Name: (please print name here $\rightarrow$)

MATH 334: MISSION 2: n-TH ORDER DEQNS [50PTS]

You write the solution neatly in the box provided and show work on your own, standard sized, non-fuzzy edged, paper. The work must be labeled with problem number and part as appropriate and if a problem is skipped it must be mentioned in the attached work. Neatness is part of the score. There are more than 50pts that can be earned, but 50pts are in the Syllabus. Enjoy.

Problem 25 Determine if $\{\sin^2 x, \cos^2 x, 1\}$ is linearly independent on $\mathbb{R}$.

$\boxed{\sin^2 x + \cos^2 x - 1 = 0 \quad \therefore \ \underline{\text{not}} \ LI}$

Problem 26 Suppose $y_1 = t^2e^t$ and $y_2 = te^t$ and $y_3 = e^t$. Calculate $W(y_1, y_2, y_3; t)$.

$\boxed{W(y_1, y_2, y_3; t) = -2e^{3t}}$

Problem 27 Let us define $L[y] = y''' + y' + xy$. Let $y_1 = \sin x$ and $y_2 = x$.

(a.) Calculate $L[y_1]$ and $L[y_2]$,

$\boxed{L[y_1] = x\sin x \ , \ L[y_2] = 1 + x^2}$

(b.) Solve $L[y] = 2x\sin x - x^2 - 1$,

$\boxed{y = 2\sin x - x}$

(c.) Solve $L[y] = 4x^2 + 4 - 6x\sin x$.

$\boxed{y = 4x - 6\sin x}$

I am not asking for the general solution in the problem above

Problem 28 (IVP) Solve $y'' + 2y' + y = 0$ where $y(0) = 1$ and $y'(0) = -3$ given $y' = dy/dx$.

$\boxed{y = e^{-x} - 2xe^{-x}}$

Problem 29 (IVP) Solve $(D^2 - 2D + 2)[y] = 0$ given $y(\pi) = e^{\pi}$ and $y'(\pi) = 0$ and $D = d/dt$.

$\boxed{y = e^t(\sin t - \cos t)}$

Problem 30 (Constant Coefficient Problems) Find the general solution for each DEqn below: assume the independent variable in each solution is denoted by x.

(a.) $y'' - y' - 11y = 0$

$$\boxed{y = c_1 \exp\left(\left(\frac{1+3\sqrt{5}}{2}\right)x\right) + c_2 \exp\left(\left(\frac{1-3\sqrt{5}}{2}\right)x\right)}$$

(b.) $4w'' + 20w' + 25w = 0$

$$\boxed{W = c_1 e^{-\frac{5x}{2}} + c_2 x e^{-\frac{5x}{2}}}$$

(c.) $y'' - 8y' + 7y = 0$

$$\boxed{y = c_1 e^{x} + c_2 e^{7x}}$$

(d.) $z'' + 10z' + 25z = 0$

$$\boxed{z = c_1 e^{-5x} + c_2 x e^{-5x}}$$

(e.) $u'' + 7u = 0$

$$\boxed{u = c_1 \cos(\sqrt{7}x) + c_2 \sin(\sqrt{7}x)}$$

(f.) $y'' + 10y' + 41y = 0$

$$\boxed{y = c_1 e^{-5x} \cos(4x) + c_2 e^{-5x} \sin(4x)}$$

(g.) $y''' + 2y'' - 8y' = 0$

$$\boxed{y = c_1 + c_2 e^{-4x} + c_3 e^{2x}}$$

(h.) $u''' - 9u'' + 27u' - 27u = 0$

$$\boxed{u = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}}$$

(i.) $y^{(4)} + 4y'' + 4y = 0$

$$\boxed{\begin{gathered} y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + \\ + c_3 x \cos\sqrt{2}x + c_4 x \sin(\sqrt{2}x) \end{gathered}}$$

(j.) $(D^4 - 36)[y] = 0$

$$\boxed{\begin{gathered} y = c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x + \\ + c_3 \exp(-\sqrt{6}x) + c_4 \exp(\sqrt{6}x) \end{gathered}}$$

(k.) $((D+1)^2 + 36)^2[y] = 0$

$$\boxed{\begin{gathered} y = c_1 e^{-x} \cos 6x + c_2 e^{-x} \sin 6x + \\ + c_3 x e^{-x} \cos 6x + c_4 x e^{-x} \sin 6x \end{gathered}}$$

Problem 31 Solve $y^{(4)} + 2y''' + 10y'' + 18y' + 9y = 0$ given that $y = \sin(3x)$ is a solution.

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 \cos 3x + c_4 \sin 3x}$$

Problem 32 Use the method of annihilators to set-up (but do not determine explicitly) the particular solutions for:

$y_p = (Ax^2 + Bx^3 + Cx^4)e^x$	(a.) $y'' - 2y' + y = x^2e^x$
$y_p = e^x(A\cos 4x + B\sin 4x)$	(b.) $y'' + 16y = e^x\cos(4x)$
$y_p = Ax + Bx^2 + Cx^3 + Dx^4 + Ee^x\cos 4x + Fe^x\sin 4x.$	(c.) $y''' + y' = x^3 + e^x\cos(4x)$
$y_p = Ax + Bx\cos 6x + Cx\sin 6x + De^{2x}$	(d.) $y'''+36y' = \cos^2(3x)+e^x\cosh(x)$

Problem 33 Consider $y'' - 6y' - 4y = 4\sin(3t) - t^2e^{3t} + \frac{1}{t}$. Can we solve this via the method of undetermined coefficients ? If so, suggest a form for the particular solution.

NOT POSSIBLE.

Problem 34 (nonhomogeneous problems) Find the general solution for each DEqn below:

$y = c_1e^t + c_2e^{-t} + 11t - 1$	(a.) $y'' - y = 1 - 11t.$
$y = c_1\cos 4x + c_2\sin 4x + \frac{e^x}{65}(\cos 4x + 8\sin 4x)$	(b.) $y'' - 9y = t^2 + e^t + 1.$
$y = c_1e^{-t} + c_2e^{-2t} + \frac{t}{2} - \frac{1}{4}$	(c.) $y'' + 3y' + 2y = t + 1.$
$y = c_1\cos t + c_2\sin t + \frac{t\sin t}{2} + \frac{e^t}{2}$	(d.) $y'' + y = \cos t + e^t$

Problem 35 Solve $z'' + z = 2e^{-x}$ given $z(0) = 0$ and $z'(0) = 0$.

$z = \sin x - \cos x + e^{-x}$

Problem 36 Solve $y'' + 2y' + y = \dfrac{e^{-x}}{x+1}$.

