Name: (please print name here →)

MATH 334:	MISSION 1: FIRST ORDER DEQNS [50PTS]

You write the solution neatly in the box provided and show work on your own, standard sized, non-fuzzy edged, paper. The work must be labeled with problem number and part as appropriate and if a problem is skipped it must be mentioned in the attached work. Neatness is part of the score. There are more than 50pts that can be earned, but 50pts are in the Syllabus. Enjoy.

Problem 1 (Separation of Variables) Solve the differential equations below. If possible, find the explicit solution, otherwise find an implicit general solution.

$y = \frac{1}{3}(x+1)^3 + C$	(a.) $\frac{dy}{dx} = (x+1)^2$
$y = -2xe^{-x} - 2e^{-x} + C$	(b.) $e^x \frac{dy}{dx} = 2x$
$y = -1 + kx$	(c.) $\frac{dy}{dx} = \frac{y+1}{x}$
$\frac{1}{2}(y-1)^2 + \ln\lvert y+1\rvert = \frac{-1}{x} + C$	(d.) $x^2y^2dy = (y+1)dx$
$y = \cos^{-1}\left(k e^{-x\sin x - \cos x}\right)$	(e.) $\sec x dy = x \cot y dx$
$-\ln\lvert\cos y\rvert + \frac{1}{2}\ln\lvert\cos y+1\rvert + \frac{1}{2}\ln\lvert\cos y-1\rvert = -2\sin x + C$	(f.) $\sec y \frac{dy}{dx} + \sin(x-y) = \sin(x+y)$
$w = \frac{-1}{\tan^{-1}(e^x) + C}$	(g.) $(e^x + e^{-x})\frac{dw}{dx} = w^2$

Problem 2 (Initial Value Problems) Use separation of variables to solve the IVPs below:

$\tan^{-1}(x) = 4y - \frac{3\pi}{4}$	(a.) $\frac{dx}{dy} = 4(x^2+1)$ with $x(\pi/4) = 1$.
$y = \frac{1}{2} + 2e^{-2x}$	(b.) $y' + 2y = 1$ with $y(0) = 5/2$.

Problem 3 (substitution of form $u = ax + by + c$) Solve the following by making an appropriate substitution and using separation of variables,

$\boxed{\frac{1}{2}\left[x+y+\frac{1}{2}\sin(2(x+y))\right] = x + C}$ (a.) $\frac{dy}{dx} = \tan^2(x+y)$

$\boxed{y = x - 5 - \ln(-x - c)}$ (b.) $\frac{dy}{dx} = 1 + e^{y-x+5}$

Problem 4 (homogeneous equations, try $y = ux$ or $x = vy$ on $M(x,y)dx + N(x,y)dy = 0$ where M and N are homogeneous functions of same degree)

$\boxed{\tan^{-1}(y/x) + \frac{1}{2}\ln(1 + y^2/x^2) = -\ln|x| + c}$ (a.) $\frac{dy}{dx} = \frac{y-x}{y+x}$

$\boxed{\frac{-1}{3}\ln\left|1 - \frac{y}{x}\right| - \frac{1}{6}\ln\left|1 + \frac{2y}{x}\right| = \ln|x| + C}$ (b.) $(x^2 + xy - y^2)dx + xydy = 0$

Problem 5 (exact equations) If the DEqn below is exact then solve it, otherwise explain why the given DEqn is not exact.

$\boxed{x^2z^2 - 3x + 4z = C}$ (a.) $(2xz^2 - 3)dx + (2zx^2 + 4)dz = 0$

$\boxed{\text{not exact}}$ (b.) $\left(2y - \frac{1}{x} + \cos 3x\right)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$

$\boxed{\frac{1}{4}\theta^4 + \theta\beta^3 = C}$ (c.) $(\theta^3 + \beta^3)d\theta + 3\theta\beta^2 d\beta = 0$

$\boxed{e^y + xy^2\cosh(x) = C}$ (d.) $(e^y + 2xy\cosh x)y' + xy^2\sinh x + y^2\cosh x = 0$

(e.) $\left(\frac{1}{x} + \frac{1}{x^2} - \frac{t}{x^2+t^2}\right)dx + \left(te^t + \frac{x}{x^2+t^2}\right)dt = 0$

$\boxed{te^t - e^t + \tan^{-1}\left(\frac{t}{x}\right) + \ln|x| - \frac{1}{x} = C}$

(sorry box too small)

Problem 6 (exact equations with integrating factor) A general form of an integrating factor is suggested. Find the specific form I which serves as an integrating factor and solve the DEqn $Mdx + Ndy = 0$ by solving the exact equation $IMdx + INdy = 0$)

$(xy + y^2)e^x = c$	(a.) $y(x+y+1)dx + (x+2y)dy = 0$ given $I = e^{Ax}$
$x^4y^{-2} + x^3y^{-7} = C$	(b.) $y(4xy^5+3)dx - x(2xy^5+7)dy = 0$ given $I = x^Ay^B$

Problem 7 (linear first order DEqn) Solve the linear first order ODEqn given below and state the interval on which the solution is defined. If given an initial value, then fit the given data to the explicit solution.

$y = \frac{1}{3} + ce^{-x^3}$	(a.) $y' + 3x^2y = x^2$
$y = x + c/x^2$	(b.) $x\frac{dy}{dx} + 2y = 3$
$y = \frac{1}{7}x^3 - \frac{1}{5}x + \frac{c}{x^4}$	(c.) $x\frac{dy}{dx} + 4y = x^3 - x$
$y = \frac{\tan x + c}{\sin x} = \sec x + c\csc x$	(d.) $\cos^2 x \sin x dy + (y\cos^3 x - 1)dx = 0$
$r = \frac{\theta - \cos\theta + C}{\sec\theta + \tan\theta}$	(e.) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$

Problem 8 (Bernoulli's Equation). If the DEqn has form $\frac{dy}{dx} + P(x)y = f(x)y^n$ for some real n then it is called a Bernoulli Equation. These can be solved by a $w = y^{1-n}$ substitution, we assume $n \neq 0, 1$. Solve the following:

$y = \frac{1}{ce^{-x} - \frac{1}{2}e^x}$	(a.) $\frac{dy}{dx} - y = e^xy^2$
$y = \frac{1}{\sqrt[3]{1 + c(1+x^2)}}$	(b.) $3(1+x^2)\frac{dy}{dx} = 2xy(y^3 - 1)$

Problem 9 (Ricatti's Equation). If $\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$ then the given DEqn is a Ricatti Equation. If y_1 is a known solution then the substitution $v = y_1 + u$ turns the problem into a Bernoulli Equation with $n = 2$. Given the Ricatti Equation below with known solution y_1, solve it. Or, if no y_1 is given then figure one out then solve it.

y

$y = x + \frac{x}{-\frac{1}{2} + ce^{2x^2}}$	(a.) $\frac{dy}{dx} = 2x^2 + y/x - 2y^2$, $y_1 = x$
$y = -3 - \frac{1}{x+c}$	(b.) $\frac{dy}{dx} = 9 + 6y + y^2$

Problem 10 (Clauiraut Equation) Let f be a smooth function. The differential equation $y = xy' + f(y')$ is known as a Clairaut Equation. Show that $y = cx + f(c)$ serves as a solution to Clairaut Equation for any $c \in \mathbb{R}$. Furthermore, show

$$x = -f'(t), \qquad \& \qquad y = f(t) - tf'(t)$$

give a parametric solution to Clairaut Equation. If $f''(t) \neq 0$ then the parametric solution describes a solution not found in the linear family and as such it is known as the **singular solution**. Solve the Clairaut Equations below by finding both their linear solutions and the singular solution.

$y = cx + c^2 + 4c$	$y = \frac{-1}{4}(x+4)^2$	(a.) $y = (x+4)y' + (y')^2$

Problem 11 Solve $\frac{dy}{dx} = e^{x-y}\cosh x$

$$\boxed{y = \ln\left(c + \frac{x}{2} + \frac{1}{4}e^{2x}\right)}$$

Problem 12 Solve $\frac{dy}{dx} = \frac{y^2 + 4y + 5}{x^2 - 3x - 4}$

$$\boxed{y = -2 + \tan\left(c + \ln\left|\frac{x-4}{x+1}\right|^{1/5}\right)}$$

Problem 13 Solve $(y + \sin^{-1}(x))dx + \left(x + \frac{1}{1+y^2}\right)dy = 0$

$$\boxed{xy + x\sin^{-1}(x) + \sqrt{1-x^2} + \tan^{-1}(y) = c}$$

Problem 14 Find the explicit solution of $\frac{dy}{dx} = \frac{e^x}{y}$ for which $y(0) = -2$.

$$\boxed{y = -\sqrt{2e^x + 2}}$$

Problem 15 Find the implicit solution of:

$$\left(1 + 2xy^2 - \frac{1}{x^2+4}\right) dx + \left(2y + 2x^2y - \frac{1}{1-y^2}\right) dy = 0.$$

$$\boxed{x + x^2y^2 - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + y^2 - \tanh^{-1}(y) = c}$$

Problem 16 A differential equation $Mdx + Ndy = 0$ is exact if there exists F for which $dF = Mdx + Ndy$. Since $d(dF) = 0$ is an identity of the exterior calculus we can check on the exactness of a given differential equation in Pfaffian form by taking its exterior derivative. Determine if the differential equations below are exact by taking the exterior derivative of the differential equation:

$\boxed{d\omega = 0 \quad \therefore \quad \omega = 0 \quad \text{exact}}$ (a.) $\underbrace{y\sin(xy)dx + x\sin(xy)dy}_{\omega} = 0$

