Problem 1 (10pts) **Paul's easy question:** Consider $T(x, y, z) = (x + y + z, y + z, z)$.
(a.) Find the standard matrix of T. **(b.)** Is T surjective? **(c.)** Is T injective?

Problem 2 (4pts) In our proof that continuously differentiable implies differentiable, we used the ____________ ____________ Theorem and continuity of the ____________ ____________ to prove the Frechet quotient has limit zero. (fill in the blanks)

Problem 3 (16pts) Consider $G(x, y) = (x^2 + y^2,\ xy,\ x + y)$.
(a.) Find the Jacobian matrix of G
(b.) Find the linearization of G at $(4, 2)$

Problem 7 (10pts)**Joe's Problem** Let $\gamma(t) = (t, 2t^2, 3t^3, 4t^4)$ for each $t \in \mathbb{R}$ defines a curve in $\mathbb{R}^4$. Find the tangent and normal space to the curve at $(1, 2, 3, 4)$.

Problem 4 (10pts) Suppose A is a symmetric real $n \times n$ matrix.
Let $R(u) = u^T A u$ for all $u \in S_{n-1}$ (meaning $u \in \mathbb{R}^n$ and $||u|| = 1$).
Show that Q defined by $Q(x) = x^T A x$ for all $x \in \mathbb{R}^n$ is the only quadratic form on $\mathbb{R}^n$ which restricts to R on the unit-$(n-1)$-sphere in $\mathbb{R}^n$.

Problem 5 (15pts) Let D be a diagonal matrix with $D_{ii} > 0$ for each $i = 1, 2, \ldots, n$.
Show that $\eta(v, w) = v^T D w$ defines an inner product on $\mathbb{R}^n$.

Problem 6 (20pts) We sketched a proof of the inverse function theorem for functions $F : \mathbb{R}^n \to \mathbb{R}^n$. Let V be a real normed linear space of dimension n

(a.) conjecture an inverse function theorem for a mapping $G : V \to V$

(b.) prove your conjecture follows the known theorem for functions on $\mathbb{R}^n$.

Problem 8 (15pts) Suppose $F : V \to V$ is a differentiable mapping of real normed linear space V with basis $\beta = \{v_1, \dots, v_n\}$. Given that the differential of F at $p \in V$ satisfies $d_pF(v_i) = (i^2 + 1)v_i$ for $i = 1, 2, \dots, n$ what can we say the function F near p.

Problem 9 (15pts) Let $F(A) = A^2$ for each $A \in \mathbb{R}^{n\times n}$. Show that F is differentiable on $\mathbb{R}^{n\times n}$.

Problem 10 (15pts) Let $F(A) = A^2$ for each $A \in \mathbb{R}^{n\times n}$. Furthermore, define $G : \mathbb{R} \to \mathbb{R}^{n\times n}$ by $G(t) = \cos(t)I + \sin(t)J$ where I is the identity matrix and J is a given constant matrix. Calculate the differential of $F \circ G$ at time t.

Problem 11 (10pts) Let $f(x, y) = 3 + 10(x-1)^2 - 4(x-1)(y-2) + 6(y-2)^2 + \cdots$. Determine if $f(1, 2) = 3$ is a minimum, maximum or a saddle point in view of the given multivariate Taylor expansion.

Problem 12 (10pts) If $D = f_{xx}f_{yy} - f_{xy}^2$ then, given a critical point p of f, explain why $D(p) > 0$ for $f(p)$ an extreme value. Your argument should take the multivariate Taylor expansion as a given starting point.

Show your work. Thanks! Choose 7, including Problem 7 and 10. 10pts a problem.

Problem 1 Let $V = \{A \in \mathbb{R}^{2\times 2} \mid \text{trace}(A) = 0\}$.
(a.) Find a basis for V
(b.) Write the formula for the coordinate map on V.

Problem 2 Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined by $T(A) = \det(A) = A_{11}A_{22} - A_{12}A_{21}$ for each $A \in \mathbb{R}^{2\times 2}$. Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ serve as the basis for 2×2 matrices. Find $d_AT(E_{ij})$ for each basis element and use these to construct $d_AT(H)$ for arbitrary $H \in \mathbb{R}^{2\times 2}$.

Problem 3 Let $F(x, y, z) = (x^2 + y^2, y^2 + z^2)$. Find a parametrization(s) of the curve(s) $F^{-1}\{(4, 1)\}$.

Problem 4 Suppose $w = x^2 + y^2 - z^2$ and $z = x^2 - y^2$. Calculate $\left(\frac{\partial w}{\partial x}\right)_y$.

Problem 5 Find extrema of $f(x, y) = x^2 + y^2$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$. $(a, b > 0)$

Problem 6 Let $x^2 + y^2 = 4$ and $z^2 + w^2 = 49$ define a space M in $\mathbb{R}^4$ with coordinates (x, y, z, w). Find the tangent and normal space to M at $(1, \sqrt{3}, 0, 7)$

Problem 7 Consider $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (e^{x^2} + y, 2 + e^{x^2} - y)$.
(a.) calculate the Jacobian of F and calculate its determinant
(b.) where can we expect F to have a local inverse ?
(c.) let G denote F restricted to $U = \{(x, y) \mid x > 0\}$. Find the formula for $G^{-1}(a, b)$
(d.) and determine the range of G.

Problem 8 Consider the system of equations below:

$$t^2 + x^2 + y^2 - z^2 - w^2 = 4,$$
$$tx + xy + yz$$

Determine what condition is needed to apply the implicit function theorem to solve for z, w as functions of t, x, y. Given that condition, calculate the partial derivatives of z and w with respect to the independent variables t, x, y.

Problem 9 Neji has a defense against attacks which is nearly impenetrable. In particular, his occular Jutsu has a strength which is roughly modelled as $J(\theta, \phi) = \exp\left(\dfrac{-1}{\theta^2 + (\phi - \pi/6)^2}\right)$. Given that θ, ϕ are the usual spherical coordinates, along which ray would a wise opponent attack as to avoid Neji's occular defenses? Assume Neji Hoogerwerf is at the origin.

Problem 10 Consider the Lagrangian $L = x^T Ax + \lambda(x^T x - 1)$ where $x = (x_1, x_2, \ldots, x_n)$ and A is a symmetric real $n \times n$ matrix. Show the Euler Lagrange equations imply that x is an eigenvector of A.