Problem 91 We define the coderivative $\delta : \Lambda^p M \to \Lambda^{p-1} M$ by:

$$\delta\alpha = (-1)^{np+n+1} \star d \star \alpha$$

Observe $\delta\alpha$ is a differential form of one-less degree than α. In contrast, $d\alpha$ is a differential form of one-more degree than α. We proved, or observed, that $d^2 = 0$ and $\star\star = \pm 1$ where the sign depends both on the signature of the metric on M as well as the degree of the form and the dimension of M. However, those details need not concern us for the following question: **Show:**

$$\delta^2\alpha = 0$$

$$\begin{aligned} \delta(\delta\alpha) &= \pm(\star d \star)(\star d \star \alpha) \\ &= \pm \star d(d(\star\alpha)) \\ &= 0. \qquad (\text{as } d^2 = 0). \end{aligned}$$

Problem 92 In the previous problem I was intentionally vague about which manifold and metric we were using. Here, I wil be precise: let us consider $M = \mathbb{R}^3$ with the Euclidean metric. Find nice formulas for: here $\omega_{\vec{F}} = Pdx + Qdy + Rdz$ as usual,

(a.) $d\delta\omega_{\vec{F}}$ where $\vec{F} = \langle P, Q, R\rangle$. $\longleftrightarrow$ $\delta\omega_{\vec{F}} = (-1)^{3+3+1} * d * \omega_{\vec{F}} = \underline{- * d * \omega_{\vec{F}}}$ ($P=1$)

(b.) $\delta d\omega_{\vec{F}}$ where $\vec{F} = \langle P, Q, R\rangle$.

(c.) is $d\delta + \delta d$ anything interesting?

$$\begin{aligned} \text{(a.)}\ d\delta\omega_{\vec{F}} &= d(- * d * \omega_{\vec{F}}) \\ &= -d(* d\,\Phi_{\vec{F}}) \\ &= -d(*(\nabla\cdot\vec{F})\, dx\wedge dy\wedge dz) \\ &= -d(\nabla\cdot\vec{F}) \\ &= \underline{-\omega_{\nabla(\nabla\cdot\vec{F})}}. = -\partial_x(\nabla\cdot\vec{F})dx - \partial_y(\nabla\cdot\vec{F})dy - \partial_z(\nabla\cdot\vec{F})dz \end{aligned}$$

$$\begin{aligned} \text{(b.)}\ \delta d\omega_{\vec{F}} &= \delta\,\Phi_{\nabla\times\vec{F}} \quad, p=2,\ np+n+1 = 3(2)+2+1 = 8. \\ &= * d * \Phi_{\nabla\times\vec{F}} \\ &= * d\,\omega_{\nabla\times\vec{F}} \\ &= * \Phi_{\nabla\times(\nabla\times\vec{F})} \\ &= \underline{\omega_{\nabla\times(\nabla\times\vec{F})}}. \end{aligned}$$

(c.) $d\delta + \delta d = \omega_{\nabla\times(\nabla\times\vec{F}) - \nabla(\nabla\cdot\vec{F})}$ $\longleftarrow$ simplifies to $\downarrow$

$-\nabla^2\vec{F} = -\langle \nabla^2 P, \nabla^2 Q, \nabla^2 R\rangle$

(some folks say minus, others +. The Laplacian.)

Problem 93 Suppose M is a manifold with boundary ∂M and suppose α, β are differential forms on M with $deg(\alpha) + deg(\beta) = dim(M) - 1$. Relate $\int_M \alpha \wedge d\beta$ and $\int_M \beta \wedge d\alpha$; that is, derive integration-by-parts for differential forms.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge d\beta$$

Therefore,

$$\int_M d(\alpha \wedge \beta) = \int_M d\alpha \wedge \beta + (-1)^{deg(\alpha)} \int_M \alpha \wedge d\beta$$

Use gen. Stokes' Th^{m} to convert $\int_M d(\alpha\wedge\beta) = \int_{\partial M} \alpha\wedge\beta$, hence,

$$\Rightarrow \boxed{\int_M \alpha \wedge d\beta = (-1)^{deg(\alpha)} \left[\int_{\partial M} \alpha \wedge \beta - \int_M d\alpha \wedge \beta \right]}$$

Problem 94 In $\mathbb{R}^4$ with metric $\eta = Diag(-1, 1, 1, 1)$ I describe in my notes how Hodge duality introduces certain signs. The basic idea is very much like the simpler context of $\mathbb{R}^3$. Because I use the metric which agrees with the euclidean metric on x, y, z components the work and flux-form correspondences naturally generalize: a general one-form on $\mathbb{R}^4_{txyz}$ space has the form:

$$\alpha = \alpha_0 dt + \alpha_1 dx + \alpha_2 dy + \alpha_3 dz = \alpha_0 dt + \omega_{\vec{\alpha}}.$$

Notice, I am encouraging the notation $\vec{\alpha}$ for the spatial vector piece of the one-form α. No such simple correspondence is possible for a generic two-form since it has six independent components:

$$\begin{aligned} \beta &= F_1 dt \wedge dx + F_2 dt \wedge dy + F_3 dt \wedge dz + G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy \\ &= dt \wedge \omega_{\vec{F}} + \Phi_{\vec{G}} \end{aligned}$$