$\boxed{d\omega \neq 0 \quad \therefore \quad \omega = 0 \quad \text{inexact.}}$ (b.) $\underbrace{-x^2dy + y^2dx}_{\omega} = 0$

Problem 17 (Orthogonal Trajectories) Find the orthogonal trajectories to the curve or family of curves described below:

$\boxed{y = \left(\frac{3}{2}(x+c)\right)^{2/3}}$ (a.) $y = (x - c_1)^2$

$\boxed{x^2 + 3y^2 = cy}$ (b.) $y^2 - x^2 = c_1x^3$

Problem 18 (Orthogonal Trajectories to Polar Curves) Find the orthogonal trajectory for the curves described below (please use polar coordinates to formulate the answer)

$\boxed{\frac{\sin^2\theta}{1+\cos\theta} = kr}$ (a.) $r = c_1(1 + \cos\theta)$

$\boxed{r = c_2 e^{-\theta}}$ (b.) $r = c_1e^{\theta}$

Problem 19 (Isogonal Families) A family of curves which intersects a given family of curves at an angle $\alpha \neq \pi/2$ are said to be **isogonal trajectories** of each other. If $\frac{dy}{dx} = f(x, y)$ describes a given family of curves then show its isogonal family are solutions of

$$\frac{dy}{dx} = \frac{f(x,y) \pm \tan\alpha}{1 \mp f(x,y)\tan\alpha}.$$

Then, find the isogonal family to $y = c_1 x$ at angle $\alpha = 30^o$.

$$\boxed{\pm\sqrt{3}\tan^{-1}\left(\frac{y}{x}\right) = C + \frac{1}{2}\ln(x^2+y^2)}$$

Problem 20 An integral curve to a vector field $\vec{F} = \langle P, Q\rangle$ can be described parametrically as a path $t \mapsto \vec{\gamma}(t) = (x(t), y(t))$ for which $\vec{F}(\vec{\gamma}(t)) = \frac{d\vec{\gamma}}{dt}$. That is, $\langle P, Q\rangle = \langle dx/dt, dy/dt\rangle$. Parametrically we need to solve $\frac{dx}{dt} = P$ and $\frac{dy}{dt} = Q$. However, if we are only interested in describing the integral curve in Cartesian coordinates then we can eliminate t via the calculus

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q}{P}$$

thus finding an integral curve for a given vector field which depends only on x, y is as simple as solving the above first order ODEqn.

(a.) Consider the vector field $\vec{F}(x,y) = \left\langle \frac{y}{(x-1)^2+y^2}, \frac{1-x}{(x-1)^2+y^2}\right\rangle$. Find the the level curve which serves as an integral curve for $\vec{F}$ through $P_o = (x_o, y_o) \neq (1,0)$.

$$\boxed{(x-1)^2 + y^2 = (x_o-1)^2 + y_o^2}$$

(b.) Find the integral curves of the vector field $\vec{F} = \langle 1, e^{x^3} - 2y/x\rangle$. Please leave your answer explicitly in terms of y as a function of x.

$$\boxed{y = \frac{1}{x^2}\left(\frac{1}{3}e^{x^3} + C\right)}$$

Problem 21 Let b be a positive constant. If a friction force of $F_f = -bv^4$ is applied to a mass m with initial position x_o and initial velocity v_o then find the velocity as a function of

$$\boxed{V(t) = \frac{V_o}{\sqrt[3]{1 + 3btV_o^3/m}}}$$

(a.) time t,

$$\boxed{V(x) = \frac{V_o}{\sqrt{1 + \frac{2b}{m}(x-x_o)V_o^2}}}$$

(b.) position x.

Problem 22 When a resistor R and inductor L are in series with a voltage source $\mathcal{E}$ then circuit analysis yields the differential equation:

$$L\frac{di}{dt} + Ri = \mathcal{E}$$

where i is the current flowing in the circuit. Given $\mathcal{E}(t) = V_o \sin\omega t$ and $i(0) = i_o$ find the current as a function of time t.

$\boxed{}$ ← Sorry ☺

$$i(t) = i_o e^{-Rt/L} + \frac{V_o}{L}\left(\frac{1}{w^2 + R^2/L^2}\left(\frac{R}{L}\sin wt - w\cos wt\right)\right) + \frac{we^{-Rt/L}}{w^2+R^2/L^2}$$

MISSION 1 SOLUTION

PROBLEM 1

(a.) $\dfrac{dy}{dx} = (x+1)^2$ $\Rightarrow$ $\int dy = \int (x+1)^2 dx$

$\Rightarrow$ $\boxed{y = \frac{1}{3}(x+1)^3 + C}$

(b.) $e^x \dfrac{dy}{dx} = 2x$ $\Rightarrow$ $\int dy = \int \underbrace{2x}_{u} \underbrace{e^{-x} dx}_{dv}$

$\Rightarrow$ $y = uv - \int v du = -2xe^{-x} + \int e^{-x} \cdot 2dx$

$\Rightarrow$ $\boxed{y = -2xe^{-x} - 2e^{-x} + C}$

(c.) $\dfrac{dy}{dx} = \dfrac{y+1}{x}$ $\Rightarrow$ $\int \dfrac{dy}{y+1} = \int \dfrac{dx}{x}$

$\ln|y+1| = \ln|x| + C$

$|y+1| = \exp(\ln|x| + c) = \exp(c)\exp(\ln|x|)$

$y+1 = \pm e^c x$ setting $k = \pm e^c$ and notice $k=0$

$\therefore$ $\boxed{y = -1 + kx}$

(d.) $x^2 y^2 dy = (y+1)dx$

$\int \dfrac{y^2 dy}{y+1} = \int \dfrac{dx}{x^2}$ $\Rightarrow$ $\int \left(y - 1 + \dfrac{1}{y+1}\right) dy = \int \dfrac{dx}{x^2}$

$\boxed{\frac{1}{2}(y-1)^2 + \ln|y+1| = \frac{-1}{x} + C}$

(I can't find explicit sol^n here)

$$
\begin{array}{r|l}
 & y - 1 \\
y+1 & y^2 \\
 & -(y^2+y) \\
 & -y \\
 & (-y-1) \\
 & 1
\end{array}
$$

P1 continued

(e.) $\sec(x)\,dy = x\cot(y)\,dx$

$$\frac{dy}{\cot y} = \frac{x\,dx}{\sec x} \Rightarrow \tan y\,dy = x\cos x\,dx$$

$$\Rightarrow \int \frac{\sin y\,dy}{\cos y} = \int x\cos x\,dx$$

Let $w = \cos y$ so $dw = -\sin y\,dy$ and $u = x$, $dv = \cos x\,dx$ so $v = \sin x$ and by IBP,

$$\int \frac{-dw}{w} = uv - \int v\,du$$

$$-\ln|w| = x\sin x - \int(\sin x)\,dx$$

$$\ln|\cos y|^{-1} = x\sin x + \cos x + C \quad \leftarrow \text{implicit sol}^n$$

$$\frac{1}{|\cos y|} = \exp(x\sin x + \cos x + C)$$

$$\boxed{y = \cos^{-1}\left(k e^{-x\sin x - \cos x}\right)}$$

(f.) $\sec y\,\frac{dy}{dx} + \sin(x-y) = \sin(x+y)$

$$\sec y\,dy = (\sin(x+y) - \sin(x-y))\,dx$$

$$\sec y\,dy = 2\cos(x)\sin(y)\,dx$$

$$\int \frac{dy}{\cos y\sin y} = \int 2\cos(x)\,dx \Rightarrow \int \frac{\sin y\,dy}{\cos y\sin^2 y} = \int 2\cos(x)\,dx$$

$$\Rightarrow \int \frac{\sin y\,dy}{\cos y(1-\cos^2 y)} = -2\sin x + C$$

$w = \cos y$, $dw = -\sin y\,dy$

partial fractions

$$\Rightarrow \int \frac{-dw}{w(1-w^2)} = \int\left(\frac{-1}{w} + \frac{1}{2(w+1)} + \frac{1}{2(w-1)}\right)dw = -2\sin x + C$$

$$\therefore \boxed{-\ln|\cos y| + \tfrac{1}{2}\ln|\cos y + 1| + \tfrac{1}{2}\ln|\cos y - 1| = -2\sin x + C}$$

P1 continued

(g.) $(e^x + e^{-x})\dfrac{dw}{dx} = w^2$

$u = e^x$

$$\int \frac{dw}{w^2} = \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{1 + (e^x)^2} = \int \frac{du}{1+u^2}$$

$$\frac{-1}{w} = \tan^{-1}(u) + C \quad \therefore \quad \boxed{w = \frac{-1}{\tan^{-1}(e^x) + C}}$$

PROBLEM 2

(a.) $\dfrac{dx}{dy} = 4(x^2+1)$ with $x(\pi/4) = 1$

$$\int \frac{dx}{x^2+1} = \int 4\,dy \Rightarrow \tan^{-1}(x) = 4y + C$$

when $y = \pi/4$ we have $x = 1$ thus $\tan^{-1}(1) = \pi + C$

then $C = \tan^{-1}(1) - \pi = \pi/4 - \pi = -3\pi/4$

$$\therefore \quad \boxed{\tan^{-1}(x) = 4y - 3\pi/4}$$

$\underline{y = \frac{1}{4}\left(\tan^{-1}(x) + \frac{3\pi}{4}\right)}$. or $\underline{x = \tan\left(4y - \frac{3\pi}{4}\right)}$.

