I will select some subset of these problems to collect. The more you work, the more you know. The ordering of topics in these problems is rather lumpy, I've more or less cut and pasted multiple old homeworks and quizzes one after the other.

PP 183 Find the first three nonzero terms in the power series solutions of

$$\frac{dy}{dx} = x^2 + y^2$$

given $y(0) = 1$.

PP 184 Find the first three nonzero terms in the power series solutions of

$$\frac{dy}{dx} = \sin y + e^x$$

given $y(0) = 0$.

PP 185 Find the first three nonzero terms in the power series solutions of

$$x'' + tx = 0$$

given $x(0) = 1$ and $x'(0) = 0$.

PP 186 **Duffing's Equation.** A nonlinear spring with periodic forcing is described by

$$y'' + ky + ry^3 = A\cos\omega t.$$

If we set $k = r = A = 1$ and $\omega = 10$ then find the first three nonzero terms in the Taylor polynomial approximations tot he solution with $y(0) = 0$ and $y'(0) = 1$.

PP 187 Express the power series $\displaystyle\sum_{n=1}^{\infty} na_n x^{n-1}$ as a power series with generic term x^k. That is, find k_o and c_k for which $\displaystyle\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{k=k_o}^{\infty} c_k x^k$.

PP 188 Express the power series $\displaystyle\sum_{n=1}^{\infty} a_n x^{n+1}$ as a power series with generic term x^k. That is, find k_o and c_k for which $\displaystyle\sum_{n=1}^{\infty} a_n x^{n+1} = \sum_{k=k_o}^{\infty} c_k x^k$.

PP 189 Find the Taylor series for $f(x) = \dfrac{1+x}{1-x}$ about $x_0 = 0$.

PP 190 Find the singular points of the differential equation $(x+1)y'' - x^2y' + 3y = 0$.

PP 191 Find the singular points of the differential equation $(t^2 - t - 2)x'' - (t+1)x' - (t-2)x = 0$.

PP 192 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$z'' - x^2 z = 0.$$

PP 193 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$y'' + (x-1)y' + y = 0.$$

PP 194 Find the complete power series solution (including a formula for the general coefficient) about $x = 0$ for:

$$y' - 2xy = 0.$$

PP 195 Find the complete power series solution (including a formula for the general coefficient) about $x = 0$ for:

$$y'' - xy' + 4y = 0.$$

PP 196 Find the complete power series solution (including a formula for the general coefficient) about $x = 0$ for:

$$z'' - x^2 z' - xz = 0.$$

PP 197 Find the minimum value for the radius of covergence of a power series solution about x_0

$$(1 + x + x^2)y'' - 3y = 0, \qquad x_0 = 1.$$

PP 198 Find the minimum value for the radius of covergence of a power series solution about x_0

$$y'' - (\tan x)y' + y = 0 = 0, \qquad x_0 = 0.$$

PP 199 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$x' + (\sin t)x = 0, \qquad x(0) = 1.$$

PP 200 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$y'' - e^{2x}y' + (\cos x)y = 0, \qquad y(0) = -1,\ y'(0) = 1.$$

PP 201 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$z'' + xz' + z = x^2 + 2x + 1.$$

PP 202 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$(1 + x^2)y'' - xy' + y = e^{-x}.$$

PP 203 If $\sum_{n=0}^{\infty} B_n x^n = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}$ then find the formula for B_n in terms of c_n. You will need to break into cases, B_0, B_1 verse B_n for $n \geq 2$.

PP 204 Find the minimum radius of convergence about $x = 0$ for the solution of

$$(x^2 - 2x + 10)y'' + xy' - 4y = 0.$$

PP 205 Solve $y'' + (x+1)y' - y = 0$ up to 4-th order. Center the solution at zero.

PP 206 Find the first three nontrivial terms in the power series solution centered at zero of the differential equation $(x^2+1)y'' + 2xy' = 0$ with $y(0) = 0$ and $y'(0) = 1$.

PP 207 Is $x = 0$ an ordinary point of $y'' + 5xy' + \sqrt{x}y = 0$?

PP 208 Find all singularties of the following differential equations, or state no singularities:

(a.) $y'' + xy' + 3y = 0,$

(b.) $(x^2 - 3x_2)y'' + \sqrt{x}y' + x^2 y = 0$

(c.) $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$

(d.) $(x^2 - x)y'' + x^2 y' - 3xy = 0$

(e.) $e^x - 1)y'' + xy = 0$

(f.) $x(x^2 + 2x + 2)y'' + (x^2+1)y' + 3y = 0$

PP 209 Find the complete Frobenius solution of

$$x^2 y'' + x\left(x - \frac{1}{2}\right)y' + \frac{1}{2}y = 0$$

(this one has real exponents $r = 1$ and $r = 1/2$)