$y = c_1e^{-x} + c_2xe^{-x} - xe^{-x} + e^{-x}(1+x)\ln|1+x|$

Problem 37 Solve $y'' + 4y = \tan(2x)$.

$$\boxed{y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{1}{4}\cos(2x)\ln|\sec 2x + \tan 2x|}$$

Problem 38 Solve $y'' + 3y' + 2y = \sin(e^x)$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin(e^x)}$$

Problem 39 Solve the following cauchy euler problems. Give your solution as a real linear combination of the real-valued functions in the fundamental solution set.

$$\boxed{y = c_1\sqrt{x} + c_2 \ln(x)\sqrt{x}}$$ (a.) $4x^2y'' + y = 0$

$$\boxed{y = c_1 x^2\cos(\ln x) + c_2 x^2 \sin(\ln(x))}$$ (b.) $x^2y'' - 3xy' + 5y = 0$

$$\boxed{y = c_1\sqrt{x} + \frac{c_2}{x}}$$ (c.) $2x^2y'' + 3xy' - y = 0$

$$\boxed{y = c_1 x + c_2 x \ln x + \frac{c_3}{x}}$$ (d.) $x^3y''' + 2x^2y'' - xy' + y = 0$

Problem 40 Find an integral solution for $x > 0$ to the Cauchy Euler problem $x^2y'' + xy' + 9y = g$ where g is a continuous function.

$$\boxed{y = c_1\cos(3\ln x) + c_2 \sin(3\ln x) + \cos(3\ln(x))\int \frac{-\sin(3\ln x)\, g\, dx}{3x} + \sin(3\ln x)\int \frac{\cos 3\ln x\, g\, dx}{3x}}$$

Problem 41 (based on Cook section 3.7) Suppose $T = D$ and $S = 3 - x^2D$. Solve

$$\boxed{y = c + \int k \exp\left(\frac{-3}{x}\right)dx}$$ (a.) $ST[y] = 0$,

$$\boxed{y = k + Me^{-3/x}}$$ (b.) $TS[y] = 0$.

Problem 42 A spring has mass $m = 1$, coefficient of damping $\beta = 4$ and a spring constant $k = 5$. Find the general solution of Newton's Second Law.

$$\boxed{x = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t}$$

Problem 43 Newton's Law for a retarded spring-mass system with external force f yield

$$m\ddot{x} + \beta\dot{x} + kx = f$$

Given $m = 2$, $b = 0$, $k = 32$ and $f = 68e^{-2t}\cos(4t)$ find the equation of motion given the system has initial conditions $x(0) = \dot{x}(0) = 0$.

$$\boxed{x = \frac{-\cos 4t}{2} + \frac{9\sin 4t}{4} + \left(\frac{\cos 4t}{2} - 2\sin 4t\right)e^{-2t}}$$

Problem 44 Consider Newton's Second Law for mass-spring system under a sinusoidal force:

$$\ddot{x} + \omega^2 x = F_o \cos\gamma t$$

given $x(0) = \dot{x}(0) = 0$. Here $F_o, \omega\gamma$ are nonzero constants.

(a.) Find $x(t)$ given that $\gamma \neq \omega$

$$\boxed{x = \frac{F_o}{\omega^2 - \gamma^2}(\cos\gamma t - \cos\omega t)}$$

(b.) Calculate $x_r(t) = \lim_{\gamma\to\omega} x(t)$

$$\boxed{x_r(t) = \frac{t F_o \sin(\omega t)}{2\omega}}$$

(c.) Contrast the motion of $x(t)$ and $x_r(t)$ as $t \to \infty$

$x(t)$ bounded as $t \to \infty$
$x_r(t)$ unbounded as $t \to \infty$

Problem 45 Kirchoff's Voltage Law for an RLC-circuit with voltage source $\mathcal{E}$ is given by

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = \mathcal{E}$$

Since $I = \frac{dQ}{dt}$ we find $L\ddot{Q} + R\dot{Q} + Q/C = \mathcal{E}$. Given that $L = 1$ and $R = 2$ and $C = 0.25$ and $\mathcal{E} = 50\cos t$ find the charge Q as a function of time t given the initial charge and current are both zero for $t = 0$.

$$\boxed{Q = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t) + \frac{100}{13}\sin(t) + \frac{150}{13}\cos t}$$

Problem 46 If we study the motion of an spring

$$m\ddot{x} + \beta\dot{x} + kx = F$$

such that $\beta^2 - 4mk < 0$ then it is known as **underdamped motion**. If the external force $F = F_o\cos(\gamma t)$ then we find the motion is dominated by the particular solution as $t \to \infty$. Let $\omega = \frac{\sqrt{4mk-\beta^2}}{2m}$, then the homogeneous solution $x_h(t) =$

$e^{\frac{-\beta t}{2m}}(c_1 \cos(\omega t) + c_2 \sin(\omega t)) \to 0$ as $t \to \infty$. It can be shown that the particular solution of such a system is given by

$$x_p = \frac{F_o \sin(\gamma t + \phi)}{\sqrt{(k - m\gamma^2)^2 + \beta^2\gamma^2}}$$

where ϕ is a constant. Find the frequency γ which maximizes the magnitude of x_p in the following cases:

(a.) $m = 1/2$ and $k = 19$ and $\beta = 1$

$$\boxed{\gamma_c = 6}$$

(b.) $m = 1$ and $k = 2$ and $\beta = \sqrt{6}$.

$$\boxed{\gamma_c = 0}$$

Problem 47 Solve the integral $\int(x^3 + 2x)e^x\,dx = y$ by solving $\frac{dy}{dx} = (x^{\overset{3}{\cancel{2}}} + 2x)e^x$ via the method of <u>undetermined coefficients</u>

$$\boxed{\int(x^3+2x)e^x dx = (x^3 - 3x^2 + 8x - 8)e^x + C_1 \quad \text{or} \quad \int(x^2+2x)e^x = x^2e^x + C_1}$$

Problem 48 Find the general solution of

(a.) $y'' + y' - 2y = x^2 + 3x$

$$\boxed{y = c_1e^{-2x} + c_2e^x - \frac{1}{2}x^2 - 2x - \frac{3}{2}}$$

(b.) $y'' + y' - 2y = x\cosh x$

$$\boxed{y = c_1e^{-2x} + c_2e^x + \frac{x^2e^x}{12} - \frac{xe^{-x}}{4} - \frac{xe^x}{18} + \frac{e^{-x}}{8}}$$

(c.) $y'' + y' - 2y = 2(x^2 + 3x) + 10\cosh(x) \cdot x$

$$\boxed{y = c_1e^{-2x} + c_2e^x - x^2 - 4x - 3 + 10\left(\frac{x^2e^x}{12} - \frac{xe^{-x}}{4} - \frac{xe^x}{18} + \frac{e^{-x}}{8}\right)}$$

Mission 2 Solution

P25 $\{\sin^2 x, \cos^2 x, 1\}$ is not LI on $\mathbb{R}$ since we know $\cos^2 x + \sin^2 x = 1$.

P26

$$W(t^2e^t, te^t, e^t; t) = \det\begin{bmatrix} t^2e^t & te^t & e^t \\ (t^2+2t)e^t & (t+1)e^t & e^t \\ (t^2+4t+2)e^t & (t+2)e^t & e^t \end{bmatrix}$$

$$= (e^t)^3 \det\begin{bmatrix} t^2 & t & 1 \\ t^2+2t & t+1 & 1 \\ t^2+4t+2 & t+2 & 1 \end{bmatrix} \quad \begin{matrix} r_2 - r_1 \\ r_3 - r_1 \end{matrix}$$

$$= e^{3t} \det\begin{bmatrix} t^2 & t & 1 \\ 2t & 1 & 0 \\ 4t+2 & 2 & 0 \end{bmatrix}$$

$$= e^{3t} \det\begin{bmatrix} 2t & 1 \\ 4t+2 & 2 \end{bmatrix}$$

$$= e^{3t}\left(4t - (4t+2)\right)$$

$$= \boxed{-2e^{3t}}$$

Remark: this shows that $\{t^2e^t, te^t, e^t\}$ is LI on $\mathbb{R}$.