(b.) $y' + 2y = 1$ with $y(0) = 5/2$

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{2x} \qquad \left(\mu = \exp\left(\int 2dx\right) = e^{2x} \text{ int. factor}\right)$$

$$\frac{d}{dx}\left(e^{2x}y\right) = e^{2x}$$

$$e^{2x}y = \frac{1}{2}e^{2x} + C \longrightarrow e^0\frac{5}{2} = \frac{1}{2}e^0 + C \therefore \underline{C=2}.$$

$$\Rightarrow y = e^{-2x}\left(\frac{1}{2}e^{2x} + 2\right)$$

$$\Rightarrow \boxed{y = \frac{1}{2} + 2e^{-2x}}$$

PROBLEM 3

(a.) $\frac{dy}{dx} = \tan^2(x+y)$

$$u = x+y$$

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \quad \therefore \quad \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \tan^2(u) \Rightarrow \frac{du}{dx} = \tan^2(u) + 1$$

$$\int \frac{du}{\tan^2(u)+1} = \int dx \Rightarrow \int \frac{du}{\sec^2 u} = x + C$$

$$\Rightarrow \int \cos^2(u)\,du = \int \tfrac{1}{2}(1+\cos(2u))\,du = x+C$$

$$\Rightarrow \frac{1}{2}\left(u + \frac{1}{2}\sin(2u)\right) = x + C$$

$$\therefore \boxed{\frac{1}{2}\left[x+y + \frac{1}{2}\sin(2(x+y))\right] = x+C}$$

(b.) $\frac{dy}{dx} = 1 + e^{y-x+5}$

$$u = y - x + 5 \longrightarrow \frac{du}{dx} = \frac{dy}{dx} - 1 \quad \therefore \quad \frac{dy}{dx} = 1 + \frac{du}{dx}$$

$$1 + \frac{du}{dx} = 1 + e^u$$

$$\Rightarrow \frac{du}{dx} = e^u \Rightarrow e^{-u}\,du = dx$$

$$\Rightarrow \int e^{-u}\,du = \int dx$$

$$-e^{-u} = x + C \Rightarrow e^{-u} = -x - C$$

$$\Rightarrow -u = \ln(-x-C)$$

$$\therefore -(y-x+5) = \ln(-x-C)$$

$$\boxed{y = x - 5 - \ln(-x-C)}$$

P4 Try $y = ux$ or $x = vy$ substitution

(a.) $\frac{dy}{dx} = \frac{y-x}{y+x} = \frac{y/x - 1}{y/x + 1} = \frac{u-1}{u+1}$ for $y = ux$ or $u = y/x$

Note $y = ux$ gives $\frac{dy}{dx} = x\frac{du}{dx} + u$ thus

$$x\frac{du}{dx} + u = \frac{u-1}{u+1}$$

$$x\frac{du}{dx} = \frac{u-1}{u+1} - u = \frac{u-1-u(u+1)}{u+1} = \frac{u-1-u^2-u}{u+1} = \frac{-1-u^2}{1+u}$$

$$\int \frac{(1+u)\,du}{1+u^2} = \int \frac{-dx}{x}$$

$$\tan^{-1}(u) + \tfrac{1}{2}\ln(1+u^2) = -\ln|x| + C$$

$$\boxed{\tan^{-1}(y/x) + \tfrac{1}{2}\ln(1+y^2/x^2) = -\ln|x| + C}$$

$\Updownarrow$

$2\tan^{-1}\left(\frac{y}{x}\right) + \ln(x^2+y^2) = C$.

(b.) $(x^2 + xy - y^2)\,dx + xy\,dy = 0$ (*)

Let's try $x = vy$ then $dx = y\,dv + v\,dy$

and $y = \frac{x}{v}$ thus $dy = \frac{1}{v}dx - \frac{x}{v^2}dv$ thus,

$$(x^2 + xy - y^2)(y\,dv + v\,dy) + xy\left(\tfrac{1}{v}dx - \tfrac{x}{v^2}dv\right) = 0$$

Nope. Let's try something else. Divide (*) by x^2,

$$(1 + y/x - y^2/x^2)\,dx + (y/x)\,dy = 0$$

Or,

$$\frac{dy}{dx} = \frac{1 + y/x - y^2/x^2}{y/x} = \frac{1}{y/x} + 1 - \frac{y}{x}$$

better use $u = y/x$ again, thus $\frac{dy}{dx} = x\frac{du}{dx} + u$

$$x\frac{du}{dx} + u = \frac{1}{u} + 1 - u$$

$$x\frac{du}{dx} = \frac{1}{u} + 1 - 2u$$

$$\int \frac{du}{\frac{1}{u} + 1 - 2u} = \int \frac{dx}{x}$$

p 4 continued

$$\frac{1}{\frac{1}{u}+1-2u} = \frac{u}{1+u-2u^2} = \frac{u}{(1-u)(1+2u)} = \frac{A}{1-u} + \frac{B}{1+2u}$$

$$u = A(1+2u) + B(1-u)$$

$\underline{u=1}$, $1 = 3A \quad \therefore \quad \underline{A = 1/3}$.

$\underline{u=-1/2}$, $-\frac{1}{2} = B\left(+\frac{1}{2}+1\right) = \frac{B}{2} \quad \therefore \quad \underline{B = -3}$.

$$\frac{1}{\frac{1}{u}+1-2u} = \frac{1}{3(1-u)} - \frac{1}{3(1+2u)}$$

from last page.

$$\int \frac{du}{\frac{1}{u}+1-2u} = \frac{1}{3}\int\left(\frac{1}{1-u} - \frac{1}{1+2u}\right) du = \int \frac{dx}{x}$$

$$= -\frac{1}{3}\ln|1-u| - \frac{1}{6}\ln|1+2u| = \ln|x| + C$$

$$\boxed{-\frac{1}{3}\ln\left|1-\frac{y}{x}\right| - \frac{1}{6}\ln\left|1+\frac{2y}{x}\right| = \ln|x| + C}$$

after some algebra, I believe this simplifies to $\underline{y + x = kx^2 e^{y/x}}$.

PROBLEM 5 If the DEq^n is exact then solve it, otherwise explain why the given DEq^n is not exact.

(a.) $$\underbrace{(2xz^2 - 3)}_{M}dx + (\underbrace{2zx^2 + 4}_{N})dz = 0 \quad (*)$$

$\partial_z M = 4xz = \partial_x N$ so the DEq^n is exact.

$$\frac{\partial F}{\partial x} = 2xz^2 - 3 \Rightarrow F = x^2z^2 - 3x + C_1(z)$$

$$\frac{\partial F}{\partial z} = 2zx^2 + 4 \Rightarrow 2x^2z + \frac{dC_1}{dz} = 2zx^2 + 4 \Rightarrow \underline{C_1 = 4z}.$$

$$\therefore \quad F(x,z) = \boxed{x^2z^2 - 3x + 4z = C}$$ solution to $*$.

(b.) $$\underline{\left(2y - \frac{1}{x} + \cos(3x)\right)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin(3x) = 0} \quad **$$

$$\underbrace{\left(\frac{y}{x^2} - 4x^3 + 3y\sin(3x)\right)}_{M}dx + \left(\underbrace{2y - \frac{1}{x} + \cos(3x)}_{N}\right)dy = 0$$

$$\partial_y M = \frac{1}{x^2} + 3\sin(3x) \qquad \partial_x N = \frac{1}{x^2} - 3\sin(3x)$$

Thus $\partial_y M \neq \partial_x N \Rightarrow$ $\underline{** \text{ not an exact DEq}^n}$.

(c.) $$\underbrace{(\theta^3 + \beta^3)}_{M}d\theta + \underbrace{3\theta\beta^2}_{N}d\beta = 0$$

exactness?) $\partial_\beta M = 3\beta^2 \overset{\checkmark}{=} \partial_\theta N$

I used guessing / integration to write down answer.

Notice $d\left(\frac{1}{4}\theta^4 + \theta\beta^3\right) = M\,d\theta + N\,d\beta = 0$ so

we obtain solution $\boxed{\frac{\theta^4}{4} + \theta\beta^3 = C}$

P5 continued

(d.) $(e^y + 2xy\cosh x)\frac{dy}{dx} + xy^2\sinh x + y^2\cosh x = 0$

$$\underbrace{(xy^2\sinh x + y^2\cosh x)}_{M}dx + \underbrace{(e^y + 2xy\cosh x)}_{N}dy = 0$$

$$\left.\begin{aligned}\partial_y M &= 2xy\sinh x + 2y\cosh x \\ \partial_x N &= 2y\cosh x + 2xy\sinh x\end{aligned}\right\} \underbrace{\partial_y M = \partial_x N}_{\text{closed condition}}$$

satisfied $\therefore$ exact.