Clearly the formula in terms of the work and flux-form correspondance will make it easier for us to follow calculus and algebra for β. Next, a three-form has the general form:

$$\begin{aligned} \gamma &= G_0 dx \wedge dy \wedge dz + G_1 dt \wedge dy \wedge dz + G_2 dt \wedge dz \wedge dx + G_3 dt \wedge dx \wedge dy \\ &= G_0 dx \wedge dy \wedge dz + dt \wedge \Phi_{\vec{\gamma}} \end{aligned}$$

where I am encouraging use of the notation $\vec{\gamma} = \langle G_1, G_2, G_3 \rangle$ to emphasize the correspondence between spatial 3-vectors and those components of γ. Continuting, there is just one 4-form:

$$\zeta = f dt \wedge dx \wedge dy \wedge dz.$$

Please notice that all the coefficients of the forms are in fact 0-forms on $\mathbb{R}^4$, that is, functions of t, x, y, z. This introduces time derivative terms in the formulas you are to find below. Use the notation given above to calculate:

(a.) df where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $d\alpha$

(c.) $d\beta$

(d.) $d\gamma$

(e.) $d\zeta$

(a.) $$df = \underbrace{\frac{\partial f}{\partial t}dt}_{(\partial_t f)dt} + \underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}_{\omega_{\nabla f}} = \underline{(\partial_t f)dt + \omega_{\nabla f}}.$$

(b.) $$\begin{aligned} d\alpha &= d\alpha_0 \wedge dt + d\alpha_1 \wedge dx + d\alpha_2 \wedge dy + d\alpha_3 \wedge dz \\ &= \omega_{\nabla\alpha_0} \wedge dt + \sum_{i=1}^{3} \left[(\partial_t \alpha_i)\, dt + \omega_{\nabla \alpha_i}\right] \wedge dx^i \\ &= \omega_{\nabla\alpha_0} \wedge dt - \sum_{i=1}^{3} (\partial_t \alpha_i)\, dx^i \wedge dt + \sum_{i=1}^{3} \omega_{\nabla\alpha_i} \wedge dx^i \\ &= \omega_{\nabla\alpha_0} \wedge dt - \omega_{\partial_t \vec{\alpha}} \wedge dt + \sum_{i=1}^{3} \omega_{\nabla\alpha_i} \wedge dx^i \\ &= \left(\omega_{\nabla\alpha_0} - \frac{\partial \vec{\alpha}}{\partial t}\right) \wedge dt + \sum_{i=1}^{3} \omega_{\nabla\alpha_i} \wedge \omega_{e_i} \\ &= \underline{\omega_{\nabla\alpha_0 - \frac{\partial\vec{\alpha}}{\partial t}} \wedge dt + \underbrace{\sum_{i=1}^{3} \Phi_{\nabla\alpha_i \times e_i}}}. \\ &= \underline{\omega_{\nabla\alpha_0 - \frac{\partial\vec{\alpha}}{\partial t}} \wedge dt + \Phi_{\nabla\times\vec{\alpha}}} \end{aligned}$$

Curious, I think I missed this in Lecture.

oh, duh.

(b.) again, but, to the point,

$$
\begin{aligned}
d\alpha &= d(\alpha_0\, dt + \omega_{\vec{\alpha}}) \\
&= d\alpha_0 \wedge dt + d\omega_{\vec{\alpha}} \\
&= (\partial_t \alpha_0\, dt + \omega_{\nabla \alpha_0}) \wedge dt + \Phi_{\nabla \times \vec{\alpha}} + dt \wedge \omega_{\frac{\partial \vec{\alpha}}{\partial t}} \\
&= \underline{\omega_{\nabla \alpha_0 - \frac{\partial \vec{\alpha}}{\partial t}} \wedge dt + \Phi_{\nabla \times \vec{\alpha}}}.
\end{aligned}
$$

(c.)

$$
\begin{aligned}
d\beta &= d(dt \wedge \omega_{\vec{F}} + \Phi_{\vec{G}}) \\
&= -d\omega_{\vec{F}} \wedge dt + d\Phi_{\vec{G}} \\
&= -\Phi_{\nabla \times \vec{F}} \wedge dt + (\nabla \cdot \vec{G})\, dx \wedge dy \wedge dz + dt \wedge \Phi_{\frac{\partial \vec{G}}{\partial t}} \\
&= \underline{(\nabla \cdot \vec{G})\, dx \wedge dy \wedge dz + dt \wedge \Phi_{\nabla \times \vec{F} + \frac{\partial \vec{G}}{\partial t}}}.
\end{aligned}
$$

(d.)

$$
\begin{aligned}
d\gamma &= d(G_0\, dx \wedge dy \wedge dz + dt \wedge \Phi_{\vec{\gamma}}) \\
&= dG_0 \wedge dx \wedge dy \wedge dz + d\Phi_{\vec{\gamma}} \wedge dt \\
&= (\partial_t G_0)\, dt \wedge dx \wedge dy \wedge dz + (\nabla \cdot \vec{\gamma})\, dx \wedge dy \wedge dz \wedge dt \\
&= \underline{\left(\frac{\partial G_0}{\partial t} - \nabla \cdot \vec{\gamma} \right) dt \wedge dx \wedge dy \wedge dz}.
\end{aligned}
$$

(e.)

$$
\begin{aligned}
d\zeta &= df \wedge dt \wedge dx \wedge dy \wedge dz \\
&= \underline{0.} \quad (\text{no non-zero 5-form in } \mathbb{R}^4)
\end{aligned}
$$

Problem 95 Again, using the notation ~~introduced in the previous~~ problem, find the explicit (and as nice as possible) formulas for: (use the results on page 221 of my notes)

(a.) $\star f$ where f is a real-valued function on $\mathbb{R}^4_{txyz}$.