P27 $L[y] = y''' + y' + xy$, $y_1 = \sin x$, $y_2 = x$

(a.) $L[y_1] = y_1''' + y_1' + xy_1 = -\cos x + \cos x + x\sin x = \boxed{x\sin x}$.

$L[y_2] = y_2''' + y_2' + xy_2 = 0 + 1 + x(x) = \boxed{1 + x^2}$.

(b.) $L[y] = 2x\sin x - x^2 - 1 = 2(x\sin x) - (x^2+1) = 2L[y_1] - L[y_2]$

$= L[2y_1 - y_2]$

select $\boxed{y = 2\sin x - x}$.

(c.) $L[y] = 4x^2 + 4 - 6x\sin x$, Let $y = 4y_2 - 6y_1$

then $L[y] = 4L[y_2] - 6L[y_1] = 4(x^2+1) - 6x\sin x \overset{\checkmark}{=} 4x^2 + 4 - 6x\sin x$

$\therefore$ $\boxed{y = 4x - 6\sin x}$

P28 $\quad y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = -3, \quad y' = \frac{dy}{dx}$

$$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \quad \therefore \quad y = c_1 e^{-x} + c_2 x e^{-x}.$$

$$y'(x) = -c_1 e^{-x} + c_2 (1-x) e^{-x}$$

$$y(0) = c_1 = 1.$$

$$y'(0) = -1 + c_2(1-0) = -3 \Rightarrow c_2 = -2.$$

$$\therefore \boxed{y = e^{-x} - 2xe^{-x}} = (1-2x)e^{-x}.$$

P29 $\quad (D^2 - 2D + 2)[y] = 0$ given $y(\pi) = e^{\pi}$ and $y'(\pi) = 0$, $D = \frac{d}{dt}$

$$[(D-1)^2 + 1][y] = 0 \quad \therefore \quad y = c_1 e^t \cos t + c_2 e^t \sin t$$

$$y' = c_1 e^t(\cos t - \sin t) + c_2 e^t(\sin t + \cos t)$$

$$y(\pi) = c_1 e^{\pi} \underbrace{\cos \pi}_{-1} + c_2 e^{\pi} \underbrace{\sin \pi}_{0} \Rightarrow -c_1 e^{\pi} = e^{\pi} \Rightarrow c_1 = -1.$$

$$y'(\pi) = -e^{\pi}(-1) + c_2 e^{\pi}(0-1) = 0 \Rightarrow +1 - c_2 = 0 \Rightarrow c_2 = 1.$$

$$\therefore \boxed{y = e^t(\sin t - \cos t)}$$

P30 Solve each DEqn using x as indep. variable.

(a.) $y'' - y' - 11y = 0$

$$\lambda^2 - \lambda - 11 = \left(\lambda - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - \tfrac{44}{4} =$$

$$= \left(\lambda - \tfrac{1}{2}\right)^2 - \tfrac{45}{4} \Rightarrow \lambda = \tfrac{1}{2} \pm \tfrac{\sqrt{45}}{2}$$

$$= \lambda = \tfrac{1}{2} \pm \tfrac{3\sqrt{5}}{2}$$

$$\boxed{y = c_1 \exp\left(\left(\tfrac{1}{2} + \tfrac{3\sqrt{5}}{2}\right)x\right) + c_2 \exp\left(\left(\tfrac{1}{2} - \tfrac{3\sqrt{5}}{2}\right)x\right)}$$

P30 continued

(b.) $4w'' + 20w' + 25w = 0$

$$4\lambda^2 + 20\lambda + 25 = 0$$

$$4\left(\lambda^2 + 5\lambda + \frac{25}{4}\right) = 0$$

$$4\left(\left(\lambda + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{25}{4}\right) = 0 \quad \therefore \lambda = \frac{-5}{2} \text{ twice}$$

$$\boxed{w = c_1 \exp\left(\frac{-5x}{2}\right) + c_2 x \exp\left(\frac{-5x}{2}\right)}$$

(c.) $y'' - 8y' + 7y = 0$

$$\lambda^2 - 8\lambda + 7 = (\lambda - 1)(\lambda - 7) = 0 \quad \therefore \lambda_1 = 1, \quad \lambda_2 = 7.$$

$$\boxed{y = c_1 e^{x} + c_2 e^{7x}}$$

(d.) $z'' + 10z' + 25z = 0$

$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0 \quad \therefore \boxed{z = c_1 e^{-5x} + c_2 x e^{-5x}}$$

(e.) $u'' + 7u = 0$

$$\lambda^2 + 7 = 0 \quad \therefore \lambda = \pm i\sqrt{7} \quad \therefore \boxed{u = c_1 \cos(\sqrt{7}\,x) + c_2 \sin(\sqrt{7}\,x)}$$

(f.) $y'' + 10y' + 41y = 0$

$$\lambda^2 + 10\lambda + 41 = (\lambda + 5)^2 - 25 + 41 = (\lambda + 5)^2 + 16 = 0$$

$$\lambda = -5 \pm 4i \quad \therefore \boxed{y = c_1 e^{-5x} \cos 4x + c_2 e^{-5x} \sin 4x}$$

(g.) $y''' + 2y'' - 8y' = 0$

$$\lambda^3 + 2\lambda^2 - 8\lambda = \lambda(\lambda^2 + 2\lambda - 8) = \lambda(\lambda + 4)(\lambda - 2) = 0$$

$$\therefore \boxed{y = c_1 + c_2 e^{-4x} + c_3 e^{2x}}$$

P30 continued

(h.) $u''' - 9u'' + 27u' - 27u = 0$

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = (\lambda - 3)^3 = 0$$

$$\therefore \boxed{u = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}}$$

(i.) $y^{(4)} + 4y'' + 4y = 0$

$$\lambda^4 + 4\lambda^2 + 4 = (\lambda^2 + 2)^2 = 0 \therefore \lambda = \pm i\sqrt{2} \text{ twice.}$$

$$\boxed{y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + c_3 x \cos(\sqrt{2}x) + c_4 x \sin(\sqrt{2}x)}$$

(j.) $(D^4 - 36)[y] = 0$

$$\lambda^4 - 36 = (\lambda^2 + 6)(\lambda^2 - 6) = (\lambda^2 + 6)(\lambda + \sqrt{6})(\lambda - \sqrt{6}) = 0$$

$$\therefore \boxed{y = c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x + c_3 e^{-\sqrt{6}x} + c_4 e^{\sqrt{6}x}}$$

(k.) $((D+1)^2 + 36)^2 [y] = 0$

$\lambda = -1 \pm 6i$ twice

$$\boxed{y = c_1 e^{-x} \cos 6x + c_2 e^{-x} \sin 6x + c_3 x e^{-x} \cos 6x + c_4 x e^{-x} \sin 6x}$$