It is easier to integrate $\frac{\partial F}{\partial y} = N$,

$$\frac{\partial F}{\partial y} = e^y + 2xy\cosh x$$

$$\Rightarrow F(x,y) = e^y + xy^2\cosh(x) + c_1(x)$$

Then $\frac{\partial F}{\partial x} = y^2\cosh x + xy^2\sinh x + \frac{dc_1}{dx} = y^2\cosh x + xy^2\sinh x$

thus c_1 = constant as $\frac{dc_1}{dx} = 0$. Consequently, solution is

$$\boxed{e^y + xy^2\cosh(x) = c}$$

(e.) $$\underbrace{\left(\frac{1}{x} + \frac{1}{x^2} - \frac{t}{x^2+t^2}\right)}_{M}dx + \underbrace{\left(te^t + \frac{x}{x^2+t^2}\right)}_{N}dt = 0 \quad ⓔ$$

$$\partial_t M = \frac{\partial}{\partial t}\left[\frac{-t}{x^2+t^2}\right] = \frac{-1(x^2+t^2) + 2t^2}{(x^2+t^2)^2} = \frac{t^2-x^2}{(x^2+t^2)^2}$$

$$\partial_x N = \frac{\partial}{\partial x}\left[\frac{x}{x^2+t^2}\right] = \frac{x^2+t^2-2x^2}{(x^2+t^2)^2} = \frac{t^2-x^2}{(x^2+t^2)^2}$$

Apparently ⓔ is exact, the closed condition $\partial_t M = \partial_x N$ is satisfied. So we solve it

PS continued

(e.) $$\frac{\partial F}{\partial t} = te^t + \frac{x}{x^2+t^2}$$

$$\int \underbrace{t}_{u}\, \underbrace{e^t dt}_{dV} = uv - \int V du = te^t - \int e^t dt = te^t - e^t + C.$$

$$\int \frac{x\,dt}{t^2+x^2} = \frac{1}{x^2}\int \frac{x\,dt}{1+(t/x)^2} \overset{w = t/x}{=} \frac{1}{x}\int \frac{x\,dw}{1+w^2} = \tan^{-1}(w) + c = \tan^{-1}(t/x) + c$$

Hence, $F(x,t) = te^t - e^t + \tan^{-1}\left(\frac{t}{x}\right) + C_1(x)$.

Then we also need $\frac{\partial F}{\partial x} = M = \frac{1}{x} + \frac{1}{x^2} - \frac{t}{x^2+t^2}$

from which we find

$$\frac{dC_1}{dx} = \frac{1}{x} + \frac{1}{x^2} \rightarrow C_1 = \frac{-1}{x} + \ln|x|$$

Thus the sol^{n} to (e) is given by

$$\boxed{\underbrace{te^t - e^t + \tan^{-1}\left(\frac{t}{x}\right) + \ln|x| - \frac{1}{x}}_{F} = C}$$

F has $dF = Mdx + Ndy$

thus $F(x,t) = C$ solves

$Mdx + Ndy = 0$.

$\boxed{\text{PROBLEM 6}}$ Given type of integrating factor I find I which makes $IM\,dx + IN\,dy = 0$ exact. Then solve it.

(a.) $y(x+y+1)\,dx + (x+2y)\,dy = 0, \quad I = e^{Ax}$

$$\underbrace{y(x+y+1)e^{Ax}}_{M}\,dx + \underbrace{(x+2y)e^{Ax}}_{N}\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= (x+y+1+y)e^{Ax} = (1+2y+x)e^{Ax} \\ \frac{\partial N}{\partial x} &= (1+A(x+2y))e^{Ax} = (1+2Ay+Ax)e^{Ax} \end{aligned}\right\} \underline{A=1}.$$

Consider then

$$(xy + y^2 + y)e^x\,dx + (x+2y)e^x\,dy = 0$$

$$\Rightarrow \quad F(x,y) = \boxed{(xy + y^2)e^x = C}$$

You can check, $dF = (ye^x + xye^x + y^2e^x)\,dx + (x+2y)e^x\,dy.$

(b.) $y(4xy^5 + 3)\,dx - x(2xy^5 + 7)\,dy = 0, \quad I = x^A y^B$

$$x^A y^B(4xy^6 + 3y)\,dx - x^A y^B(2x^2y^5 + 7x)\,dy = 0$$

$$\Big(\underbrace{4x^{A+1}y^{B+6} + 3x^A y^{B+1}}_{M}\Big)\,dx - \Big(\underbrace{2x^{A+2}y^{B+5} + 7x^{A+1}y^B}_{N}\Big)\,dy = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 4(B+6)x^{A+1}y^{B+5} + 3(B+1)x^A y^B \\ \frac{\partial N}{\partial x} &= -2(A+2)x^{A+1}y^{B+5} - 7(A+1)x^A y^B \end{aligned}\right\} \boxed{\begin{aligned} 4(B+6) &= -2(A+2) \\ 3(B+1) &= -7(A+1) \end{aligned}} \; *$$

Algebra reveals $*$ gives $A = 2$ and $B = -8$

$$\Big(\underbrace{4x^3y^{-2} + 3x^2y^{-7}}_{\frac{\partial F}{\partial x}}\Big)\,dx - \Big(\underbrace{2x^4y^{-3} + 7x^3y^{-8}}_{\frac{\partial F}{\partial y}}\Big)\,dy = 0$$

$$\therefore \quad F(x,y) = \boxed{x^4y^{-2} + x^3y^{-7} = C}$$

PROBLEM 7

(a.) $\frac{dy}{dx} + 3x^2y = x^2$

$$I = \exp\int 3x^2dx = e^{x^3}$$

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = x^2e^{x^3}$$

$$\frac{d}{dx}\left(e^{x^3}y\right) = x^2e^{x^3} \Rightarrow e^{x^3}y = \int x^2e^{x^3}dx$$

$$\therefore e^{x^3}y = \frac{1}{3}e^{x^3} + C \qquad \therefore \boxed{y = \frac{1}{3} + ce^{-x^3}}$$

(b.) $x\frac{dy}{dx} + 2y = 3$

$$\frac{dy}{dx} + \frac{2}{x}y = 3 \Rightarrow \underbrace{x^2\frac{dy}{dx} + 2xy}_{} = 3x^2$$

$$\left(I = \exp\left(\int\frac{2dx}{x}\right) = \exp(2\ln|x|) = \exp(\ln|x|^2) = x^2\right)$$

$$\frac{d}{dx}(x^2y) = 3x^2$$

$$x^2y = x^3 + C$$

$$\therefore \boxed{y = x + c/x^2}$$

(c.) $x\frac{dy}{dx} + 4y = x^3 - x$

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 - 1 \longrightarrow I = \exp\left(\int\frac{4dx}{x}\right) = \exp(4\ln|x|) = x^4$$

$$x^4\frac{dy}{dx} + 4x^3y = x^4(x^2-1)$$

$$\frac{d}{dx}(x^4y) = x^6 - x^4$$

$$\Rightarrow x^4y = \frac{1}{7}x^7 - \frac{1}{5}x^5 + C$$

$$\therefore \boxed{y = \frac{1}{7}x^3 - \frac{1}{5}x + \frac{C}{x^4}}$$

P7 continued

(d.) $\cos^2 x \sin x \, dy + (y \cos^3 x - 1)\, dx = 0$

$$\cos^2 x \sin x \frac{dy}{dx} + \cos^3(x)\, y = 1$$

$\cancel{I = \exp\left(\int \cos^3(x)\, dx\right)}$ no, not yet need to isolate $\frac{dy}{dx}$

$$\frac{dy}{dx} + \frac{\cos(x)}{\sin(x)} y = \frac{1}{\cos^2 x \sin x}$$

$$I = \exp\left(\int \frac{\cos x}{\sin x} dx\right) = \exp(\ln|\sin x|) = |\sin x|$$

$$(\sin x)\frac{dy}{dx} + (\cos x)\, y = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}\left(\sin(x)\, y\right) = \sec^2 x \Rightarrow \sin(x)\, y = \int \sec^2 x \, dx$$

$$\therefore \sin(x)\, y = \tan(x) + C$$

$$\boxed{y = \frac{\tan(x) + C}{\sin x}}$$

$$\boxed{y = \sec(x) + C\csc(x)}$$

(e.) $\frac{dr}{d\theta} + r\sec\theta = \cos\theta$

$$I = \exp\left(\int \sec\theta \, d\theta\right) = \exp\left(\ln|\sec\theta + \tan\theta|\right) = |\sec\theta + \tan\theta|$$

$$\underbrace{(\sec\theta + \tan\theta)\frac{dr}{d\theta} + (\sec^2\theta + \sec\theta\tan\theta)\, r} = (\sec\theta + \tan\theta)\cos\theta$$

$$\frac{d}{d\theta}\left((\sec\theta + \tan\theta)\, r\right) = 1 + \sin\theta$$

$$(\sec\theta + \tan\theta)\, r = \int (1 + \sin\theta)\, d\theta = \theta - \cos\theta + C$$

$$\boxed{r = \frac{\theta - \cos\theta + C}{\sec\theta + \tan\theta}}$$

PROBLEM 8

(a.) $\boxed{\dfrac{dy}{dx} - y = e^x y^2}$ $\quad w = y^{1-2} = \dfrac{1}{y} \quad \therefore \quad y = \dfrac{1}{w}$

$$yw = 1 \hookrightarrow \frac{dy}{dx} w + y \frac{dw}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = \frac{-y}{w}\frac{dw}{dx} = -y^2 \frac{dw}{dx}$$

$$-y^2 \frac{dw}{dx} - y = e^x y^2 \longrightarrow \frac{dw}{dx} + \frac{1}{y} = -e^x$$

$$\frac{dw}{dx} + w = -e^x \Rightarrow e^x \frac{dw}{dx} + e^x w = -e^{2x}$$

$$\frac{d}{dx}(e^x w) = -e^{2x}$$

$$e^x w = \frac{-1}{2} e^{2x} + C$$

$$w = \frac{1}{y} = \frac{-1}{2} e^x + Ce^{-x} \quad \therefore \quad \boxed{y = \frac{1}{Ce^{-x} - e^x/2}}$$

(b.) $\boxed{3(1+x^2)\dfrac{dy}{dx} = 2xy(y^3 - 1)}$ $\qquad n = 4$

$$\frac{dy}{dx} - \frac{2xy(y^3-1)}{3(1+x^2)} = 0 \longrightarrow \frac{dy}{dx} + \frac{2x}{3(1+x^2)} y = \frac{2x}{3(1+x^2)} y^4$$