PP 210 Find the Frobenius solution near $x = 0$ for $x > 0$ up to order x^2 for

$$x^2 y'' + \sin(x)y' - \cos(x)y = 0.$$

PP 211 Solve $x^3 y'' - x^2 y' - y = 0$ for $x >> 0$ by making the substitution $z = 1/x$ and solving the resulting differential equation in z about the regular singular point $z = 0$. Find the first four nonzero terms in the series expansion about ∞ (once upon a time this was Problem 41 in §8.6 of Nagle, Saff and Snider, 5th edition)

PP 212 Find the complete (summation-notation) power series solution of the following integral:

$$\int x^6 \sin(x^2)\, dx$$

PP 213 Find the first **TWO** nontrivial terms in a power series solution of $e^x y'' + xy' + y = 0$ given that $y(0) = 1$ and $y'(0) = 2$.

PP 214 Find the singularities of $x(x^2 + 2x + 2)y'' + (x^2+1)y' + 3y = 0$ and determine the largest open interval of convergence for a solution of the form $y = \sum_{n=0} a_n (x+2)^n$.
Think. Do not try to solve this, I'm asking you about the interval of convergence, I'm **not** *asking for what a_n are in particular*

PP 215 Find the complete power series solution of $y'' - 9x^2y = 0$ given that $y(0) = 1$ and $y'(0) = 0$ by explicit substitution of a series solution into the given differential equation.

PP 216 Suppose $y'' + \frac{x}{(x-2)(x^2-6x+10)}y' + \left(\frac{1}{(x+3)^3} + \frac{1}{x^2}\right)y = 0$.

(a) find all singular points

(b) classify each real singular point as either regular or irregular (not regular)

(c) plot the singularities in a complex plane

(d) find the largest possible open and real domain of the solution

$$y = \sum_{n=0}^{\infty} a_n(x - 0.5)^2$$

(e) find the largest possible open and real domain of the solution

$$y = \sum_{n=0}^{\infty} a_n(x - 4)^2$$

PP 217 Suppose $y(0) = 1$ and $y'(0) = 2$. Find the solution up to order 5 in x for the differential equation

$$y'' + (x^2 - 1)\cos(x)y' + \sinh(3x)y = 0.$$

PP 218 Find the complete power series solution centered at zero for $\frac{dy}{dx} - 2xy = 0$.

PP 219 Find the first two nontrivial terms in the frobenius expansions for the fundamental solutions y_1 and y_2 of

$$3xy'' + (2 - x)y' - y$$

PP 220 Find the complete power series solution of $y'' + x^2y' + 2xy = 0$ about the ordinary point $x = 0$. Your answer should include nice formulas for arbitrary coefficients in each of the fundamental solutions. You need to both set-up and solve the reccurrence relations as best you can.

PP 221 Find the first four nonzero terms in the power series solution about zero for the initial value problem $y'' + \sin(x)y' + (x - 1)y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

PP 222 Find the complete Frobenius solution of

$$x^2y'' + x(x - \tfrac{1}{2})y' + \tfrac{1}{2}y = 0.$$

(it turns out this one has real exponents)

PP 223 Solve $x^3y'' - x^2y' - y = 0$ for $x >> 0$ by making the subsitution $z = 1/x$ and solving the resulting differential equation in z about the regular singular point $z = 0$. Find the first four nonzero terms in the series expansion about infinity.

PP 224 Consider $y'' + e^xy' + \sin(3x)y = 0$. Find the first 3 nontrivial terms in a series solution centered about $x = 0$ given that $y(0) = 1$ and $y'(0) = 6$.

PP 225 Find the complete power series solution of $y'' + 6x^2y = 0$ centered at $x = 0$.

PP 226 Suppose we define $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$. Show that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

PP 227 Suppose $\sum_{k=0}^{\infty} (a_{2k}x^{2k} + b_{2k+1}x^{2k+1} = e^x + \cos(x+2)$. Find explicit formulas for a_{2k} and b_{2k+1} via Σ-notation algebra.

PP 228 Find a power series solution to the integrals below:

(a.) $\int \frac{x^3 + x^6}{1 - x^3}\, dx$

(b.) $\int x^8 e^{x^3+2}\, dx$

PP 229 Calculate the 42^{nd}-derivative of $x^2\cos(x)$ at $x = 1$. (use power series techniques)

PP 230 Find the complete power series solution of $y'' + x^2y' + 2xy = 0$ about the ordinary point $x = 0$. Your answer should include nice formulas for arbitrary coefficients in each of the fundamental solutions. You need to both set-up and solve the reccurrence relations as best you can.

PP 231 (Ritger & Rose 7-2 problem 7 part c) Find the first four nonzero terms in the power series solution about zero for the initial value problem $(x + 2)y'' + 3y = 0$ with $y(0) = 0$ and $y'(0) = 1$.