P31 Solve $y^{(4)} + 2y''' + 10y'' + 18y' + 9y = 0$ given $y = \sin(3x)$ is solution.

$$L = D^4 + 2D^3 + 10D^2 + 18D + 9 \quad \text{and} \quad \underbrace{L[\sin 3x] = 0}$$

tells us D^2+9 is a factor of L

$$\begin{array}{r l}
 & D^2 + 2D + 1 \\
D^2+9 \,\big) & D^4 + 2D^3 + 10D^2 + 18D + 9 \\
 & -(D^4 \qquad + 9D^2) \\
 & \qquad 2D^3 + D^2 + 18D + 9 \\
 & \qquad -(2D^3 \qquad + 18D) \\
 & \qquad\qquad D^2 + \qquad 9 \\
 & \qquad\qquad -(D^2 \qquad +9) \\
 & \qquad\qquad\qquad 0
\end{array}$$

Hence, $L = (D^2 + 2D + 1)(D^2+9) = (D+1)^2(D^2+9)$

and we find $\boxed{y = c_1e^{-x} + c_2xe^{-x} + c_3\cos 3x + c_4 \sin 3x}$

is the general sol$^{\underline{n}}$ to $L[y] = 0$.

P32 Use method of annihilators to set-up y_p for,

(a.) $\underbrace{y'' - 2y' + y} = x^2e^x$

$$(D^2 - 2D + 1)[y] = x^2e^x$$

$$(D-1)^2[y] = x^2e^x \qquad (y_h = c_1e^x + c_2xe^x)$$

Let $A = (D-1)^3$ to kill off x^2e^x.

$$(D-1)^3(D-1)^2[y] = (D-1)^3[x^2e^x] = 0$$

$$\therefore \ (D-1)^5[y] = 0$$

$$y = \underbrace{c_1e^x + c_2xe^x}_{y_h} + \underbrace{c_3x^2e^x + c_4x^3e^x + c_5x^4e^x}_{y_p}$$

$$\therefore \ \boxed{y_p = (Ax^2 + Bx^3 + Cx^4)e^x}$$

P32 continued

(b.) $\underbrace{y'' + 16y}_{(D^2+16)[y]} = \underbrace{e^x \cos(4x)}_{A = (D-1)^2 + 16}$

$y_h = C_1 \cos 4x + C_2 \sin 4x$

$A(D^2+16)[y] = A[e^x \cos 4x] = 0$

$((D-1)^2+16)(D^2+16)[y] = 0$

$y = \underbrace{C_1 e^x \cos 4x + C_2 e^x \sin 4x}_{y_p} + \underbrace{C_3 \cos 4x + C_4 \sin 4x}_{y_h}$

$\therefore \quad \boxed{y_p = e^x (A \cos 4x + B \sin 4x)}$

(c.) $y''' + y' = \cancel{x}^3 + e^x \cos(4x)$

$\underbrace{(D^3 + D)}_{L}[y] = \underbrace{x^3 + e^x \cos 4x}_{A = D^4((D-1)^2+16)} = g \qquad (\underbrace{y_1 = 1, y_2 = \cos x, y_3 = \sin x}_{\text{homog. sol}^{\text{ns}}})$

$AL[y] = A[g] = 0$

$D(D^2+1)D^4((D-1)^2+16)[y] = 0$

$D^5(D^2+1)((D-1)^2+16)[y] = 0$

$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4 + C_6 \cos x + C_7 \sin x + C_8 e^x \cos 4x + C_9 e^x \sin 4x$

Thus,

$\boxed{y_p = Ax + Bx^2 + Cx^3 + Dx^4 + Ee^x \cos 4x + Fe^x \sin 4x}$

P32 continued

(d.) $y''' + 36y' = \underbrace{\cos^2(3x) + e^x \cosh(x)}_{g}$

$\underbrace{D(D^2+36)[y]}_{L} = g$

$g = \frac{1}{2}(1+\cos(6x)) + e^x \frac{1}{2}(e^x + e^{-x})$

$g = 1 + \frac{1}{2}\cos(6x) + \frac{1}{2}e^{2x}$

$\Rightarrow \quad A = D(D^2+36)(D-2)$

$AL[y] = A[g] = 0$

$D(D^2+36)(D-2)D(D^2+36)[y] = 0$

$D^2(D^2+36)^2(D-2)[y] = 0$

$y = \underbrace{c_1 + c_2 x + c_3\cos 6x + c_4 \sin 6x}_{y_h} + c_5 x\cos 6x + c_6 x \sin 6x + c_7 e^{2x}$

Hence,

$$\boxed{y_p = Ax + Bx\cos 6x + Cx\sin 6x + De^{2x}}$$

P33 Consider $y'' - 6y' - 4y = 4\sin(3t) - t^2 e^{3t} + \frac{1}{t}$

Can we solve this via the method of undetermined coeff?

No. It is not possible to find $A = P(D)$ for some polynomial P and $D = d/dt$ such that $A[\frac{1}{t}] = 0$. The function $y = \frac{1}{t}$ is not a solution to a constant coeff. ODE.

$\boxed{P34}$ Find the general solution

(a.) $y'' - y = 1 - 11t$

$\lambda^2 - 1 = 0 \therefore \lambda = \pm 1 \Rightarrow \underline{y_h = c_1 e^t + c_2 e^{-t}}$. (no overlap)

Then $y_p = A + Bt$ so $y_p' = B$ and $y_p'' = 0$

$y_p'' - y_p = 1 - 11t \Rightarrow -y_p = 1 - 11t \therefore \underline{y_p = 11t - 1}$.

$y = y_h + y_p$

$\therefore \boxed{y = c_1 e^t + c_2 e^{-t} + 11t - 1}$

(b.) $y'' + 16y = e^x \cos(4x)$

$\lambda^2 + 16 = 0 \therefore \lambda = \pm 4i \Rightarrow \underline{y_h = c_1 \cos 4x + c_2 \sin 4x}$.

(no over lap)

$y_p = e^x(A\cos 4x + B\sin 4x)$

$y_p' = e^x(A\cos 4x + B\sin 4x) + e^x(-4A\sin 4x + 4B\cos 4x)$

$y_p' = e^x((A+4B)\cos 4x + (B-4A)\sin 4x)$

$y_p'' = e^x((A+4B)\cos 4x + (B-4A)\sin 4x) + e^x(-4(A+4B)\sin 4x + 4(B-4A)\cos 4x)$

$y_p'' = e^x([A+4B+4(B-4A)]\cos 4x + [B-4A-4(A+4B)]\sin 4x)$

$y_p'' = e^x([8B-15A]\cos 4x + [-8A-15B]\sin 4x)$

Thus,

$y_p'' + 16y_p = e^x([8B-15A+16A]\cos 4x + [-8A-15B+16B]\sin 4x) = e^x \cos 4x$

$\therefore \underbrace{8B + A = 1}_{\text{from } e^x\cos 4x}$ and $\underbrace{-8A + B = 0}_{\text{from } e^x \sin 4x}$ (equating coefficients)

Then $B = 8A \Rightarrow 8B + A = 8(8A) + A = 1 \therefore \underline{A = 1/65}$ & $\underline{B = 8/65}$.