(b.) $\star\alpha$

(c.) $\star\beta$

(d.) $\star\gamma$

(e.) $\star\zeta$

used to do these $e^0 \to dt$, $e^1 \to dx$, $e^2 \to dy$, etc.

(a.) $* f = \underline{f\, dt \wedge dx \wedge dy \wedge dz}$.

(b.)
$$\begin{aligned} *\alpha &= \alpha_0 * dt + \alpha_1 * dx + \alpha_2 * dy + \alpha_3 * dz \\ &= -\alpha_0\, dx\wedge dy\wedge dz - \alpha_1\, dy \wedge dz \wedge dt - \alpha_2\, dz \wedge dx \wedge dt - \alpha_3\, dx \wedge dy \wedge dt \\ &= \underline{-\alpha_0\, dx \wedge dy \wedge dz - \Phi_{\vec{\alpha}} \wedge dt}. \end{aligned}$$

(c.)
$$\begin{aligned} *\beta &= *\left(dt \wedge \omega_{\vec{F}} + \Phi_{\vec{G}}\right) \\ &= -*(\omega_{\vec{F}} \wedge dt) + *\Phi_G \\ &= \underline{-\Phi_{\vec{F}} + dt \wedge \omega_{\vec{G}}}. \end{aligned}$$

(d.)
$$\begin{aligned} *\gamma &= G_0 * (dx\wedge dy \wedge dz) + *(dt \wedge \Phi_{\vec{\gamma}}) \\ &= \underline{-G_0\, dt - \omega_{\vec{\gamma}}}. \end{aligned}$$

(e.)
$$\begin{aligned} *\zeta &= *(f\, dt\wedge dx \wedge dy \wedge dz) \\ &= \underline{-f}. \end{aligned}$$

Problem 96 Derive differential identities from $d^2 = 0$ on $\mathbb{R}^4_{txyz}$. In other words, work out the analog of Problem 80 for four-dimensional Minkowski space.

In order for d^2 to be interesting $d(d\alpha)$ cannot be more ~~than a~~ a 4-form. Thus, consider α a $p = 0, 1, 2$ form.

$\underline{p=0}$ $d(df) = d(\partial_t f \, dt + \omega_{\nabla f}) \rightarrow \alpha_0 = \partial_t f$, $\vec{\alpha} = \nabla f$ part b. of P. 94

$= \underbrace{(\omega_{\nabla \partial_t f - \frac{\partial}{\partial t}\nabla f})}_{} \wedge dt + \Phi_{\nabla \times \nabla f}$

zero provided $\nabla \partial_t = \partial_t \nabla$.

$= \Phi_{\nabla \times \nabla f} \Rightarrow \boxed{\nabla \times \nabla f = 0}$

$\underline{p=1}$ $d(d\alpha) = d\left(\omega_{\nabla \alpha_0 - \frac{\partial \vec{\alpha}}{\partial t}} \wedge dt + \Phi_{\nabla \times \vec{\alpha}}\right)$

$= d\left(dt \wedge \omega_{\underbrace{\frac{\partial \vec{\alpha}}{\partial t} - \nabla \alpha_0}_{\vec{F}}} + \Phi_{\underbrace{\nabla \times \vec{\alpha}}_{\vec{G}}}\right)$ of part c from 94

$= (\nabla \cdot (\nabla \times \vec{G})) dx \wedge dy \wedge dz$

$+ dt \wedge \Phi_{\nabla \times \left(\frac{\partial \vec{\alpha}}{\partial t} - \nabla \alpha_0\right) + \frac{\partial}{\partial t}(\nabla \times \vec{\alpha})}$

$= \nabla \cdot (\nabla \times \vec{G}) dx \wedge dy \wedge dz + dt \wedge \Phi_{-\nabla \times \nabla \alpha_0}$ as $\frac{\partial}{\partial t}\nabla = \nabla \frac{\partial}{\partial t}$ So the other terms cancel.

$\Rightarrow \boxed{\nabla \cdot (\nabla \times \vec{G}) = 0 \text{ and } \nabla \times \nabla \alpha_0 = 0}$

$\underline{p=2}$ $d(d\beta) = d\left(\underbrace{(\nabla \cdot \vec{G})}_{G_0} dx \wedge dy \wedge dz + dt \wedge \Phi_{\underbrace{\nabla \times G + \frac{\partial \vec{G}}{\partial t}}_{\vec{\gamma}}}\right)$ part d. of 94

$= \left[\frac{\partial}{\partial t}(\nabla \cdot \vec{G}) - \nabla \cdot \left(\nabla \times \vec{G} + \frac{\partial \vec{G}}{\partial t}\right)\right] dt \wedge dx \wedge dy \wedge dz$

$= -\nabla \cdot (\nabla \times \vec{G}) dt \wedge dx \wedge dy \wedge dz \Rightarrow \boxed{\nabla \cdot (\nabla \times \vec{G}) = 0}$

Problem 97 Suppose $L(y, \dot{y}, t) = y^3 + y\dot{y}^2$ is the Lagrangian for some model. Find the Euler-Lagrange equations for y.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \implies \boxed{\frac{d}{dt}\left(2y\dot{y}\right) = 3y^2 + \dot{y}^2}$$

Guess I should put one of these on the final ☺.

