end{array}\right\}$$ slight abuse of notation as x is being used both as $\mathbb{R}^3$ and $\mathbb{R}^4$ coordinate here.

Consider then,

$$\Phi^*(F) = \Phi^*(\omega_{\vec{E}} \wedge \underbrace{dt}_{0} + \Phi_{\vec{B}}) = \Phi_{\vec{B}}$$

$$\Phi^*(*F) = \Phi^*(\Phi_{\vec{E}} - \omega_{\vec{B}} \wedge \underbrace{dt}_{0}) = \Phi_{\vec{E}}$$

$$\Phi^*(*J) = \Phi^*(\rho\, dx \wedge dy \wedge dz - \Phi_{\vec{J}} \wedge \underbrace{dt}_{0}) = \rho\, dx \wedge dy \wedge dz$$

Thus, $dF = 0 \Rightarrow \Phi^*(dF) = \Phi^*(0)$ $\qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$$\Rightarrow d(\Phi^*(F)) = 0$$

$$\Rightarrow d\Phi_{\vec{B}} = 0$$

$$\Rightarrow \underline{(\nabla \cdot \vec{B})\, dx \wedge dy \wedge dz = 0}.$$

$$d(*F) = \mu_0 * J \Rightarrow \Phi^*(d(*F)) = \Phi^*(\mu_0 * J)$$

$$\Rightarrow d(\Phi^*(*F)) = \mu_0 \Phi^*(*J)$$

$$\Rightarrow d\Phi_{\vec{E}} = \mu_0 \rho\, dx \wedge dy \wedge dz$$

$$\Rightarrow \underline{\nabla \cdot \vec{E} = \mu_0 \rho}$$

Well, there is some dimensional problem with my J. We should get $\nabla \cdot \vec{E} = \rho/\epsilon_0$