Name: (please print name here →) .

MATH 334:	MISSION 1: FIRST ORDER DEQNS [50PTS]

You write the solution neatly in the box provided and show work on your own, standard sized, non-fuzzy edged, paper. The work must be labeled with problem number and part as appropriate and if a problem is skipped it must be mentioned in the attached work. Neatness is part of the score. There are more than 50pts that can be earned, but 50pts are in the Syllabus. Enjoy.

Problem 1 (Separation of Variables) Solve the differential equations below. If possible, find the explicit solution, otherwise find an implicit general solution.

(a.) $\dfrac{dy}{dx} = (x+1)^2$

(b.) $e^x \dfrac{dy}{dx} = 2x$

(c.) $\dfrac{dy}{dx} = \dfrac{y+1}{x}$

(d.) $x^2y^2dy = (y+1)dx$

(e.) $\sec x dy = x \cot y dx$

(f.) $\sec y \dfrac{dy}{dx} + \sin(x-y) = \sin(x+y)$

(g.) $(e^x + e^{-x})\dfrac{dw}{dx} = w^2$

Problem 2 (Initial Value Problems) Use separation of variables to solve the IVPs below:

(a.) $\dfrac{dx}{dy} = 4(x^2+1)$ with $x(\pi/4) = 1$.

(b.) $y' + 2y = 1$ with $y(0) = 5/2$.

Problem 3 (substitution of form $u = ax + by + c$) Solve the following by making an appropriate substitution and using separation of variables,

(a.) $\dfrac{dy}{dx} = \tan^2(x+y)$

(b.) $\dfrac{dy}{dx} = 1 + e^{y-x+5}$

Problem 4 (homogeneous equations, try $y = ux$ or $x = vy$ on $M(x,y)dx + N(x,y)dy = 0$ where M and N are homogeneous functions of same degree)

(a.) $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

(b.) $(x^2 + xy - y^2)dx + xydy = 0$

Problem 5 (exact equations) If the DEqn below is exact then solve it, otherwise explain why the given DEqn is not exact.

(a.) $(2xz^2 - 3)dx + (2zx^2 + 4)dz = 0$

(b.) $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$

(c.) $(\theta^3 + \beta^3)d\theta + 3\theta\beta^2 d\beta = 0$

(d.) $(e^y + 2xy\cosh x)y' + xy^2\sinh x + y^2\cosh x = 0$

(e.) $\left(\dfrac{1}{x} + \dfrac{1}{x^2} - \dfrac{t}{x^2+t^2}\right)dx + \left(te^t + \dfrac{x}{x^2+t^2}\right)dt = 0$

Problem 6 (exact equations with integrating factor) A general form of an integrating factor is suggested. Find the specific form I which serves as an integrating factor and solve the DEqn $Mdx + Ndy = 0$ by solving the exact equation $IMdx + INdy = 0$)

(a.) $y(x + y + 1)dx + (x + 2y)dy = 0$ given $I = e^{Ax}$

(b.) $y(4xy^5 + 3)dx - x(2xy^5 + 7)dy = 0$ given $I = x^A y^B$

Problem 7 (linear first order DEqn) Solve the linear first order ODEqn given below and state the interval on which the solution is defined. If given an initial value, then fit the given data to the explicit solution.

(a.) $y' + 3x^2 y = x^2$

(b.) $x\dfrac{dy}{dx} + 2y = 3$

(c.) $x\dfrac{dy}{dx} + 4y = x^3 - x$

(d.) $\cos^2 x \sin x dy + (y \cos^3 x - 1)dx = 0$

(e.) $\dfrac{dr}{d\theta} + r \sec\theta = \cos\theta$

Problem 8 (Bernoulli's Equation). If the DEqn has form $\frac{dy}{dx} + P(x)y = f(x)y^n$ for some real n then it is called a Bernoulli Equation. These can be solved by a $w = y^{1-n}$ substitution, we assume $n \neq 0, 1$. Solve the following:

(a.) $\dfrac{dy}{dx} - y = e^x y^2$

(b.) $3(1 + x^2)\dfrac{dy}{dx} = 2xy(y^3 - 1)$

Problem 9 (Ricatti's Equation). If $\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$ then the given DEqn is a Ricatti Equation. If y_1 is a known solution then the substitution $v = y_1 + u$ turns the problem into a Bernoulli Equation with $n = 2$. Given the Ricatti Equation below with known solution y_1, solve it. Or, if no y_1 is given then figure one out then solve it.