$$\boxed{y = c_1\cos 4x + c_2 \sin 4x + e^x\left(\frac{\cos 4x}{65} + \frac{8\sin 4x}{65}\right)}$$

P34 continued

(c.) $y'' + 3y' + 2y = t+1$

$\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) \Rightarrow y_h = c_1 e^{-t} + c_2 e^{-2t}$ (no overlap)

$$\left.\begin{aligned} y_p &= At + B \\ y_p' &= A \\ y_p'' &= 0 \end{aligned}\right\} \quad \begin{aligned} y_p'' + 3y_p' + 2y_p &= t+1 \\ 3A + 2(At+B) &= t+1 \end{aligned}$$

$$\left.\begin{aligned} 1\,|\ & 3A + 2B = 1 \\ t\,|\ & 2A = 1 \end{aligned}\right\} \text{ Equating coefficients of } 1 \,\&\, t$$

Thus $\underline{A = 1/2}$ and $2B = 1 - 3A = 1 - 3/2 = -1/2 \therefore \underline{B = -1/4}$.

$$\therefore \boxed{y = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}t - \frac{1}{4}}$$

(d.) $y'' + y = \cos t + e^t$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \therefore \underbrace{y_h = c_1 \cos t + c_2 \sin t}$

overlap, so I'll use method of annihilators to set-up y_p

$$(D^2+1)[y] = \underbrace{\cos t + e^t}_{A = (D^2+1)(D-1)}$$

$$(D^2+1)(D-1)(D^2+1)[y] = (D^2+1)(D-1)[\cos t + e^t] = 0$$

$$(D^2+1)^2(D-1)[y] = 0$$

$$y = \underbrace{c_1 \cos t + c_2 \sin t}_{y_h} + \underbrace{c_3 t \cos t + c_4 t \sin t + c_5 e^t}_{y_p}$$

$$y_p = t(A\cos t + B\sin t) + Ce^t$$

now I'm <u>not</u> done

I have to plug-in y_p to specify A, B, C

P34 continued

(d.) $y'' + y = \cos t + e^t$

$$y_p = t(A\cos t + B\sin t) + Ce^t$$

$$y_p' = A\cos t + B\sin t + t(-A\sin t + B\cos t) + Ce^t$$

$$y_p' = (A + tB)\cos t + (B - tA)\sin t + Ce^t$$

$$y_p'' = B\cos t - (A+tB)\sin t + (-A)\sin t + (B - tA)\cos t + Ce^t$$

$$+\begin{pmatrix} y_p'' = (2B - tA)\cos t + (-2A - tB)\sin t + Ce^t \\ y_p = tA\cos t + tB\sin t + Ce^t \end{pmatrix}$$

$$y_p'' + y_p = 2B\cos t + 2A\sin t + 2Ce^t = \cos t + e^t$$

$\cos t$	$2B = 1$	$\therefore B = 1/2$
$\sin t$	$-2A = 0$	$\therefore A = 0$
e^t	$2C = 1$	$\therefore C = 1/2$

Thus,

$$\boxed{y = c_1\cos t + c_2\sin t + \tfrac{1}{2}t\sin t + \tfrac{1}{2}e^t}$$

$\boxed{P35}$ $z'' + z = 2e^{-x}$ given $z(0) = z'(0) = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \quad \therefore \quad \underline{z_h = c_1 \cos x + c_2 \sin x}$ (no overlap)

$$\left.\begin{aligned} z_p &= Ae^{-x} \\ z_p' &= -Ae^{-x} \\ z_p'' &= Ae^{-x} \end{aligned}\right\} \qquad \begin{aligned} z_p'' + z_p &= 2e^{-x} \\ 2Ae^{-x} &= 2e^{-x} \\ \underline{A = 1} . \end{aligned}$$

$$z = c_1 \cos x + c_2 \sin x + e^{-x}$$

$$z' = -c_1 \sin x + c_2 \cos x - e^{-x}$$

$$z(0) = c_1 + 1 = 0 \Rightarrow c_1 = -1.$$

$$z'(0) = c_2 - 1 = 0 \Rightarrow c_2 = 1.$$

$$\therefore \quad \boxed{z = \sin x - \cos x + e^{-x}}$$

$\boxed{P36}$ $y'' + 2y' + y = \dfrac{e^{-x}}{x+1} = f$

$\lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0 \Rightarrow y_1 = e^{-x}$ and $y_2 = xe^{-x}$

Hence $y_1 y_2' - y_2 y_1' = e^{-x}(1-x)e^{-x} - xe^{-x}(-e^{-x}) = e^{-2x} = W$

Use variation of parameters, $y_p = V_1 y_1 + V_2 y_2$ where,

$$V_1 = \int \frac{-f y_2}{W} dx = \int \frac{-e^{-x} x e^{-x}}{e^{-2x}} dx$$

$$= \int \frac{-e^{-x} x e^{-x} dx}{(x+1)e^{-2x}}$$

$$= \int \frac{-x}{x+1} dx$$

$$= \int \left(-1 + \frac{1}{1+x}\right) dx$$

$$= \underline{-x + \ln|1+x|}.$$

$$V_2 = \int \frac{f y_1 dx}{W}$$

$$= \int \frac{e^{-x} e^{-x} dx}{(1+x)e^{-2x}}$$

$$= \int \frac{dx}{1+x}$$

$$= \ln|1+x|$$

$$\therefore \quad y = c_1 e^{-x} + e^{-x}(-x + \ln|1+x|) + xe^{-x} \ln|1+x| + c_2 x e^{-x}$$

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} - xe^{-x} + e^{-x}(1+x)\ln|1+x|}$$

Problem 37 Solve $y'' + 4y = \tan(2x)$.

$$\lambda^2 + 4 = 0 \quad \therefore \quad \lambda = \pm 2i$$

$$y_1 = \cos(2x), \quad y_2 = \sin(2x), \quad g = \tan(2x), \quad a = 1$$

$$\therefore \quad y_1 y_2' - y_1' y_2 = 2\cos^2(2x) + 2\sin^2(2x) = 2. = W$$

Variation of parameters is way to go here,

$$V_1 = \int \frac{-g y_2}{aW}\,dx = \int \frac{-\tan(2x)\sin(2x)}{2}\,dx$$

$$= \frac{-1}{2}\int \frac{\sin^2(2x)}{\cos(2x)}\,dx$$

$$= \frac{-1}{2}\int \left[\frac{1 - \cos^2(2x)}{\cos(2x)}\right]dx$$

$$= -\frac{1}{2}\int \left[\sec(2x) - \cos(2x)\right]dx$$

$$= \underline{\frac{-1}{4}\ln|\sec(2x) + \tan(2x)| + \frac{1}{4}\sin(2x)}.$$

$$V_2 = \int \frac{g y_1}{aW}\,dx = \int \frac{1}{2}\tan(2x)\cos(2x)\,dx = \int \frac{1}{2}\sin(2x)\,dx = \underline{\frac{-1}{4}\cos(2x)}.$$