Use $w = y^{1-4} = y^{-3}$ substitution, $y = w^{-1/3}$, $y^4 = w^{-4/3}$

$$\frac{dy}{dx} = \frac{-1}{3} w^{-4/3}\frac{dw}{dx} = \frac{-1}{3 w w^{1/3}} \frac{dw}{dx}$$

$$\frac{-1}{3} w^{-4/3} \frac{dw}{dx} + \frac{2x}{3(1+x^2)} w^{-1/3} = \frac{2x}{3(1+x^2)} w^{-4/3}$$

$$\frac{dw}{dx} - \frac{2x}{1+x^2} w = \frac{-2x}{1+x^2} \quad ; \quad I = \exp\left(\int \frac{-2x}{1+x^2}\right) = \exp\left(-\ln(1+x^2)\right) = \frac{1}{1+x^2}$$

$$\underbrace{\left(\frac{1}{1+x^2}\right)\frac{dw}{dx} - \frac{2x}{(1+x^2)^2} w}_{\frac{d}{dx}\left[\frac{1}{1+x^2} w\right]} = \frac{-2x}{(1+x^2)^2}$$

$$\frac{d}{dx}\left[\frac{1}{1+x^2} w\right] = \frac{-2x}{(1+x^2)^2} \longrightarrow \frac{w}{1+x^2} = \int \frac{-2x\,dx}{(1+x^2)^2} = \frac{1}{1+x^2} + C$$

$$w = 1 + C(1+x^2) = \frac{1}{y^3} \quad \therefore \quad \boxed{y = \frac{1}{\sqrt[3]{1 + C(1+x^2)}}}$$

$\boxed{\text{PROBLEM 9}}$ Ricatti's Eq^n

(a.) $\dfrac{dy}{dx} = 2x^2 + y/x - 2y^2 \quad , \quad y_1 = x$

$y = x + u$

$$\frac{dy}{dx} = 1 + \frac{du}{dx} = 2x^2 + \frac{1}{x}(x+u) - 2(x+u)^2$$

$$1 + \frac{du}{dx} = 2x^2 + 1 + \frac{1}{x}u - 2(x^2 + 2xu + u^2)$$

$$\frac{du}{dx} = \cancel{2x^2} + \frac{1}{x}u - \cancel{2x^2} - 4xu - 2u^2$$

$$\frac{du}{dx} + \left(4x - \frac{1}{x}\right)u = -2u^2 \quad w = \frac{1}{u} \text{ or } u = \frac{1}{w}$$

$$\frac{du}{dx} = \frac{-1}{w^2}\frac{dw}{dx}$$

$$\frac{-1}{w^2}\frac{dw}{dx} + \left(4x - \frac{1}{x}\right)\frac{1}{w} = \frac{-2}{w^2}$$

$$\frac{dw}{dx} + \left(\frac{1}{x} - 4x\right)w = 2$$

$$I = \exp\left(\int\left(\frac{1}{x} - 4x\right)dx\right) = \exp\left(\ln|x| - 2x^2\right) = |x|\,e^{-2x^2}$$

$$xe^{-2x^2}\frac{dw}{dx} + \left(\frac{1}{x} - 4x\right)xe^{-2x^2}w = 2xe^{-2x^2}$$

$$\underbrace{xe^{-2x^2}\frac{dw}{dx} + (1 - 4x^2)e^{-2x^2}w}_{} = 2xe^{-2x^2}$$

$$\frac{d}{dx}\left(xe^{-2x^2}w\right) = 2xe^{-2x^2}$$

$$xe^{-2x^2}w = -\frac{1}{2}e^{-2x^2} + C$$

$$w = \frac{-1}{2x} + \frac{C}{xe^{-2x^2}} = \frac{1}{u} = \frac{1}{y - x}$$

$$\therefore \quad y = x + \frac{1}{\frac{-1}{2x} + \frac{C}{xe^{-2x^2}}} \Rightarrow \boxed{y = x + \frac{x}{\frac{-1}{2} + Ce^{2x^2}}}$$

P9 continued

(16.) $\boxed{\dfrac{dy}{dx} = y^2 + 6y + 9 = (y+3)^2}$ note $y_1 \equiv -3$ solves DEq^{n}

Substitute $y = -3 + u$ thus $u = y + 3$ and $\dfrac{du}{dx} = \dfrac{dy}{dx}$

$$\frac{du}{dx} = u^2 \Rightarrow \int \frac{du}{u^2} = \int dx$$

$$\Rightarrow \frac{-1}{u} = x + C$$

$$\Rightarrow u = \frac{-1}{x+C} = y + 3$$

$$\therefore \boxed{y = -3 - \frac{1}{x+C}}$$

PROBLEM 10 $y = (x+4)y' + (y')^2$ * is Clairaut Eq^{n} with $f(y') = (y')^2 + 4y'$ or $f(t) = t^2 + 4t$ then by problem statement,

$$y = cx + f(c) \longrightarrow \boxed{y = cx + c^2 + 4c \text{ solves } *}$$

Next, following the problem statement,

$$x = -f'(t) = -2t - 4$$

$$y = f(t) - t f'(t) = t^2 + 4t - t(2t+4) = -t^2$$

Note $t = \dfrac{x+4}{-2}$ thus,

$$y = -\left(\frac{x+4}{-2}\right)^2$$

$$\boxed{y = -\frac{1}{4}(x+4)^2}$$

(singular sol^{n})

$$y' = \frac{-1}{2}(x+4)$$

$$(x+4)y' + (y')^2 = \frac{-1}{2}(x+4)^2 + \frac{(x+4)^2}{4} = \frac{-1}{4}(x+4)^2 = y$$

(so the singular sol^{n} is a sol^{n} here)

PROBLEM 11

$$\frac{dy}{dx} = e^{x-y}\cosh(x) = e^x e^{-y}\cosh(x) = \frac{1}{2}e^x e^{-y}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}e^{-y}(e^{2x}+1) \longrightarrow \int e^y\,dy = \int \frac{1}{2}(e^{2x}+1)\,dx$$

$$e^y = \frac{1}{4}e^{2x} + \frac{x}{2} + C$$

$$\boxed{y = \ln\left(C + \frac{x}{2} + \frac{1}{4}e^{2x}\right)}$$

PROBLEM 12

$$\frac{dy}{dx} = \frac{y^2+4y+5}{x^2-3x-4} \longrightarrow \frac{dy}{y^2+4y+5} = \frac{dx}{x^2-3x-4}$$

$$\frac{1}{x^2-3x-4} = \frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-4) \quad \xrightarrow{x=-1} \quad 1 = -5B,\ B = \frac{-1}{5}$$

$$\xrightarrow[x=4]{} \quad 1 = 5A,\ A = \frac{1}{5}$$

$$\int \frac{dy}{(y+2)^2+1} = \frac{1}{5}\int\left(\frac{1}{x-4} - \frac{1}{x+1}\right)dx$$

$$\tan^{-1}(y+2) = \frac{1}{5}\left(\ln|x-4| - \ln|x+1|\right) + C$$

$$\boxed{y = -2 + \tan\left(C + \ln\left|\frac{x-4}{x+1}\right|^{1/5}\right)}$$

PROBLEM 13

$$\boxed{\overbrace{\left(y + \sin^{-1}(x)\right)}^{M}dx + \overbrace{\left(x + \frac{1}{1+y^2}\right)}^{N}dy = 0}$$

$$\int \underbrace{\sin^{-1}(x)}_{u}\,\underbrace{dx}_{dv} = uv - \int v\,du = x\sin^{-1}(x) - \int\frac{x\,dx}{\sqrt{1-x^2}}$$

$u = 1-x^2$

$du = -2x\,dx$

$$= x\sin^{-1}(x) + \int\frac{du}{2\sqrt{u}}$$

$$= x\sin^{-1}(x) + \sqrt{1-x^2} + C$$

$$\boxed{\underbrace{xy + x\sin^{-1}(x) + \sqrt{1-x^2} + \tan^{-1}(y)}_{F} = C}$$

you can check, $dF = M\,dx + N\,dy$.

PROBLEM 14

$$\frac{dy}{dx} = \frac{e^x}{y} \quad \text{with } y(0) = -2 \quad \text{solve explicitly.}$$

$$\int y\,dy = \int e^x\,dx \longrightarrow \frac{1}{2}y^2 = e^x + C$$

$$y = \pm\sqrt{2e^x + 2C}$$

$$y(0) = -2 = -\sqrt{2e^0 + 2C} \longrightarrow 4 = 2 + 2C \therefore \underline{C = 1}.$$

$$\boxed{y = -\sqrt{2e^x + 2}}$$

PROBLEM 15

$$\Big(\underbrace{1 + 2xy^2 - \frac{1}{x^2+4}}_{M}\Big)dx + \Big(\underbrace{2y + 2x^2y - \frac{1}{1-y^2}}_{N}\Big)dy = 0 \quad (*)$$

$$\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{1 + x^2/4} = \frac{1}{4}\int \frac{2\,du}{1+u^2} \qquad \frac{x}{2} = u$$

$$= \frac{1}{2}\tan^{-1}(u) + c$$

$$= \frac{1}{2}\tan^{-1}(x/2) + c.$$

$$\frac{d}{dy}\left(\tanh^{-1}(y)\right) = \frac{1}{1-y^2}$$

Thus, we solve * by,

$$F(x,y) = \boxed{x + x^2y^2 - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + y^2 - \tanh^{-1}(y) = C}$$

Note $dF = M\,dx + N\,dy$ as needed.