Problem 98 The three-dimensional free-particle Lagrangian for mass m is simply

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right].$$

We saw in lecture the Euler-Lagrange equations for x, y, z were simply:

$$m\ddot{x} = 0, \qquad m\ddot{y} = 0, \qquad m\ddot{z} = 0.$$

Find the L in spherical coordinates:

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi$$

and derive the Euler Lagrange equations for r, θ, ϕ.

$$\dot{x} = \cos\theta\sin\varphi\,\dot{r} - r\sin\theta\sin\varphi\,\dot{\theta} + r\cos\theta\cos\phi\,\dot{\varphi}$$

$$\dot{y} = \sin\theta\sin\varphi\,\dot{r} + r\cos\theta\sin\varphi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\varphi}$$

$$\dot{z} = \cos\varphi\,\dot{r} - r\sin\varphi\,\dot{\varphi}$$

Thus, as the cross-terms cancel, well, most of them,

$$\dot{x}^2 + \dot{y}^2 = (\cos^2\theta + \sin^2\theta)\sin^2\varphi\,\dot{r}^2 + r^2(\sin^2\theta + \cos^2\theta)\sin^2\varphi\,\dot{\theta}^2 + r^2\cos^2\phi\,\dot{\varphi}^2 +$$

$$= \sin^2\varphi\,\dot{r}^2 + r^2\sin^2\varphi\,\dot{\theta}^2 + r^2\cos^2\phi\,\dot{\varphi}^2 + \underline{2r\cos\varphi\sin\varphi\,\dot{r}\dot{\varphi}}$$

Also, notice,

$$\dot{z}^2 = \cos^2\varphi\,\dot{r}^2 - \underline{2r\cos\varphi\sin\varphi\,\dot{r}\dot{\varphi}} + r^2\sin^2\varphi\,\dot{\varphi}^2$$

Then, after using $\sin^2\varphi + \cos^2\varphi = 1$ a couple times,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\sin^2\phi\,\dot{\theta}^2 + r^2\dot{\varphi}^2$$

$$\implies \underline{L = \frac{m}{2}\left[\dot{r}^2 + r^2\sin^2\phi\,\dot{\theta}^2 + r^2\dot{\varphi}^2\right]}.$$

PROBLEM 98 continued

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = mr\sin^2\phi\,\dot{\theta}^2 + mr\dot{\phi}^2$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\sin^2\phi\,\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \qquad \frac{\partial L}{\partial \varphi} = mr^2\sin\phi\cos\phi\,\dot{\theta}^2$$

Thus, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$ and the same for θ, ϕ reveal,

$$\boxed{\begin{aligned} \frac{d}{dt}\left(m\dot{r}\right) &= mr\sin^2\phi\,\dot{\theta}^2 + mr\dot{\varphi}^2 \\ \frac{d}{dt}\left(mr^2\sin^2\phi\,\dot{\theta}\right) &= 0 \\ \frac{d}{dt}\left(mr^2\dot{\varphi}\right) &= mr^2\sin\phi\cos\phi\,\dot{\theta}^2 \end{aligned}}$$

Euler - Lagrange Eq$^{\underline{n}}$'s for r, θ, ϕ.

Problem 99 Find the geodesic equations for the paraboloid $z = x^2 + y^2$. If it is within your power, solve them. (I'm not sure how horrible this suggestion is, but where effort is made, credit will likely flow)

$$L = V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2\dot{\theta}^2 + \dot{r}^2 + \dot{z}^2 \quad : \quad \text{where } V = \frac{ds}{dt}$$

$$= r^2\dot{\theta}^2 + \dot{r}^2 + (2r\dot{r})^2 \quad : \quad z = r^2 \hookrightarrow \dot{z} = 2r\dot{r}$$

$$= \underline{r^2\dot{\theta}^2 + (1+4r^2)\dot{r}^2}.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow \underline{\frac{d}{dt}\left(2r^2\dot{\theta}\right) = 0}. \quad \therefore \quad \underbrace{l = 2r^2\dot{\theta}}_{\text{constant.}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow \underline{\frac{d}{dt}\left[2(1+4r^2)\dot{r}\right] = 2r\dot{\theta}^2 + 8r\dot{r}^2}.$$

Usually, imposing $\dot{\theta} = l/2r^2$ to eliminate θ helps,

$$\frac{d}{dt}\left[2\dot{r} + 8r^2\dot{r}\right] = 2r\left(\frac{l}{2r^2}\right)^2 + 8r\dot{r}^2$$

$$\underline{2\ddot{r} + 16r\dot{r}^2 + 8r^2\ddot{r} = \frac{l^2}{2r} + 8r\dot{r}^2}$$

YEP. WELL, IT DOESN'T LOOK TOO HARD...