Thus,

$$y_p = V_1 y_1 + V_2 y_2 = \cos(2x)\left[\frac{-1}{4}\ln|\sec 2x + \tan 2x| + \underline{\underline{\frac{1}{4}\sin 2x}}\right] + \sin(2x)\underline{\underline{\left(\frac{-\cos 2x}{4}\right)}}$$

cancel ☺

$$\therefore \quad \boxed{y = C_1\cos(2x) + C_2\sin(2x) - \frac{1}{4}\cos(2x)\ln|\sec 2x + \tan 2x|}$$

$\boxed{P38}$ Solve $y'' + 3y' + 2y = \sin(e^x) = f$

$$\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) \Rightarrow y_1 = e^{-x}, \quad y_2 = e^{-2x}$$

$$W = y_1 y_2' - y_2 y_1' = e^{-x}(-2e^{-2x}) - e^{-2x}(-e^{-x}) = \underline{-e^{-3x} = W}.$$

Use variation of parameters, $y_p = V_1 y_1 + V_2 y_2$ where

$$V_1 = \int \frac{-y_2 f}{W} dx =$$

$$= \int e^{3x} e^{-2x} \sin(e^x)\, dx$$

$$= \int e^x \sin(e^x)\, dx$$

$$= \int \sin(e^x)\, de^x$$

$$= \underline{-\cos(e^x) = V_1}.$$

$$V_2 = \int \frac{y_1 f}{W} dx$$

$$= -\int e^{3x} e^{-x} \sin(e^x)\, dx$$

$$= -\int e^x \sin(e^x)\, e^x dx \qquad y = e^x$$

$$= -\int \underbrace{y}_{u} \underbrace{\sin(y)\, dy}_{dV}$$

$$= -uV + \int V du$$

$$= +y\cos(y) + \int -\cos y\, dy$$

$$= y\cos y - \sin y \Rightarrow \underline{V_2 = e^x \cos(e^x) - \sin(e^x)}.$$

Therefore,

$$y = c_1 e^{-x} + c_2 e^{-2x} - e^{-x}\cos(e^x) + e^{-2x}\left(e^x \cos(e^x) - \sin(e^x)\right)$$

$$\therefore \boxed{y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin(e^x)}$$

Problem 39 Solve the following cauchy euler problems. Give your solution as a real linear combination of the real-value functions in the fundamental solution set.

a. $4x^2y'' + y = 0$

$$4R(R-1) + 1 = 0$$
$$4R^2 - 4R + 1 = 0$$
$$R^2 - R + \tfrac{1}{4} = (R - \tfrac{1}{2})^2 = 0$$

$$\boxed{y = c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x)}$$

we derived this in lecture via the 2^{nd} LI sol$^{\underline{n}}$ formula. could also from [illegible] in retrospect

b. $x^2y'' - 3xy' + 5y = 0$

$$R(R-1) - 3R + 5 = 0$$
$$R^2 - 4R + 5 = 0$$
$$(R-2)^2 + 1 = 0$$

$$R = 2 \pm i$$
$$x^{2+i} = x^2x^i$$
$$= x^2[\cos(\ln(x)) + i\sin \ln x]$$
$$\Rightarrow \boxed{y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x)}$$

c. $2x^2y'' + 3xy' - y = 0$

$$2R(R-1) + 3R - 1 = 0$$
$$2R^2 + R - 1 = 0$$
$$(2R-1)(R+1) = 0$$
$$R_1 = 1/2, \quad R_2 = -1$$

$$\boxed{y = c_1\sqrt{x} + c_2/x}$$

d. $x^3y''' + 2x^2y'' - xy' + y = 0$

$$R(R-1)(R-2) + 2R(R-1) - R + 1 = 0$$
$$(R-1)[R(R-2) + 2R - 1] = 0$$
$$(R-1)[R^2 - 1] = 0$$
$$(R-1)^2(R+1) = 0$$

$$\boxed{y = c_1x + c_2x\ln(x) + c_3/x}$$

e. $x^2y'' + 5xy' + 4y = 0$ with $y(1) = 2$ and $y'(1) = -3$

$$R(R-1) + 5R + 4 = 0$$
$$R^2 + 4R + 4 = 0$$
$$(R+2)^2 = 0$$
$$y = c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\right)\ln(x)$$

$$y'(x) = \frac{-2c_1}{x^3} + c_2\left(\frac{-2}{x^3}\ln(x) + \frac{1}{x^3}\right)$$

$$y(1) = c_1 = 2$$
$$y'(1) = -2c_1 + c_2 = -3 \Rightarrow c_2 = 1$$

$$\boxed{y = \frac{2}{x^2} + \frac{1}{x^2}\ln(x)}$$

Problem 34 Derive a formula to rewrite x^4D^4 as a polynomial in xD. Use the result to solve $x^4D^4[y] = 0$. Please use my notes for formulas for x^3D^3 and x^2D^2, also, use Leibniz product rule for best results.

$$x^4D^4(g) = x^4g''''$$

$$(xD)^4[g] = (xD)^2[xD[xg']]$$
$$= (xD)^2[x(g' + xg'')]$$
$$= (xD)[xD[xg' + x^2g'']]$$
$$= (xD)[x(g' + xg'' + 2xg'' + x^2g''')]$$
$$= (xD)[xg' + 3x^2g'' + x^3g''']$$
$$= x(g' + xg'' + 6xg'' + 3x^2g''' + 3x^2g''' + x^3g'''')$$

$$= (xD + 7x^2D^2 + 6x^3D^3 + x^4D^4)[g]$$

continued

Problem 40 11 [15pts] Find an integral solution for $x > 0$ to the Cauchy Euler problem $x^2y'' + xy' + 9y = g$ where g is a continuous function.

$R(R-1) + R + 9 = R^2 + 9 = 0 \Rightarrow R = \pm 3i$ for $y = x^R$ sol^n

Hence $y = x^{3i} = \cos(3\ln(x)) + i\sin(3\ln(x))$ provides fund. sol^n set of $y_1 = \cos(3\ln(x))$ and $y_2 = \sin(3\ln(x))$ (note, $x>0$ so no $|x|$ needed)

Observe $a = x^2$ and set $W[y_1, y_2] = W$.

$$W = y_1y_2' - y_2y_1' = \cos(3\ln(x))\cdot\frac{3}{x}\cos(3\ln(x)) - \sin(3\ln x)\cdot\frac{-3}{x}\sin(3\ln x) = \frac{3}{x}.$$

Thus $aW = x^2(3/x) = 3x$. Variation of parameters provides,

$$\boxed{y = c_1\cos(3\ln(x)) + c_2\sin(3\ln(x)) + \cos(3\ln x)\int\frac{-\sin(3\ln x)\,g\,dx}{3x} + \sin(3\ln x)\int\frac{\cos(3\ln x)\,g\,dx}{3x}}$$

P41 $T = D = d/dx$ and $S = 3 - x^2D$

(a.) Solve $ST[y] = 0$.