PROBLEM 16 Given $M\,dx + N\,dy = 0$ determine if the DEq^{n} is exact by calculating $dM \wedge dx + dN \wedge dy$ and checking if it is zero. Notice, if $dF = M\,dx + N\,dy$ then $d(dF) = 0$.

(a.) $y\sin(xy)\,dx + x\sin(xy)\,dy = \omega$ has

$$d\omega = d(y\sin(xy)) \wedge dx + d(x\sin(xy)) \wedge dy$$

$$= \left(\partial_x[y\sin xy]\,dx + \partial_y[y\sin xy]\,dy\right) \wedge dx$$

$$+ \left(\partial_x[x\sin(xy)]\,dx + \partial_y[x\sin(xy)]\,dy\right) \wedge dy$$

$$= (\sin(xy) + xy\cos(xy))\,dy \wedge dx + (\sin(xy) + xy\cos(xy))\,dx \wedge dy$$

$= 0$. (since $dy \wedge dx = -dx \wedge dy$) $\therefore$ $\omega = 0$ is exact equation.

(b.) $-x^2\,dy + y^2\,dx = \omega$

$$d(-x^2) \wedge dy + d(y^2) \wedge dx = d\omega$$

$$d\omega = -2x\,dx \wedge dy + 2y\,dy \wedge dx = -2(x+y)\,dx \wedge dy \neq 0$$

$\therefore$ $-x^2\,dy + y^2\,dx = 0$ is <u>inexact</u>.

<u>Remark</u>: for (a) observe

$$d(-\cos(xy)) = \partial_x[-\cos xy]\,dx + \partial_y[-\cos(xy)]\,dy$$

$$= y\sin(xy)\,dx + x\sin(xy)\,dy$$

$\hookrightarrow$ $\boxed{-\cos(xy) = C}$ solves DEq^{n} given in (a).

PROBLEM 17 Find orthogonal trajectories to curve(s)

(a.) $y = (x - c_1)^2 \Rightarrow x - c_1 = \pm\sqrt{y}$

$$\frac{dy}{dx} = 2(x - c_1) = \pm 2\sqrt{y}$$

O.T.) $\dfrac{dy}{dx} = \dfrac{-1}{\pm 2\sqrt{y}} \hookrightarrow \mp\sqrt{y}\,dy = dx$

$$\mp\frac{2}{3}y^{3/2} = x + C$$

$$y^{3/2} = \mp\frac{3}{2}(x + C)$$

$$\boxed{y = \left(\frac{3}{2}(x + c)\right)^{2/3}}$$

(b.) $y^2 - x^2 = c_1 x^3$

$$\frac{y^2 - x^2}{x^3} = c_1 \longrightarrow \frac{y^2}{x^3} - \frac{1}{x} = c_1$$

$$\frac{2y}{x^3}\frac{dy}{dx} + y^2\left(\frac{-3}{x^4}\right) + \frac{1}{x^2} = 0$$

$$\frac{dy}{dx} = \frac{x^3}{2y}\left[\frac{3y^2}{x^4} - \frac{1}{x^2}\right] = \frac{3y}{2x} - \frac{x}{2y}$$

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} \hookrightarrow \text{O.T. } \frac{dy}{dx} = \frac{2xy}{x^2 - 3y^2}$$

$$\frac{dx}{dy} = \frac{x^2 - 3y^2}{2xy} = \frac{1x}{2y} - \frac{3y}{2x}$$

$\dfrac{dx}{dy} - \dfrac{1}{2y}x = \dfrac{-3y}{2}x^{-1}$ Bernoulli, $n = -1$, $w = x^{1-n} = x^2$

$$\frac{dw}{dy} = 2x\frac{dx}{dy}$$

$$\frac{1}{2x}\frac{dw}{dy} - \frac{1}{2y}x = \frac{-3y}{2}\frac{1}{x} \rightarrow \frac{1}{2}\frac{dw}{dy} - \frac{1}{2y}x^2 = \frac{-3y}{2}$$

$$\frac{dw}{dy} - \frac{1}{y}w = -3y$$

P17 continued

$$\frac{dW}{dy} - \frac{1}{y}W = -3y$$

$$I = \exp\left(\int \frac{-dy}{y}\right) = \exp\left(-\ln|y|\right) = \frac{1}{|y|}$$

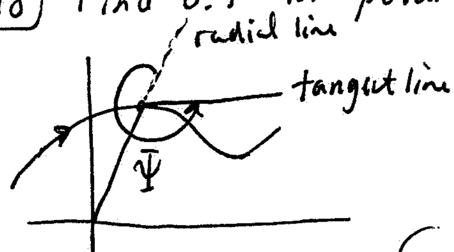
$$\underbrace{\frac{1}{y}\frac{dW}{dy} - \frac{1}{y^2}W}_{} = -3$$

$$\frac{d}{dy}\left[\frac{1}{y}W\right] = -3$$

$$\frac{1}{y}W = -3y + C \longrightarrow \frac{x^2}{y} = -3y + C$$

$$\text{or } \boxed{x^2 + 3y^2 = cy}$$

$\boxed{P18}$ Find O.T for polar curves below.



$$r\frac{d\theta}{dr} = \tan\Psi$$

$\tan\psi_1 \tan\psi_2 = -1$ for C_1 & C_2 O.T.

$$\hookrightarrow r_1\frac{d\theta_1}{dr_1} r_2\frac{d\theta_2}{dr_2} = -1 \quad \text{(from PP 40)}$$

(a.) $r = c_1(1 + \cos\theta)$

$$c_1 = \frac{r}{1+\cos\theta} \Rightarrow 0 = \frac{1}{1+\cos\theta} - \left(\frac{r(-\sin\theta)}{(1+\cos\theta)^2}\right)\frac{d\theta}{dr}$$

$$\frac{d\theta}{dr} = \left(\frac{1}{1+\cos\theta}\right)\left(\frac{(1+\cos\theta)^2}{-r\sin\theta}\right) = \frac{1+\cos\theta}{-r\sin\theta} \quad \therefore \quad r\frac{d\theta}{dr} = \frac{1+\cos\theta}{-\sin\theta}$$

$$\text{O.T.} \quad r\frac{d\theta}{dr} = \frac{-1}{\frac{1+\cos\theta}{-\sin\theta}} = \frac{\sin\theta}{1+\cos\theta} \Rightarrow \int\left(\frac{1+\cos\theta}{\sin\theta}\right)d\theta = \int\frac{dr}{r}$$

$$\therefore -\ln|\csc\theta + \cot\theta| + \ln|\sin\theta| = \ln|r| + C$$

P18 continued

$$\ln\left|\frac{\sin\theta}{\csc\theta+\cot\theta}\right| = \ln|kr|$$

$$\frac{\sin\theta}{\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}} = \boxed{\frac{\sin^2\theta}{1+\cos\theta} = kr}$$

(16.) $\boxed{r = c_1 e^{\theta}}$

$$c_1 = re^{-\theta} \longrightarrow \frac{d}{dr}(c_1) = \frac{d}{dr}(re^{-\theta}) = e^{-\theta} - re^{-\theta}\frac{d\theta}{dr} = 0$$

$$r\frac{d\theta}{dr} = 1 \Rightarrow r\frac{d\theta}{dr} = -1 \quad \text{for O.T.}$$

$$d\theta = \frac{-dr}{r}$$

$$\theta = -\ln r + c = \ln\left(\frac{1}{r}\right) + c$$

$$\ln\left(\frac{1}{r}\right) = \theta - c$$

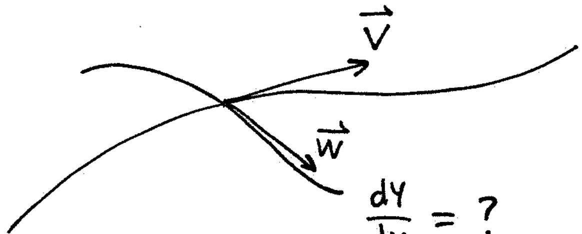
$$\frac{1}{r} = e^{\theta - c}$$

$$r = \frac{1}{e^{\theta-c}} \Rightarrow \boxed{r = c_2 e^{-\theta}}$$

$\boxed{\text{P19}}$ A family of curves which intersects a given curve or family of curves at angle $\alpha \neq \pi/2$ is said to be isogonal w.r.t. the given curve(s). Show such curves solve

$$\frac{dy}{dx} = \frac{f(x,y) \pm \tan\alpha}{1 \mp f(x,y)\tan\alpha}$$

for a given family of solutions to $\frac{dy}{dx} = f(x,y)$



we want $\vec{V}$ rotated $\pm\alpha$ to yield $\vec{W}$ tangent to the isogonal curve.