(a.) $\dfrac{dy}{dx} = 2x^2 + y/x - 2y^2,\ y_1 = x$

(b.) $\dfrac{dy}{dx} = 9 + 6y + y^2$

Problem 10 (Clauiraut Equation) Let f be a smooth function. The differential equation $y = xy' + f(y')$ is known as a Clairaut Equation. Show that $y = cx + f(c)$ serves as a solution to Clairaut Equation for any $c \in \mathbb{R}$. Furthermore, show

$$x = -f'(t), \qquad \& \qquad y = f(t) - tf'(t)$$

give a parametric solution to Clairaut Equation. If $f''(t) \neq 0$ then the parametric solution describes a solution not found in the linear family and as such it is known as the **singular solution**. Solve the Clairaut Equations below by finding both their linear solutions and the singular solution.

(a.) $y = (x+4)y' + (y')^2$

Problem 11 Solve $\dfrac{dy}{dx} = e^{x-y}\cosh x$

Problem 12 Solve $\dfrac{dy}{dx} = \dfrac{y^2+4y+5}{x^2-3x-4}$

Problem 13 Solve $(y + \sin^{-1}(x))dx + \left(x + \dfrac{1}{1+y^2}\right)dy = 0$

Problem 14 Find the explicit solution of $\dfrac{dy}{dx} = \dfrac{e^x}{y}$ for which $y(0) = -2$.

Problem 15 Find the implicit solution of:

$$\left(1+2xy^2-\frac{1}{x^2+4}\right)dx+\left(2y+2x^2y-\frac{1}{1-y^2}\right)dy=0.$$

Problem 16 A differential equation $Mdx + Ndy = 0$ is exact if there exists F for which $dF = Mdx + Ndy$. Since $d(dF) = 0$ is an identity of the exterior calculus we can check on the exactness of a given differential equation in Pfaffian form by taking its exterior derivative. Determine if the differential equations below are exact by taking the exterior derivative of the differential equation:

(a.) $y\sin(xy)dx + x\sin(xy)dy = 0$

(b.) $-x^2dy + y^2dx = 0$

Problem 17 (Orthogonal Trajectories) Find the orthogonal trajectories to the curve or family of curves described below:

(a.) $y = (x - c_1)^2$

(b.) $y^2 - x^2 = c_1x^3$

Problem 18 (Orthogonal Trajectories to Polar Curves) Find the orthogonal trajectory for the curves described below (please use polar coordinates to formulate the answer)

(a.) $r = c_1(1 + \cos\theta)$

(b.) $r = c_1e^{\theta}$

Problem 19 (Isogonal Families) A family of curves which intersects a given family of curves at an angle $\alpha \neq \pi/2$ are said to be **isogonal trajectories** of each other. If $\frac{dy}{dx} = f(x,y)$ describes a given family of curves then show its isogonal family are solutions of

$$\frac{dy}{dx} = \frac{f(x,y) \pm \tan\alpha}{1 \mp f(x,y)\tan\alpha}.$$

Then, find the isogonal family to $y = c_1 x$ at angle $\alpha = 30^o$.

Problem 20 An integral curve to a vector field $\vec{F} = \langle P, Q \rangle$ can be described parametrically as a path $t \mapsto \vec{\gamma}(t) = (x(t), y(t))$ for which $\vec{F}(\vec{\gamma}(t)) = \frac{d\vec{\gamma}}{dt}$. That is, $\langle P, Q \rangle = \langle dx/dt, dy/dt \rangle$. Parametrically we need to solve $\frac{dx}{dt} = P$ and $\frac{dy}{dt} = Q$. However, if we are only interested in describing the integral curve in Cartesian coordinates then we can eliminate t via the calculus

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q}{P}$$

thus finding an integral curve for a given vector field which depends only on x, y is as simple as solving the above first order ODEqn.

(a.) Consider the vector field $\vec{F}(x,y) = \left\langle \frac{y}{(x-1)^2+y^2}, \frac{1-x}{(x-1)^2+y^2} \right\rangle$. Find the the level curve which serves as an integral curve for $\vec{F}$ through $P_o = (x_o, y_o) \neq (1, 0)$.

(b.) Find the integral curves of the vector field $\vec{F} = \langle 1, e^{x^3} - 2y/x \rangle$. Please leave your answer explicitly in terms of y as a function of x.

Problem 21 Let b be a positive constant. If a friction force of $F_f = -bv^4$ is applied to a mass m with initial position x_o and initial velocity v_o then find the velocity as a function of