Let $T[y] = w$ then $Sw = 0 \Rightarrow 3w - x^2\frac{dw}{dx} = 0$

hence $\frac{dw}{dx} = \frac{3w}{x^2} \Rightarrow \int\frac{dw}{3w} = \int\frac{dx}{x^2}$

$$\Rightarrow \frac{1}{3}\ln|w| = \frac{-1}{x} + C/3$$

$$\Rightarrow \ln|w| = -3/x + C$$

$$\Rightarrow w = k\exp(-3/x) = T[y] = \frac{dy}{dx}$$

$$\therefore \boxed{y = c + \underbrace{\int k\exp\left(\frac{-3}{x}\right)dx}_{\text{integral solution is best I can do here.}}}$$

P41 continued

(b.) $TS[y]=0$

Let $S[y]=Q$ then $TQ=0 \Rightarrow \frac{dQ}{dx}=0$

Hence $Q=c_1=S[y]=3y-x^2\frac{dy}{dx}$

$$-x^2\frac{dy}{dx}=c_1-3y$$

$$\frac{dy}{dx}=\frac{3y-c_1}{x^2}=\frac{3(y-k)}{x^2}$$

$$\int\frac{dy}{y-k}=\int\frac{3dx}{x^2}$$

$$\ln|y-k|=\frac{-3}{x}+c_2$$

$$y=k+\exp(c_2-3/x)$$

$$\boxed{y=k+Me^{-3/x}}$$

P42 $m=1,\ \beta=4,\ k=5$ solve $m\ddot{x}+\beta\dot{x}+kx=0$

$$m\lambda^2+\beta\lambda+k=\lambda^2+4\lambda+5$$
$$=(\lambda+2)^2+1 \quad \therefore\ \lambda=-2\pm i.$$

$$\boxed{x(t)=c_1e^{-2t}\cos t+c_2e^{-2t}\sin t}$$

or as may be better for application ?

$$x(t)=Ae^{-2t}\sin(t+\phi).$$

(same sol$^{\underline{n}}$ different presentation of constants)

Problem 43 (Zill 5.1 springs and RLC-circuits) Work out problem 33 of section 5.1

$m = 2$, $k = 32$, $f = 68e^{-2t}\cos(4t)$, $b = 0$

$x(0) = x'(0) = 0$ Solve,

$$2\ddot{x} + 32x = 68e^{-2t}\cos 4t$$

$$\ddot{x} + 16x = 34e^{-2t}\cos 4t$$

$$\lambda^2 + 16 = 0 \Rightarrow \underline{x_h = c_1\cos 4t + c_2\sin 4t}$$

$$x_p = (A\cos 4t + B\sin 4t)e^{-2t}$$

$$\dot{x}_p = (-4A\sin 4t + 4B\cos 4t - 2A\cos 4t - 2B\sin 4t)e^{-2t}$$

$$\dot{x}_p = \big((-4A - 2B)\sin 4t + (4B - 2A)\cos 4t\big)e^{-2t}$$

$$\ddot{x}_p = \big(4(-4A - 2B)\cos 4t - 4(4B - 2A)\sin 4t\big)e^{-2t} + \left[-2(-4A - 2B)\sin 4t - 2(4B - 2A)\cos 4t\right]e^{-2t}$$

$$\ddot{x}_p = e^{-2t}\Big(\cos 4t\,[-16A - 8B - 8B + 4A] + \sin 4t\,[-16B + 8A + 8A + 4B]\Big)$$

Thus, $\ddot{x}_p + 16x_p = 34e^{-2t}\cos 4t$

$\underline{e^{-2t}\cos 4t}$ | $-12A - 16B + 16A = 34 \Rightarrow 4A - 16B = 34$

$\underline{e^{-2t}\sin 4t}$ | $-12B + 16A + 16B = 0 \Rightarrow \underline{16A + 4B = 0}$

$B = -4A.$

$4A - 16(-4A) = 34$

$68A = 34$

$\Rightarrow A = \frac{1}{2}, \quad B = -2$

Thus,

$$x(t) = c_1\cos 4t + c_2\sin 4t + e^{-2t}\left(\tfrac{1}{2}\cos 4t - 2\sin 4t\right)$$

Now fit $x(0) = 0$, $x'(0) = 0$ to obtain $c_1 + \frac{1}{2} = 0$, $4c_2 - 1 - 8 = 0$

$c_1 = -1/2$, $c_2 = 9/4$

$$\boxed{x(t) = \frac{-1}{2}\cos 4t + \frac{9}{4}\sin 4t + \left(\tfrac{1}{2}\cos 4t - 2\sin 4t\right)e^{-2t}}$$

Problem 44 (Zill 5.1 springs and RLC-circuits) Work out problem 39 of section 5.1

(a.) $x'' + \omega^2 x = F_0 \cos \gamma t$, $x(0) = x'(0) = 0$.

$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$

$\Rightarrow x_1 = \cos \omega t$, $x_2 = \sin \omega t$

$x_p = A\cos \gamma t + B \sin \gamma t$ (for $\gamma \neq \omega$)

$x_p'' = -\gamma^2 x_p$

$\Rightarrow x_p'' + \omega^2 x_p = F_0 \cos \gamma t$

$\Rightarrow (-\gamma^2 + \omega^2) x_p = F_0 \cos \gamma t$

$\therefore x_p = \dfrac{F_0 \cos \gamma t}{\omega^2 - \gamma^2}$

$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \dfrac{F_0 \cos \gamma t}{\omega^2 - \gamma^2}$

$x(0) = c_1 + \dfrac{F_0}{\omega^2 - \gamma^2}$ $\therefore$ $c_1 = \dfrac{-F_0}{\omega^2 - \gamma^2}$

$x'(0) = \omega c_2 = 0$ $\therefore c_2 = 0$

Thus, $\boxed{x(t) = \dfrac{F_0}{\omega^2 - \gamma^2}(\cos \gamma t - \cos \omega t)}$ (for $\gamma \neq \omega$)

(b.) Calculate $x(t) \longrightarrow ?$ as $\gamma \longrightarrow \omega$

$$\lim_{\gamma \to \omega} \left(\frac{F_0(\cos \gamma t - \cos \omega t)}{\omega^2 - \gamma^2} \right) \underset{\left(\frac{0}{0}\right)}{\overset{\mathcal{L}}{=}} \lim_{\gamma \to \omega} \left(\frac{-t F_0 \sin \gamma t}{-2\gamma} \right)$$

$\mathcal{L}$ is done by $d/d\gamma$ as γ is variable.

$$= \boxed{\frac{t F_0 \sin(\omega t)}{2\omega} = x_\gamma(t)}$$

(c.) $x(t)$ motion is <u>bounded</u> as $t \longrightarrow \infty$.

$x_\gamma(t)$ motion is <u>unbounded</u> since t blows up as $t \longrightarrow \infty$.