Notice $\vec{r}(x) = \langle x, y_1 \rangle$ has $\frac{d\vec{r}}{dx} = \langle 1, \frac{dy_1}{dx} \rangle = \langle 1, f(x,y) \rangle$

complex notation

I'll use complex variables to rotate $\vec{V} = 1 + if$

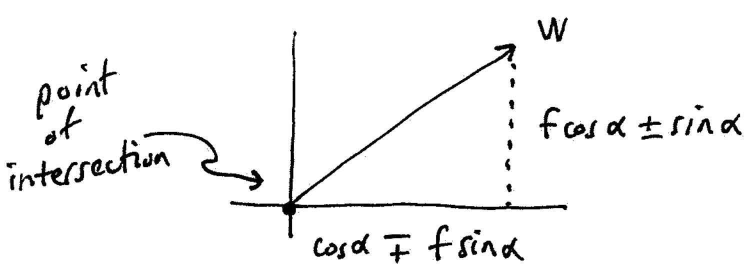
We simply rotate by $\pm\alpha$ by multiplying be $e^{\pm i\alpha}$

[see over for extended comment]

$$\vec{W} = e^{\pm i\alpha} V = (\cos\alpha \pm i\sin\alpha)(1 + if)$$

$$= \cos\alpha \mp f\sin\alpha + i(f\cos\alpha \pm \sin\alpha)$$

Geometrically,

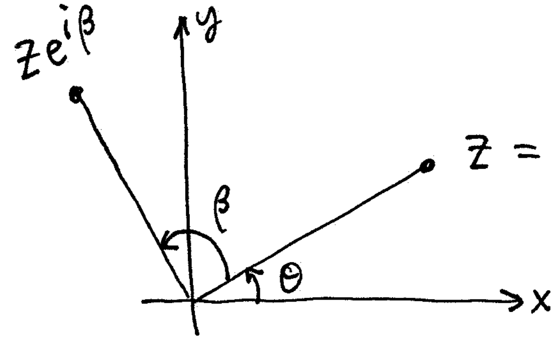


For the isogonal trajectory,

$$\frac{dy}{dx} = \frac{f\cos\alpha \pm \sin\alpha}{\cos\alpha \mp f\sin\alpha} = \frac{f \pm \tan\alpha}{1 \mp f\tan\alpha}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{f(x,y) \pm \tan\alpha}{1 \mp \tan\alpha\, f(x,y)}}$$

p 19 continued



$$z = re^{i\theta} = \underbrace{r\cos\theta}_{x} + i\underbrace{r\sin\theta}_{y}$$

Multiply z by $e^{i\beta}$ we get

$$ze^{i\beta} = re^{i\theta}e^{i\beta} = re^{i(\theta+\beta)}$$

Rotating a vector in the plane is nicely done in complex notation.

//

Find isogonal family to $y = c_1 x$ at angle $\alpha = 30°$

note $\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$ so $\tan 30° = \frac{\sin 30°}{\cos 30°} = \frac{1}{\sqrt{3}}$

$$c_1 = \frac{y}{x} \longrightarrow 0 = \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y \quad \therefore \quad \frac{dy}{dx} = x\left(\frac{y}{x^2}\right) = \frac{y}{x}$$

identify $f(x,y) = \frac{y}{x}$ for the given family. Hence solve,

$$\frac{dy}{dx} = \frac{f(x,y) \pm \tan\alpha}{1 \mp f(x,y)\tan\alpha} = \frac{(y/x) \pm (1/\sqrt{3})}{1 \mp (y/x)(1/\sqrt{3})}$$

Let's try a $W = y/x$ substitution, $y = xW$

then $\frac{dy}{dx} = W + x\frac{dW}{dx}$ and so we face,

$$W + x\frac{dW}{dx} = \frac{\sqrt{3}W \pm 1}{\sqrt{3} \mp W}$$

$c = \pm 1$

$$x\frac{dW}{dx} = \frac{\sqrt{3}W \pm 1}{\sqrt{3} \mp W} - W = \frac{\sqrt{3}W + C - W(\sqrt{3} - CW)}{\sqrt{3} - CW}$$

P19 continued

$$x \frac{dw}{dx} = \frac{c + cw^2}{\sqrt{3} - cw} \qquad c = \pm 1 = \frac{1}{c}$$

$$\left(\frac{\sqrt{3} - cw}{c(1+w^2)}\right) dw = \frac{dx}{x}$$

$$\int \left(\frac{c\sqrt{3}}{1+w^2} - \frac{w}{1+w^2}\right) dw = \int \frac{dx}{x}$$

$$\pm\sqrt{3}\tan^{-1}(w) - \frac{1}{2}\ln(1+w^2) = \ln|x| + C$$

Recall $w = y/x$ thus,

$$\pm\sqrt{3}\tan^{-1}(y/x) - \frac{1}{2}\ln(1 + y^2/x^2) - \frac{1}{2}\ln(x^2) = C$$

$$\boxed{\pm\sqrt{3}\tan^{-1}(y/x) = C + \frac{1}{2}\ln(x^2 + y^2)}$$

(I don't see solving for y going well here)

P20

(a.) find level curve which is integral curve through $P_o = (x_o, y_o) \neq (1,0)$,
for $\vec{F}(x,y) = \left\langle \frac{y}{(x-1)^2+y^2}, \frac{1-x}{(x-1)^2+y^2} \right\rangle = \langle P, Q \rangle$

$$\frac{dy}{dx} = \frac{Q}{P} = \frac{1-x}{y}$$ since the denominators cancel!

$$\int y\,dy = \int (1-x)\,dx$$

$$\frac{1}{2}y^2 = x - \frac{1}{2}x^2 + C$$

$$\frac{1}{2}y_o^2 = x_o - \frac{1}{2}x_o^2 + C \quad \therefore \quad C = \frac{1}{2}(y_o^2 - x_o^2) - x_o$$

$$\therefore \quad \frac{1}{2}y^2 = x - \frac{1}{2}x^2 + \frac{1}{2}(y_o^2 - x_o^2) - x_o \quad (*)$$

Which can be written as a circle at $(1,0)$ with radius $\sqrt{(x_o-1)^2+y_o^2}$

$$\boxed{(x-1)^2 + y^2 = (x_o-1)^2 + y_o^2}$$

(take * multiply by two, rearrange terms, complete square)

(b.) $\boxed{F = \langle 1, e^{x^3} - 2y/x \rangle}$ solve $\frac{dy}{dx} = \frac{Q}{P}$ for $\int$- curve

$$\frac{dy}{dx} = e^{x^3} - \frac{2y}{x} \longrightarrow \frac{dy}{dx} + \frac{2}{x}y = e^{x^3}$$

Notice $\exp\left(\int \frac{2}{x}\right) = \exp(2\ln|x|) = \exp(\ln|x|^2) = |x|^2 = x^2$ serves as integrating factor,

$$x^2\frac{dy}{dx} + 2xy = x^2e^{x^3}$$

$$\frac{d}{dx}(x^2y) = x^2e^{x^3} \Rightarrow x^2y = \frac{1}{3}e^{x^3} + C$$

$$\therefore \quad \boxed{y = \frac{\frac{1}{3}e^{x^3} + C}{x^2}}$$

$\boxed{P21}$ $F_f = -bv^4 = ma = m\frac{dV}{dt} = m\frac{dx}{dt}\frac{dV}{dX} = mV\frac{dV}{dX}$

thus consider, given $V = V_o$ and $X = X_o$ at $t = 0$,

(a.) $m\frac{dV}{dt} = -bV^4$

$$\int_{V_o}^{V_f}\frac{dV}{V^4} = \int_0^{t_f}\frac{-b}{m}\,dt \Rightarrow \frac{-1}{3V_f^3} + \frac{1}{3V_o^3} = \frac{-bt_f}{m}$$

Setting $V_f = V$ and $t_f = t$ we obtain

$$\frac{1}{3V^3} = \frac{1}{3V_o^3} + \frac{bt}{m} \longrightarrow \frac{1}{V^3} = \frac{1}{V_o^3} + \frac{3bt}{m}$$

$$\therefore \boxed{V(t) = \frac{1}{\sqrt[3]{\frac{1}{V_o^3} + \frac{3bt}{m}}}}$$

etc...

$$\boxed{V(t) = \frac{V_o}{\sqrt[3]{1 + 3btV_o^3/m}}}$$

(b.) $mV\frac{dV}{dX} = -bV^4$

$$\int_{V_o}^{V_f}\frac{dV}{V^3} = \int_{X_o}^{X_f}\frac{-b}{m}\,dX$$

$$\frac{-1}{2V_f^2} + \frac{1}{2V_o^2} = \frac{-b}{m}(X_f - X_o)$$

Setting $V_f = V$ and $X_f = X$ we obtain,

$$\frac{1}{V^2} = \frac{1}{V_o^2} + \frac{2b}{m}(X - X_o) = \frac{1}{V_o^2}\left(1 + \frac{2b(X-X_o)V_o^2}{m}\right)$$

$$\underbrace{\boxed{V = \frac{V_o}{\sqrt{1 + \frac{2b(X-X_o)V_o^2}{m}}}}}_{\text{reflects sign of } V_o \text{ as is physically natural}} = \underbrace{\frac{1}{\sqrt{\frac{1}{V_o^2} + \frac{2b}{m}(X - X_o)}}}_{\text{assuming } V_o > 0}$$

$\boxed{P22}$ Given $\mathcal{E}(t) = V_o \sin \omega t$ and $i(0) = i_o$ solve

$$L \frac{di}{dt} + Ri = \mathcal{E}(t)$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{V_o}{L} \sin(\omega t) \qquad (\text{we assume } L > 0)$$

then the integrating factor is $I = \exp\left(\int \frac{R}{L} dt\right) = e^{\frac{Rt}{L}}$

$$\underbrace{e^{\frac{Rt}{L}} \frac{di}{dt} + \frac{R}{L} e^{\frac{Rt}{L}} i}_{\frac{d}{dt}\left[e^{\frac{Rt}{L}} i\right]} = \frac{V_o}{L} \sin(\omega t) e^{\frac{Rt}{L}}$$

$$\frac{d}{dt}\left[e^{\frac{Rt}{L}} i\right] = \frac{V_o}{L} e^{\frac{Rt}{L}} \sin(\omega t)$$

