Name: (please print name here →) .

MATH 334: MISSION 3: SERIES SOLUTIONS & SYSTEMS OF DEQNS [50PTS]

While complex solutions may be useful as in-between work the solutions requested on this assignment are real solutions. Please write answers in space provided, do not write scratch work on these pages. However, for Problems 69, 70, 71, 72, please write your solution below the problem statement printed out Thanks!

Problem 49 Find the first 4 nonzero terms in the power series solution about $x = 0$ for $z'' - x^2 z = 0$.

Problem 50 Find the complete power series solution (including a formula for the general coefficient) about $x = 0$ for:

$$y' - 2xy = 0.$$

Problem 51 Find the complete power series solution (including a formula for the general coefficient) about $x = 0$ for:

$$y'' - xy' + 4y = 0.$$

Problem 52 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$y'' - e^{2x}y' + (\cos x)y = 0, \qquad y(0) = -1,\ y'(0) = 1.$$

Problem 53 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$z'' + xz' + z = x^2 + 2x + 1.$$

Problem 54 Find the singularities of $x(x^2 + 2x + 2)y'' + (x^2 + 1)y' + 3y = 0$ and determine the largest open interval of convergence for a solution of the form $y = \sum_{n=0} a_n(x + 2)^n$. *Think. Do not try to solve this, I'm asking you about the interval of convergence, I'm* **not** *asking for what a_n are in particular*

Problem 55 Suppose we define $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n$. Show that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Problem 56 Suppose $\displaystyle\sum_{k=0}^{\infty} (a_{2k}x^{2k} + b_{2k+1}x^{2k+1} = e^x + \cos(x + 2)$. Find explicit formulas for a_{2k} and b_{2k+1} via Σ-notation algebra.

Problem 57 Find a power series solution to the integrals below:

(a.) $\displaystyle\int \frac{x^3 + x^6}{1 - x^3}\, dx$

(b.) $\displaystyle\int x^8 e^{x^3+2}\, dx$

Problem 58 Suppose $\frac{dx}{dt} = x + 4y$ and $\frac{dy}{dt} = x + y$. Find the general real solution via the e-vector method. Also, calculate e^{tA} for $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

Problem 59 Suppose $\frac{dx}{dt} = 2x + y$ and $\frac{dy}{dt} = 2y$. Find the general real solution via the generalized e-vector method.

Problem 60 Suppose $\frac{dx}{dt} = 4x - 3y$ and $\frac{dy}{dt} = 3x + 4y$. Find the general real solution via the e-vector method.

Problem 61 Suppose $\frac{dx}{dt} = 5x - 6y - 6z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 6y - 4z$. Find the general real solution via the e-vector method.

Problem 62 Suppose $\frac{dx}{dt} = 5x - 5y - 5z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 5y - 3z$. Find the general real solution via the e-vector method.

Problem 63 Suppose $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = 3y + z$ and $\frac{dz}{dt} = 3z$. Find the general real solution via the generalized e-vector method.

Problem 64 Suppose A is a 3×3 matrix with nonzero vectors $\vec{u}, \vec{v}, \vec{w}$ such that

$$A\vec{u} = 3\vec{u}, \qquad (A - 3I)\vec{v} = \vec{u}, \qquad A\vec{w} = 0.$$

Write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors.

Problem 65 Suppose $(A - \lambda I)\vec{u}_1 = 0$ and $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 3 + i\sqrt{2}$ and $\vec{u}_1 = [3 + i, 4 + 2i, 5 + 3i, 6 + 4i]^T$ and $\vec{u_2} = [i, 1, 2, 3 - i]^T$.

(a.) find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

(b.) extract four real solutions to write the general real solution (c_1, c_2, c_3, c_4 should be real in this answer)

Problem 66 Consider A is a 3×3 matrix for which there exist nonzero vectors v_1, v_2, v_3 such that:

$$Av_1 = 10v_1, \quad Av_2 = 10v_2, \quad Av_3 = 10v_3 + v_1$$

derive the general solution for $\frac{d\vec{r}}{dt} = A\vec{r}$ with appropriate arguments based on the matrix exponential.

Problem 67 Suppose $A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$. Calculate e^{tA}.
Also, solve $\frac{d\vec{r}}{dt} = A\vec{r}$ given that $\vec{r}(0) = (1, 2)$.

Problem 68 work out problem 15 of section 8.3.2 in Zill. That is, solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$

Problem 69 Consider the constant coefficient problem $ay'' + by' + cy = 0$ where a, b, c are real constants and $a \neq 0$. Use reduction of order with $x_1 = y$ and $x_2 = y'$ to rewrite the given second order ODE as a system of first order ODEs. Calculate the characteristic equation for your system and comment on how it compares to the usual characteristic equation for the given second order ODE.

The method outlined below is most meaningful in a larger discussion involving coordinate change for linear transformations. The coordinates $\vec{y} = P^{-1}\vec{x}$ are **eigencoordinates**. *A matrix is said to be* **diagonalizable** *iff there exists some coordinate change matrix P such that $P^{-1}AP = D$ where D is diagonalizable. Not all matrices are diagonalizable. We've seen this.*

Problem 70 To solve $\frac{d\vec{x}}{dt} = A\vec{x}$ in the case $A = \begin{bmatrix} -3 & 0 & -3 \\ 1 & -2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ by the following calculations:

(a) find the e-values and corresponding e-vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$. (you may use technology)

(b) construct $P = [\vec{u_1}|\vec{u_2}|\vec{u_3}]$ and calculate $P^{-1}AP$. (you may use technology)

(c) note the solution of $AP\vec{y} = \frac{d}{dt}[P\vec{y}] = P\frac{d\vec{y}}{dt}$ is easily found since multiplying by P^{-1} yields $P^{-1}AP\vec{y} = P^{-1}P\frac{d\vec{y}}{dt} = I\frac{d\vec{y}}{dt} = \frac{d\vec{y}}{dt}$. Solve $P^{-1}AP\vec{y} = \frac{d\vec{y}}{dt}$. (this should be really easy, just solve 3 first order problems, one at a time)

(d) $AP\vec{y} = \frac{d}{dt}[P\vec{y}]$ means $\vec{x} = P\vec{y}$ solves $\frac{d\vec{x}}{dt} = A\vec{x}$. Solve the original system by multiplying the solution from (3.) by P.

Problem 71 An ice tray has tiny holes between each of its three partititions such that the water can flow from one partition to the next. Let x, y, z denote the height of water in the three water troughs. The holes are designed such that the flow rate is proportional to the height of water above the adjacent trough. For example, supposing x and z are the edge troughs whereas y is in the middle we have $\frac{dx}{dt} = k(y - x)$. For simplicity of discussion suppose $k = 1$. Write the corresponding differential equations to find the water-level in the y and z troughs. If initially there is 3.0 cm of water in the x trough and none in the other two troughs then find the height in all three troughs as a function of time t. Discuss the steady state solution, is it reasonable?

Problem 72 Let a, b be constants which are some measure of the trust between two nations. Furthermore, let x be the military expenditure of Bobslovakia and let y be the military expenditure of the Leaf Village. Detailed analysis by strategically gifted ninjas reveal that

$$\frac{dx}{dt} = -x + 2y + a$$

$$\frac{dy}{dt} = 4x - 3y + b$$

Analyze possible outcomes for various initial conditions and values of a, b. Consider drawing an ab-plane to explain your solution(s). Is a stable peace without a run-away arms race possible given the analysis thus far?