Problem 45 (Zill 5.1 springs and RLC-circuits) Work out problem 49 of section 5.1

$L = 1, \quad R = 2, \quad C = 0.25, \quad E(t) = 50\cos t$

$$Q'' + 2Q' + 4Q = 50\cos t$$

$$\lambda^2 + 2\lambda + 4 = 0$$

$$(\lambda+1)^2 + 3 = 0 \Rightarrow \lambda = -1 \pm i\sqrt{3}$$

It follows that,

$$Q_h(t) = c_1 e^{-t}\cos\sqrt{3}t + c_2 e^{-t}\sin\sqrt{3}t$$

and you can work out

$$Q_p = A\cos t + B\sin t$$

By our usual method and find,

$$\boxed{Q(t) = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(-\sqrt{3}t) + \frac{100}{13}\sin t + \frac{150}{13}\cos t}$$

P46 In my notes we derive the formula for critical frequency which maximizes the magnitude of $x_p = \dfrac{F_0\sin(\gamma t+\phi)}{\sqrt{(k-m\gamma^2)^2+\beta^2\gamma^2}}$

$$\gamma_c = \sqrt{\frac{k}{m} - \frac{\beta^2}{2m^2}}$$

(a.) $m = 1/2, \; k = 19, \; \beta = 1$

$$\gamma_c = \sqrt{2(19) - \tfrac{1}{2}(4)} = \sqrt{36} = \boxed{6 = \gamma_c}$$

(b.) $m = 1, \; k = 2, \; \beta = \sqrt{6}$

$$\gamma_c = \sqrt{2 - 6/2} = \sqrt{-1} \Rightarrow \boxed{\gamma = 0}$$ gives max amplitude

(were optimizing $\gamma \in [0,\infty)$, no critical frequency $\Rightarrow$ use end point) typo or typo

Problem 47 Solve the integral $\int (x^3 + 2x)e^x\,dx = y$ by solving $\frac{dy}{dx} = (x^2 + 2x)e^x$ via the method of undetermined coefficients

$$y' = \underbrace{x^2e^x + 2xe^x} \implies \lambda = 0 \quad \therefore \quad y_h = C_1$$

$$y_p = (Ax^2 + Bx + C)e^x$$

$$y_p' = (2Ax + B + Ax^2 + Bx + C)e^x = x^2e^x + 2xe^x$$

Equating Coeff:

$x^2e^x \mid \quad A = 1 \quad \therefore \quad \underline{A = 1}$ yep.

$xe^x \mid \quad 2A + B = 2 \quad \therefore \quad \underline{B = 0}.$

$e^x \mid \quad B + C = 0 \quad \therefore \quad \underline{C = -B = 0}.$

Thus,

$$y = C_1 + x^2e^x$$

$$\boxed{\int (x^2 + 2x)e^x\,dx = x^2e^x + C_1}$$

$$\neq$$

$$\int (x^3 + 2x)e^x\,dx = y \implies \frac{dy}{dx} = (x^3 + 2x)e^x$$

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^x$$

$$y_p' = y_p + (3Ax^2 + 2Bx + C)e^x = x^3e^x + 2xe^x$$

$x^3e^x \mid \quad \underline{A = 1}.$

$x^2e^x \mid \quad B + 3A = 0 \quad \therefore \quad \underline{B = -3}.$

$xe^x \mid \quad C + 2B = 2 \quad \therefore \quad \underline{C = 8}.$

$e^x \mid \quad D + C = 0 \quad \therefore \quad \underline{D = -8}.$

$$\boxed{\int (x^3 + 2x)e^x\,dx = (x^3 - 3x^2 + 8x - 8)e^x + C_1}$$

P48 Find general solution

(a.) $y'' + y' - 2y = x^2 + 3x$

$\lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) \Rightarrow y_h = c_1 e^{-2x} + c_2 e^{x}$ (no overlap)

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}\right\}$$

$y_p'' + y_p' - 2y_p = x^2 + 3x$

$2A + (2Ax+B) - 2(Ax^2+Bx+C) = x^2+3x$

$x^2(-2A) + x(2A-2B) + 1(2A+B-2C) = x^2+3x$ $*$

Equate coeff. of $*$

$x^2\rfloor \quad -2A = 1$

$x\rfloor \quad 2A - 2B = 3$

$1\rfloor \quad 2A + B - 2C = 0$

Thus $A = -1/2$, and $-1-2B = 3 \therefore -2B = 4 \Rightarrow B = -2$.

Then $2C = 2A + B = -1-2 \Rightarrow C = -3/2$.

$$\boxed{y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{2}x^2 - 2x - \frac{3}{2}}$$

P48 continued

(16.) $y'' + y' - 2y = x\cosh x = \frac{1}{2}xe^{x} + \frac{1}{2}xe^{-x}$

$y_h = c_1 e^{-2x} + c_2 e^{x}$ thus overlap is present so I'll use annihilator method to set-up y_p.

$$(D+2)(D-1)[y] = \underbrace{\tfrac{1}{2}xe^{x} + \tfrac{1}{2}xe^{-x}}_{A = (D-1)^2(D+1)^2}$$

$$(D-1)^3(D+1)^2(D+2)[y] = 0$$

$$y = c_1e^{x} + c_2xe^{x} + c_3x^2e^{x} + c_4e^{-x} + c_5xe^{-x} + c_6e^{-2x} \quad y_h$$

$$y_p = e^{x}(Ax + Bx^2) + Ce^{-x} + Dxe^{-x}$$

$$y_p' = e^{x}(Ax+Bx^2) + e^{x}(A+2Bx) - Ce^{-x} + D(1-x)e^{-x}$$

$$y_p' = e^{x}((A+2B)x + Bx^2 + A) - Ce^{-x} + D(1-x)e^{-x}$$

$$y_p'' = e^{x}((A+2B)x + Bx^2 + A + A + 2B + 2Bx) + Ce^{-x} + D(x-1-1)e^{-x}$$

$$y_p'' = e^{x}[(A+4B)x + Bx^2 + 2A + 2B] + Ce^{-x} + D(x-2)e^{-x}$$

Thus,

$$y_p'' + y_p' - 2y_p = x\cosh x = \tfrac{1}{2}xe^{x} + \tfrac{1}{2}xe^{-x}$$

(good)

$$e^{x}[(A+4B)x + Bx^2 + 2A + 2B + (2A+B)x + Bx^2 + A - 2Ax - 2Bx^2]$$
$$+ e^{-x}[C - \cancel{C} + C + D - 2D] + xe^{-x}[D - D - 2D] = \frac{xe^{x}}{2} + \frac{xe^{-x}}{2}$$

$$\Rightarrow \boxed{y = c_1e^{-2x} + c_2e^{x} + \frac{x^2e^{x}}{12} - \frac{xe^{-x}}{4} - \frac{xe^{x}}{18} + \frac{e^{-x}}{8}}$$

P48 continued

(c.) $y'' + y' - 2y = 2(x^2 + 3x) + 10x\cosh(x)$ $\ast$

By superposition,

(1) from (a.) $y_{p_1} = -\frac{1}{2}x^2 - 2x - \frac{3}{2}$ gives

$$y_{p_1}'' + y_{p_1}' - 2y_{p_1} = x^2 + 3x$$

(2) from (b.) $y_{p_2} = \frac{x^2 e^x}{12} - \frac{xe^{-x}}{4} - \frac{xe^x}{18} + \frac{e^{-x}}{8}$

gives $y_{p_2}'' + y_{p_2}' - 2y_{p_2} = x\cosh x$

Hence use

$y_p = 2y_{p_1} + 10y_{p_2}$ to solve ($\ast$)

$$\boxed{y = c_1 e^{-2x} + c_2 e^x - x^2 - 4x - 3 + 10\left(\frac{x^2e^x}{12} - \frac{xe^{-x}}{4} - \frac{xe^x}{18} + \frac{e^{-x}}{8}\right)}$$