Integration of the RHS requires some calculation,

$$\mathcal{B} = \int \underbrace{e^{at}}_{u} \underbrace{\sin(\omega t)\, dt}_{dV} = \frac{-1}{\omega} e^{at} \cos(\omega t) + \int \frac{1}{\omega} \cos(\omega t)\, a e^{at}\, dt$$

$$= \frac{-1}{\omega} e^{at} \cos(\omega t) + \frac{a}{\omega} \int \underbrace{\cos(\omega t)\, dt}_{dV}\, \underbrace{e^{at}}_{u}$$

$$= \frac{-1}{\omega} e^{at} \cos(\omega t) + \frac{a}{\omega}\left[\frac{1}{\omega} e^{at} \sin(\omega t) - \int \frac{a}{\omega} \sin(\omega t) e^{at} dt\right]$$

$$= \frac{e^{at}}{\omega^2}\big(a \sin(\omega t) - \omega \cos(\omega t)\big) - \frac{a^2}{\omega^2} \underbrace{\int e^{at} \sin(\omega t)\, dt}_{\mathcal{B}}$$

Now solve for $\mathcal{B}$

$$\mathcal{B} = \frac{1}{1 + a^2/\omega^2}\left[\frac{e^{at}}{\omega^2}\big(a \sin(\omega t) - \omega \cos(\omega t)\big)\right]$$

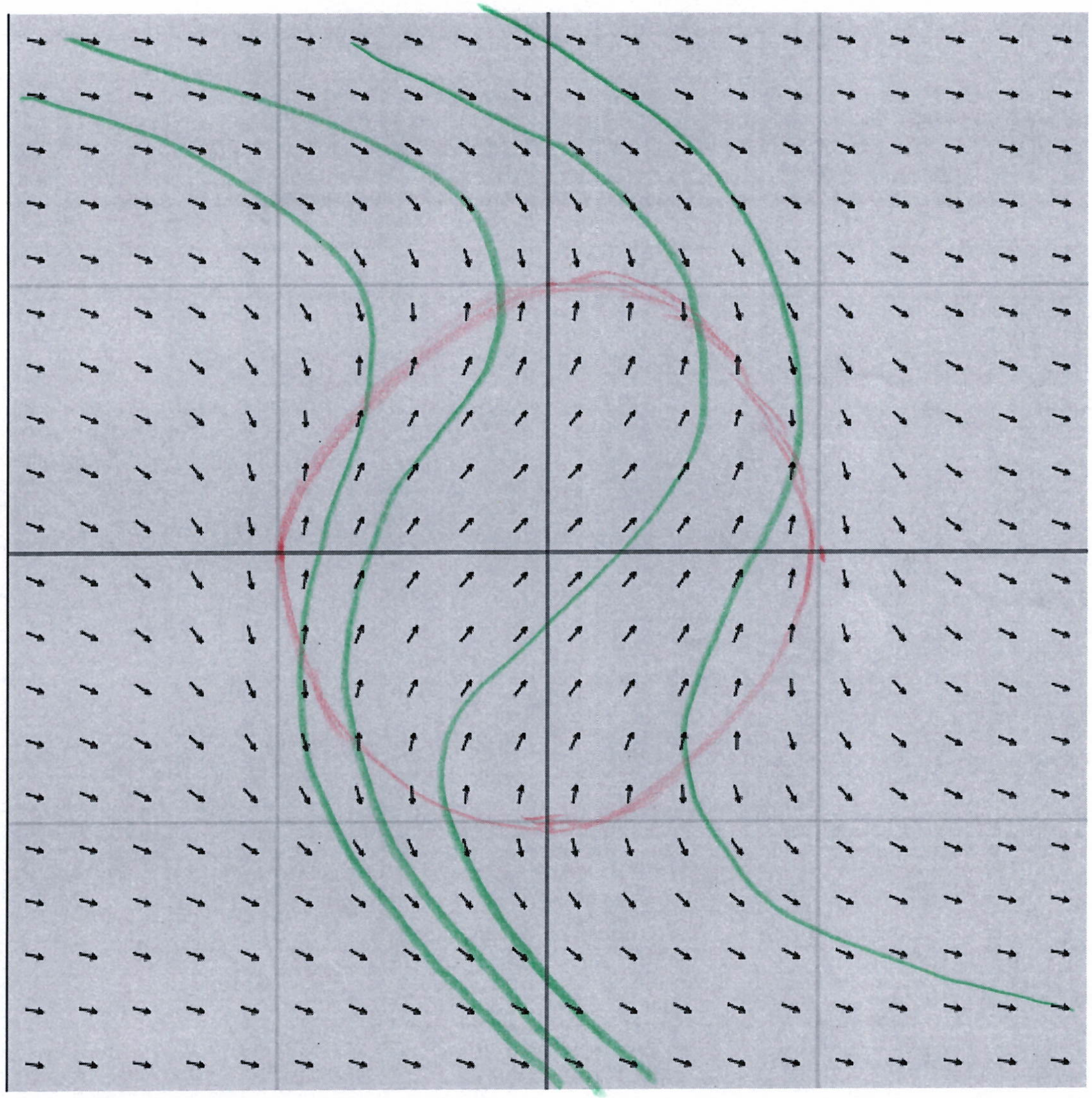
$$\therefore \int e^{\frac{Rt}{L}} \sin(\omega t)\, dt = \frac{e^{Rt/L}}{\omega^2 + R^2/L^2}\left(\frac{R}{L} \sin(\omega t) - \omega \cos(\omega t)\right) + C$$

$$\int_0^t \frac{d}{d\tau}\left[e^{\frac{R\tau}{L}} i\right] d\tau = \frac{V_o}{L} \int_0^t e^{R\tau/L} \sin(\omega \tau)\, d\tau$$

$$e^{Rt/L} i - i_o = \frac{V_o}{L}\left(\frac{e^{Rt/L}}{\omega^2 + R^2/L^2}\left(\frac{R}{L} \sin \omega t - \omega \cos(\omega t)\right) + \frac{\omega}{\omega^2 + R^2/L^2}\right)$$

$$\therefore \boxed{i(t) = i_o e^{-Rt/L} + \frac{V_o}{L}\left(\frac{1}{\omega^2 + R^2/L^2}\left(\frac{R}{L} \sin \omega t - \omega \cos \omega t\right) + \frac{\omega e^{-Rt/L}}{\omega^2 + R^2/L^2}\right)}$$

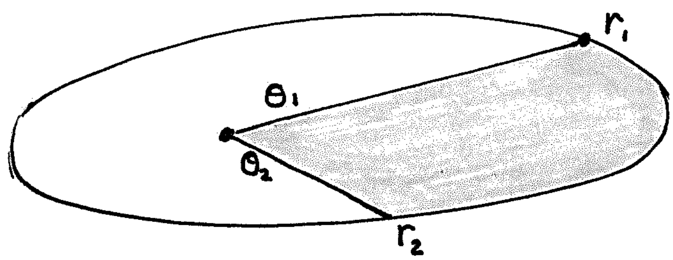
Problem 23 Plot solution curves for the direction field of $\frac{dy}{dx} = \frac{1}{1-x^2-y^2}$ given below and explain what is happening at the unit-circle for the solutions. Try it out in the pplane to check your hand-drawn answers here (maybe starting with a pencil lightly then tracing over with pen once you're sure would be wise here)



Remark: I used https://aeb019.hosted.uark.edu/pplane.html to generate the direction field above.

Problem 24 Angular momentum of a body moving in some plane is given by $L = mr^2\frac{d\theta}{dt}$ where r, θ serve as polar coordinates in the plane of motion. Assume the coordinates of the body are (r_1, θ_1) at $t = t_1$ and (r_2, θ_2) at $t = t_2$ where $t_1 < t_2$. **If L is constant then show that the area swept out by r is $A = L(t_2 - t_1)/2m$.** When the sun is taken to be at the origin and m represents a planet's mass then this proves Kepler's second law of planetary motion: the radius vector joining the sun sweeps out equal areas for equal intervals of time. **Bonus: prove L is constant in the context of the sun-planet system, you may assume $M_{sun} >> M_{planet}$. See Physics 231 for further relevant definitions.**

$\boxed{P24}$ Angular momentum L is given by $L = mr^2 \frac{d\theta}{dt}$ where r, θ are polar coordinates in the plane of motion. Assume coordinates of the body are (r_1, θ_1) at $t = t_1$ and (r_2, θ_2) at $t = t_2$ where $t_1 < t_2$. Suppose L is constant then $\frac{dL}{dt} = 0$. Also, $\frac{L}{m} dt = r^2 d\theta$,



$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta = \int_{t_1}^{t_2} \frac{L}{2m} dt = \boxed{\frac{L(t_2 - t_1)}{2m}}$$

Bonus: $\vec{L} = \vec{r} \times \vec{P} = m\vec{r} \times \vec{v}$

gravity, taking sun with mass M at origin, $\vec{F} = -\frac{GmM}{r^3}\vec{r}$

then torque $\vec{\tau} = \vec{r} \times \vec{F}$ for the planet m,

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \left(\frac{-GmM}{r^3}\vec{r}\right) = \frac{-GmM}{r^3}\vec{r} \times \vec{r} = 0$$

thus $\vec{L}$ is constant.

(Alternatively, not Physics 231 style, but the Lagrangian $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GmM}{r}$ then the Euler-Lagrange Eq^{n}'s,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}\left[mr^2\dot{\theta}\right] = 0 \quad \therefore L = mr^2\dot{\theta}$$

is constant)