Problem 100 Follow link *http : //mathoverflow.net/q/26939/24854*. Read page. We're on the honor system here, I just want you to scan through it to get a feel for how professional mathematicians think about forms.

Problem 101 Follow link *http : //mathoverflow.net/q/10574/24854*. Read page. We're on the honor system here, I just want you to scan through it to get a feel for how professional mathematicians think about forms. (if your time is limited then your time is better spend on this thread, this gives a lot of ideas for future directions and reading on differential forms)

$$df \wedge vol_M = df \wedge \frac{1}{\|f\|} * df = \frac{1}{\|f\|} df \wedge * df$$ curious.

Problem 102 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with non-vanishing ∇f. Let M be the hypersurface which is formed by the solution set of $f(x) = c$; that is $M = f^{-1}\{c\}$. Furthermore, let $n = \omega_{\vec{n}}$ be **unit-normal form** in the sense that $\text{Ker}(n_p) = T_pM$ for each $p \in M$ and $\vec{n} \bullet \vec{n} = 1$. We define the volume form vol_M on the hypersurface by $\text{vol}_M = \star n$ where $\star$ is the eulicidean Hodge dual. Show:

$$df \wedge \text{vol}_M = |\nabla f| dx^1 \wedge \cdots \wedge dx^n$$

Notice $\underbrace{f(\vec{x}) = c}_{\text{sol}^{\text{n}}\text{'s fall in the "surface" } M = f^{-1}\{c\}.}$ has normal vector field ∇f

Hence $\omega_{\nabla f} \sim n$, but need to normalize to $\frac{\nabla f}{\|\nabla f\|}$

thus,

$$n = \frac{1}{\|\nabla f\|} \omega_{\nabla f} = \frac{1}{\|\nabla f\|} df$$

Thus,

$$\text{vol}_M = * n = \frac{1}{\|\nabla f\|}\left(* \left(\frac{\partial f}{\partial x_1} dx^1 + \cdots + \frac{\partial f}{\partial x_n} dx^n \right)\right)$$

$$= \frac{1}{\|\nabla f\|}\left(\frac{\partial f}{\partial x^1} dx^2 \wedge \cdots \wedge dx^n + \frac{\partial f}{\partial x^2} dx^3 \wedge dx^4 \wedge \cdots \wedge dx^n \wedge dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \right)$$

$$= \underbrace{\frac{1}{\|\nabla f\|} \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^{j+1} \wedge \cdots \wedge dx^n \wedge dx^1 \wedge \cdots \wedge dx^{j-1}}_{\star}$$

Therefore,

$$df \wedge \text{vol}_M = \frac{1}{\|\nabla f\|}\left(\sum_{k=1}^{n} \frac{\partial f}{\partial x^k} dx^k\right) \wedge \left(\text{---} \star \text{---} \right)$$

other terms vanish due to repeated wedge.

$$= \frac{1}{\|\nabla f\|} \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x^j}\right)^2 dx^j \wedge dx^{j+1} \wedge \cdots \wedge dx^n \wedge dx^1 \wedge \cdots \wedge dx^{j-1}$$

$$= \frac{1}{\|\nabla f\|} \underbrace{\sum_{j=1}^{n} \left(\frac{\partial f}{\partial x^j}\right)^2}\, dx^1 \wedge \cdots \wedge dx^n$$

for example,
$dx^1 \wedge dx^2 \wedge dx^3 = dx^2 \wedge dx^3 \wedge dx^1$
$= dx^3 \wedge dx^1 \wedge dx^2$
etc...

$$= \frac{1}{\|\nabla f\|} \|\nabla f\|^2 dx^1 \wedge \cdots \wedge dx^n$$

$$= \underline{\|\nabla f\| dx^1 \wedge \cdots \wedge dx^n}.$$

$f = r^2, \; df = 2r\,dr$

$\|\nabla f\| = 2r$

Problem 103 Calculate, use the volume form defined in previous problem,

$$\int_{S_R} \mathrm{vol}_M = 2\pi R$$

where $S_R = F^{-1}(R)$ for $F(x, y) = x^2 + y^2$. Suggestion,

$$n = \frac{xdy - ydx}{R}$$

has $\vec{n} = \frac{\langle -y, x\rangle}{R}$ with unit-length on S_R.

$\frac{1}{\|\nabla f\|} df \wedge \mathrm{vol}_m = dx \wedge dy$

$dr \wedge \mathrm{vol}_m = r\,dr \wedge d\theta$

But, $r = R$,

$dr \wedge \mathrm{vol}_m = dr \wedge \underbrace{(R\,d\theta)}_{\mathrm{vol}_m}$

$$\int_{S_R} \mathrm{Vol}_m = \int_{S_R} R\,d\theta = R\int_0^{2\pi} d\theta = \underline{2\pi R}.$$

$$\int_{S_R} \frac{xdy - ydx}{R} = \int_0^{2\pi} \frac{(R\cos\theta)\,d(R\sin\theta) - (R\sin\theta)\,d(R\cos\theta)}{R} = \int_0^{2\pi} R\,d\theta = \underline{2\pi R}.$$

Remark: If $F = C$ is a coordinate plane for a coord. w' system then $dw' \wedge \mathrm{vol}_m = dx' \wedge dx^2 \wedge \cdots \wedge dx^n$ then if we pull-back $dx' \wedge \cdots \wedge dx^n$ to $w', \ldots, w^n$ - coord. we can read off vol_m. This is what I did in 103 and 104.