(a.) time t,

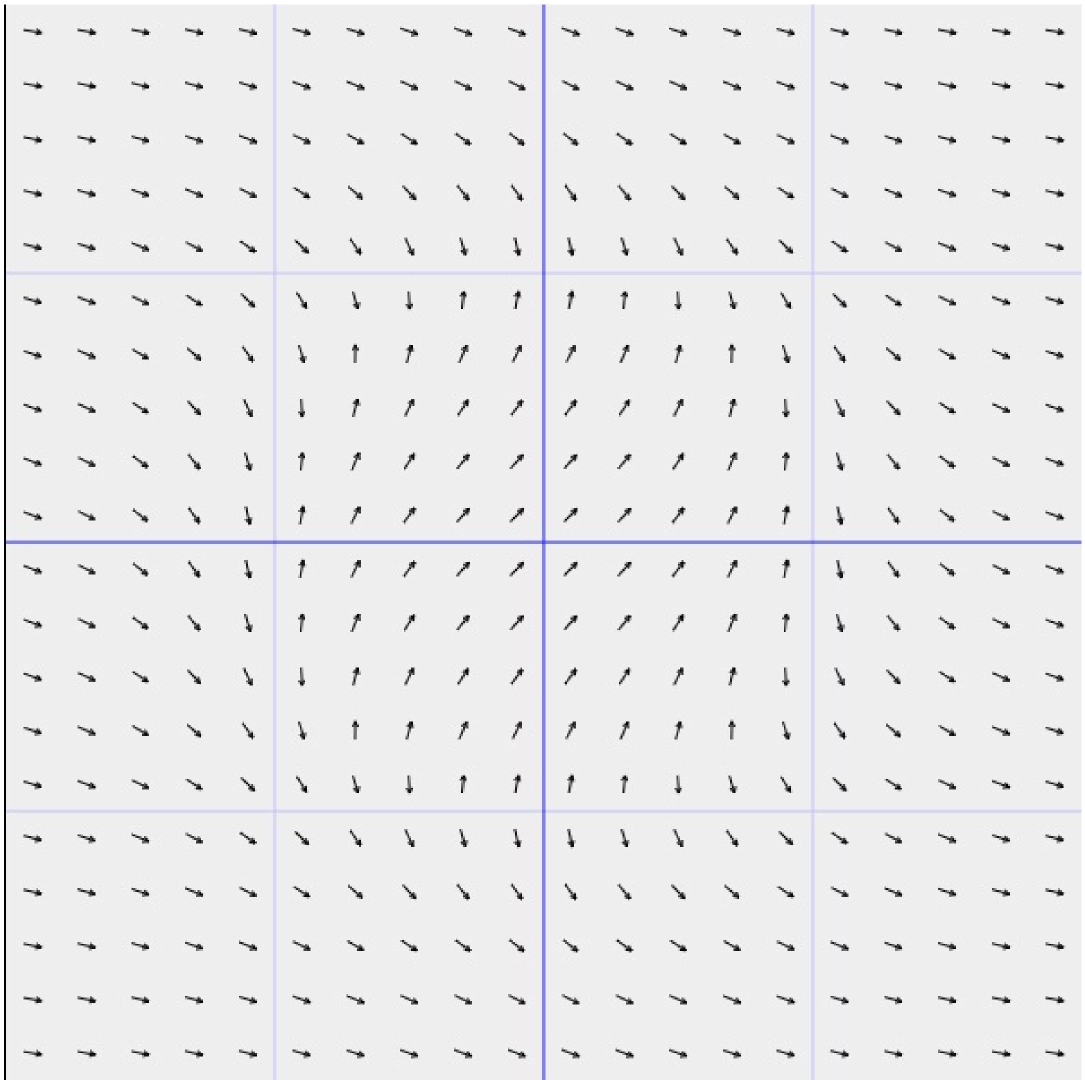
(b.) position x.

Problem 22 When a resistor R and inductor L are in series with a voltage source $\mathcal{E}$ then circuit analysis yields the differential equation:

$$L\frac{di}{dt} + Ri = \mathcal{E}$$

where i is the current flowing in the circuit. Given $\mathcal{E}(t) = V_o \sin \omega t$ and $i(0) = i_o$ find the current as a function of time t.

Problem 23 Plot solution curves for the direction field of $\frac{dy}{dx} = \frac{1}{1-x^2-y^2}$ given below and explain what is happening at the unit-circle for the solutions. Try it out in the pplane to check your hand-drawn answers here (maybe starting with a pencil lightly then tracing over with pen once you're sure would be wise here)



Remark: I used https://aeb019.hosted.uark.edu/pplane.html to generate the direction field above.

Problem 24 Angular momentum of a body moving in some plane is given by $L = mr^2\frac{d\theta}{dt}$ where r, θ serve as polar coordinates in the plane of motion. Assume the coordinates of the body are (r_1, θ_1) at $t = t_1$ and (r_2, θ_2) at $t = t_2$ where $t_1 < t_2$. **If L is constant then show that the area swept out by r is $A = L(t_2 - t_1)/2m$.** When the sun is taken to be at the origin and m represents a planet's mass then this proves Kepler's second law of planetary motion: the radius vector joining the sun sweeps out equal areas for equal intervals of time. **Bonus: prove L is constant in the context of the sun-planet system, you may assume $M_{sun} >> M_{planet}$. See Physics 231 for further relevant definitions.**