Problem 104 Calculate, use the volume form defined in previous problem,

$$\int_{S_R} \mathrm{vol}_M = 4\pi R^2$$

where $S_R = F^{-1}(R)$ for $F(x, y, z) = x^2 + y^2 + z^2$. $\longrightarrow$ $F = r^2 \Rightarrow \frac{dF}{\|\nabla F\|} = dr$

Problem 85

$dx \wedge dy \wedge dz = \frac{1}{\|\nabla F\|} dF \wedge \mathrm{Vol}_m$

$R^2 \sin\phi\, dr \wedge d\phi \wedge d\theta = dr \wedge \mathrm{vol}_m$

$dr \wedge (\underbrace{R^2 \sin\phi\, d\phi \wedge d\theta}) = dr \wedge \underbrace{\mathrm{vol}_m}$ $\therefore$ $\underline{\mathrm{vol}_m = R^2 \sin\phi\, d\phi \wedge d\theta}$

$$\int_{S_R} \mathrm{Vol}_m = \int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi\, d\phi\, d\theta = R^2\left(-\cos\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)$$

$$= \boxed{4\pi R^2}$$

Problem 105 Consider the 1-form $\alpha = xdz + ydw - (x^2+y^2+z^2+w^2)dt$ on $\mathbb{R}^5$. Calculate $\int_S d\alpha \wedge d\alpha$, where $S \subset \mathbb{R}^5$ is given by $x^2+y^2+z^2+w^2 = 1$ and $0 \le t \le 1$. Use the generalized Stokes' Theorem and the identity $d\alpha \wedge d\alpha = d(\alpha \wedge d\alpha)$ to make life easier.

$\mathbb{R}^5$ has coordinates (x, y, z, w, t) (my choice here)

$S \subset \mathbb{R}^5$: $\underbrace{x^2+y^2+z^2+w^2 = 1}$, $0 \le t \le 1$

$$S_3 = \{(x,y,z,w) \mid x^2+y^2+z^2+w^2 = 1\}$$

Thus $S = S_3 \times [0,1]$

and $\partial S = \underset{\text{BASE}}{(S_3 \times \{0\})} \cup \underset{\text{TOP}}{(S_3 \cup \{1\})}$

S

Cylinder with a hyperspherical cross-section.

Notice BASE, TOP have $t = 0, 1$ and thus $\underline{dt = 0}$ on ∂S.

Consider,

$$\int_S d\alpha \wedge d\alpha = \int_S d(\alpha \wedge d\alpha)$$

$$= \int_{\partial S} \alpha \wedge d\alpha$$

only terms w/o dt survive the integration.

$$= \int_{\partial S} (xdz + ydw) \wedge (d(xdz + ydw))$$

$$= \int_{\partial S} (xdz + ydw) \wedge (dx \wedge dz + dy \wedge dw)$$

$$= \int_{\partial S} x\, dz \wedge dy \wedge dw + y\, dw \wedge dx \wedge dz$$

$$= \boxed{0.}$$

(x, y appear symmetrically +/− over the 3-sphere hence the +/− hemi-hyperspheres integrals cancel.)

Problem 106 The rank of a one-form ω is defined, at a point, to be the largest positive integer r for which $\omega_r \neq 0$ yet $\omega_{r+1} = 0$ where we define the **auxillary forms** $\omega_1, \omega_2, \ldots$ as follows:

$$\omega_1 = \omega,\ \omega_2 = d\omega,\ \omega_3 = \omega \wedge d\omega,\ \omega_4 = d\omega \wedge d\omega,\ \omega_5 = \omega \wedge d\omega \wedge d\omega, \ldots$$

The auxillary forms $\{\omega_1, \ldots, \omega_r\}$ form a LI set of forms in the exterior algebra of the manifold at a the given point. If the rank of the one-form is constant at all points then the form is called **regular**.Find the rank of:

$$\omega = (x+y)dt + ydz$$

on $\mathbb{R}^4_{txyz}$ space.

$$w_1 = (x+y)dt + y\,dz$$

$$w_2 = dw_1 = dx \wedge dt + dy \wedge dt + dy \wedge dz$$

$$w_3 = w \wedge dw = \left[(x+y)dt + y\,dz\right] \wedge \left[dx \wedge dt + dy \wedge dt + dy \wedge dz\right]$$

$$= (x+y)\,dt \wedge dy \wedge dz + y\,dz \wedge dx \wedge dt + y\,dz \wedge dy \wedge dt$$

$$w_4 = dw \wedge dw = \left[dx \wedge dt + \cancel{dy \wedge dt}^{\,0} + dy \wedge dz\right] \wedge \left[dx \wedge dt + dy \wedge dt + dy \wedge dz\right]$$

$$= dx \wedge dt \wedge dy \wedge dz + dy \wedge dz \wedge dx \wedge dt$$

$$= -dt \wedge dx \wedge dy \wedge dz + (-dt \wedge dx \wedge dy \wedge dz)$$

$$= -2\,dt \wedge dx \wedge dy \wedge dz$$

$$w_5 = w \wedge dw \wedge dw = 0$$ thus $\boxed{\text{rank of } w \text{ is } 4}$

Problem 107 Consider the one-form $\omega = xdx + ydy + zdz$ on $\mathbb{R}^3$. Find the foliation of three dimensional space into two-dimensional submanifolds whose tangent spaces are spanned by vector fields which are found in $\ker(\omega)$. Check the condition needed to show ω is dual to a two-plane field distribution on $\mathbb{R}^3$; that is verify $\omega \wedge d\omega = 0$ (see Bachman for where I'm coming from here). Incidentally, there is one point left out, perhaps it would be more honest to say find a foliation of $\mathbb{R}^3 - \{(0,0,0)\}$.

$$\omega = x\,dx + y\,dy + z\,dz = d\Big(\underbrace{\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2}_{F}\Big)$$

thus $d\omega = d(dF) = 0$

hence $\omega \wedge d\omega = \omega \wedge 0 = 0$. Furthermore,

$\ker(\omega)$ is given by tangents to $F(x,y,z) = C$.

The foliation here is <u>spheres</u> $\underline{x^2+y^2+z^2 = R^2}$.

Notice, $\nabla F|_p = \langle x, y, z\rangle|_p$ is normal to sphere at given point p but $T_p S_R = (\nabla F|_p)^{\perp}$ and

$V_p = T_p S_R \oplus N_p S_R$ where $N_p S_R = \mathrm{span}(\nabla F|_p)$

$(p, \nabla F|_p)$ more precisely as $V_p = \{(p, v) \mid v \in \mathbb{R}^3\}$.

thus $v \in T_p S_R \Rightarrow v \cdot \nabla F|_p = 0$

$$\begin{aligned}
\Rightarrow \omega_p(v) &= (dF)_p(v) \\
&= \omega_{\nabla f(p)}(v) \\
&= (\nabla f(p)) \cdot v \\
&= 0 \quad \therefore \quad v \in \ker(\omega_p) \quad \forall v \in T_p S_R
\end{aligned}$$

this holds for each $p \in S_R$ hence $\ker(\omega)$ is made of tangents to S_R.

Problem 108 Consider $\omega = dy + dz + xydx + xzdx$. Show that $\omega \wedge d\omega = 0$ on all of $\mathbb{R}^3$. What foliation of $\mathbb{R}^3$ does ω describe. Recall, we discussed that $\omega = dz$ corresponds to foliating $\mathbb{R}^3$ into $z = c$ (a family of horizontal planes, each leaf in the foliation labeled by c). Try to find the corresponding family of surfaces for the ω given here.

$$\omega = dy + dz + xy\,dx + xz\,dx$$

$$d\omega = x\,dy \wedge dx + x\,dz \wedge dx$$

$$\begin{aligned}
\omega \wedge d\omega &= \big(dy + dz + x(y+z)dx\big) \wedge \big(-x\,dx \wedge (dy + dz)\big) \\
&= (\underline{\underline{dy+dz}}) \wedge (-x\,dx) \wedge (\underline{\underline{dy+dz}}) + x(y+z)\underline{\underline{dx}} \wedge (-x\underline{\underline{dx}}) \wedge (dy+dz) \\
&= 0.
\end{aligned}$$

Consider
$$\begin{aligned}
\omega &= dy + dz + x(y+z)\,dx \\
&= d(y+z) + x(y+z)\,dx = \ ?
\end{aligned}$$

Find S for which $T_pS = \mathrm{Ker}(\omega_p)$. If $v = \langle a, b, c\rangle \in T_pS$

or, better, $V_p = a\frac{\partial}{\partial x}\Big|_p + b\frac{\partial}{\partial y}\Big|_p + c\frac{\partial}{\partial z}\Big|_p$ as $T_pS \subset \underbrace{T_p\mathbb{R}^3}$

spanned by $\partial_x|_p, \partial_y|_p, \partial_z|_p$

$$\omega_p(V_p) = 0$$

$$\Rightarrow \quad b + c + a\big(x(y+z)\big) = 0$$

$$\Rightarrow \quad c = -ax(y+z) - b$$

$$\therefore \quad V = a\frac{\partial}{\partial x}\Big|_p + b\frac{\partial}{\partial y}\Big|_p - \big(a(x(y+z)) + b\big)\frac{\partial}{\partial z}\Big|_p$$

$$T_pS = \mathrm{span}\left\{ \frac{\partial}{\partial x}\Big|_p - x(y+z)\frac{\partial}{\partial z}\Big|_p,\ \frac{\partial}{\partial y}\Big|_p - \frac{\partial}{\partial z}\Big|_p \right\}$$

Find $F : \mathbb{R}^3 \to \mathbb{R}$ and $p \in F^{-1}(c)$ s.t. $\nabla F(p) \in T_pS^{\perp}$

$$\left.\begin{aligned}
\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle \cdot \langle 1, 0, -x(y+z)\rangle &= 0 \\
\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle \cdot \langle 0, 1, -1\rangle &= 0
\end{aligned}\right\} *$$

Can we solve $(*)$?

<u>Remark</u>: these sort of systems of PDEs is what Frobenius originally attacked... the manifolds, distributions, forms are a later development, a modernization of the original work...

PROBLEM 108 continued

Can we solve (*) directly? ← perhaps YES, but I didn't. Maybe the "method of characteristics" a very old method circa Euler etc.. will do it, see, well, I'm not sure... Maybe Thoe's Dover Text.

$$F_x - x(y+z)F_z = 0$$

$$F_y - F_z = 0$$

Let's guess $F = \ln(y+z)^2 + x^2$ and see what happens. Assuming $y+z \neq 0$,

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2x \\ \frac{\partial F}{\partial z} &= \frac{2}{y+z} \end{aligned}\right\} F_x - x(y+z)F_z = 2x - x\left(\frac{2(y+z)}{y+z}\right) \cancel{\neq} 0$$

$$\frac{\partial F}{\partial y} = \frac{2}{y+z} = \frac{\partial F}{\partial z}$$

YEP. Of course, one might wonder WHY? WHY? WHY? IS $F(x,y,z) = \ln(y+z)^2 + x^2$ the sol$^{=}$ (or are there more?). The way I found this was simply to formally solve $\omega = 0$

$$d(y+z) + x(y+z)dx = 0$$

$$\frac{d(y+z)}{y+z} = -x\,dx$$

$$\ln|y+z| = -\tfrac{1}{2}x^2 + C$$

$$\Rightarrow \underbrace{\ln(y+z)^2 + x^2}_{F} = C$$

I expect the foliation is given by $F(x,y,z) = 2\ln|y+z| + x^2 = c$ with the exceptional ~~leaves~~ leaf $y+z=0$ above and below we find $y+z>0$ or $y+z<0$.

olver's

Problem 109 Prove Cartan's Lemma. This is cleanly stated as Exercise 1.33 on page 26 of in the *Equivalence, Invariance and Symmetry* the relevant portion of which is posted as a pdf I posted in Course Content.

Let $\omega^1, \omega^2, \dots, \omega^k$ be a set of LI one-forms thus $\omega^1 \wedge \omega^2 \wedge \dots \wedge \omega^k \neq 0$. Cartan's Lemma states, one-forms $\theta^1, \dots, \theta^k$ satisfy $\sum_{i=1}^k \theta^i \wedge \omega^i = 0 \iff \theta^i = \sum_j A^i_j \omega^j$ for some symmetric $(A^i_j = A^j_i)$ matrix of functions

We assume $\omega^1, \dots, \omega^k$ is set of (pointwise) LI one-forms.

$\Rightarrow$ Assume $\theta^1, \dots, \theta^k$ are one-forms for which $\sum_{i=1}^k \theta^i \wedge \omega^i = 0$

Since $\omega^1, \omega^2, \dots, \omega^k$ LI we may extend these to a basis for one-forms at p, say $\omega^1, \dots, \omega^k, \omega^{k+1}, \dots, \omega^n$. Thus,

$$\theta^i = \sum_{j=1}^k A^i_j \omega^j + \sum_{j=k+1}^n B^i_j \omega^j$$

for some constants $A^i_j, B^i_j \in \mathbb{R}$. We're given $\sum_{i=1}^k \theta^i \wedge \omega^i = 0$,

$$0 = \sum_{i=1}^k \theta^i \wedge \omega^i = \sum_{i,j=1}^k A^i_j \underbrace{\omega^j \wedge \omega^i}_{\neq 0 \text{ for } j \neq i} + \sum_{i=1}^k \sum_{j=k+1}^n B^i_j \underbrace{\omega^j \wedge \omega^i}_{\neq 0 \text{ as } j \neq i}$$

By LI of $\omega^1, \dots, \omega^n$ we have $\omega^i \wedge \omega^j \neq 0$ for $i \neq j$ and it follows the coefficients must vanish for all such terms. In particular, $B^i_j = 0$. However, we can rewrite the condition to see more about A^i_j,

$$0 = \underbrace{\sum_{i<j} A^i_j \omega^j \wedge \omega^i + \sum_{i>j} A^i_j \omega^j \wedge \omega^i}_{\text{connected}} + \sum_{i=j} \underbrace{A^i_j \omega^i \wedge \omega^i}_{\text{no restriction}}$$

continued, relabel sums to see,

$$0 = \sum_{i<j} \left(A^i_j - A^j_i \right) \omega^j \wedge \omega^i$$

Thus, as $j \neq i$ and $\omega^j \wedge \omega^i \neq 0$ we need the coefficient expressions $A^i_j - A^j_i = 0$ and thus $A^i_j = A^j_i$. It follows that

$$\theta^i = \sum_{j=1}^{k} A^i_j \omega^j$$

and we may do this calculation at each point p hence we create a matrix of fncts as desired.

$(\Leftarrow)$ Assume $\theta^i = \sum_{j=1}^{k} A^i_j \omega^j$ for $A^i_j = A^j_i$.

Calculate,

$$\sum_{i=1}^{k} \theta^i \wedge \omega^i = \underbrace{\sum_{i,j=1}^{k} A^i_j}_{\substack{\text{sym.} \\ \text{in} \\ (ij)}} \underbrace{\omega^j \wedge \omega^i}_{\substack{\text{antisym.} \\ \text{in } (ij)}} = 0. //$$

fundamental Lemma of tensor arithmetic. Sym. contracted over antisym. is ZERO.

Remark: this Lemma appears many places as we dig deeper into differential form driven calculational techniques.