PP 232 (Ritger & Rose 7-2 problem 7 part d) Find the first four nonzero terms in the power series solution about zero for the initial value problem $y'' + \sin(x)y' + (x - 1)y = 0$ with $y(0) = 1$ and $y'(0) = 0$.

PP 233 Construct a differential equation with $y_1(x) = \frac{\sin(x)}{x}$ for $x \neq 0$ and $y_1(0) = 1$, $y_2(x) = x$ as its fundamental solution set. To accomplish this task do two tasks:

(a.) Argue from appropriate facts from the theory of determinants that $L[y] = \det \begin{bmatrix} y & y' & y'' \\ y_1 & y_1' & y_1'' \\ y_2 & y_2' & y_2'' \end{bmatrix}$ is a linear ODE with solutions y_1 and y_2.

(b.) calculate $L[y]$ explicitly as a linear ODE of the form $py'' + qy' + ry = 0$ where p, q, r are perhaps given as Taylor expansions about zero.

PP 234 (from page 103 of Boyce and DiPrima's 3rd Ed.) Consider $xy'' - (x + N)y' + Ny = 0$ for $N \in \mathbb{N}$

(a.) show $y_1 = e^x$ is a solution.

(b.) show that $y_2 = ce^x \int x^N e^{-x}\, dx$ is a second solution. (perhaps use the result of the previous problem, or the theorem from my notes or Ritger & Rose)

(c.) set $c = \frac{-1}{N!}$ and show by induction that $y_2(x) = T_n(x)$ the n-th order Taylor polynomial of e^x.

PP 235 (introduction to theory of adjoints, from page 95 of Boyce and DiPrima's 3rd Ed.) If $p(x)y''+q(x)y' + r(x)y = 0$ can be expressed as $[p(x)y']' + [f(x)y]' = 0$ then it is said to be **exact**. Omit x-dependence in p, q, r, μ for brevity, if $py''+qy'+ry = 0$ is not exact then it is possible to make it exact with multiplication by the appropriate integrating factor μ. **Show** that for μ to accomplish its stated task it must itself be the solution of the so-called **adjoint equation**

$$p\mu'' + (2p' - q)\mu' + (p'' - q' + r)\mu = 0.$$

where we have assumed p, q possess the stated derivatives. Find the adjoint equation for

a. [constant coefficient case] $ay'' + by' + cy = 0$
b. [Bessel Eqn. of order ν] $x^2y'' + xy' + (x^2 - \nu^2)y = 0$
c. [The Airy Eqn.] $y'' - xy = 0$

PP 236 Solve $LI_1'' + R_1I_1' + \frac{1}{C}(I_1' - I_2') = 0$ and $R_2I_2' + \frac{1}{C}(I_2 - I_1) = \mathcal{E}'(t)$ given that $\mathcal{E}(t) = 10\cos(2t)$, $L = 1$ and $C = 1$ and $R_1 = 2$ and $R_2 = 3$ (in volts, seconds, Henries and Farads). These differential equations stem from the circuit pictured below:

PP 237 Suppose $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$. Show that $e^D = \begin{bmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{bmatrix}$.

PP 238 Suppose $\frac{dx}{dt} = x + 4y$ and $\frac{dy}{dt} = x + y$. Find the general real solution via the e-vector method.

PP 239 Suppose $\frac{dx}{dt} = 2x + y$ and $\frac{dy}{dt} = 2y$. Find the general real solution via the generalized e-vector method.

PP 240 Suppose $\frac{dx}{dt} = 4x - 3y$ and $\frac{dy}{dt} = 3x + 4y$. Find the general real solution via the e-vector method.

PP 241 Suppose $\frac{dx}{dt} = x + 4y + e^{6t}$ and $\frac{dy}{dt} = x + y + 3$. Find the solution with $x(0) = 0$ and $y(0) = 0$. Please use matrix arguments (do not solve by the operator method, instead, use variation of parameters for systems)

PP 242 Suppose X is a fundamental matrix for $\frac{d\vec{x}}{dt} = A\vec{x}$. Suppose B is a square matrix with $det(B) \neq 0$. Show that XB is a fundamental matrix for $\frac{d\vec{x}}{dt} = A\vec{x}$.

PP 243 Calculate e^{tA} for $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$. *(Problem 238 should help)*

PP 244 Use the Cayley Hamilton Theorem to calculate e^{tA} for $A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 1 \\ 9 & 3 & -4 \end{bmatrix}$. The Cayley Hamilton Theorem simply states that a matrix solves it's own characteristic equation; that is, if $p(\lambda) = 0$ is the characteristic equation then $p(A) = 0$. For example, if $p(\lambda) = (\lambda + 2)^3 = 0$ then $(A + 2I)^3 = 0$. The proof of this theorem is easy in the diagonalizable case, however the general proof requires ideas about invariant subspaces often not covered in the undergraduate course on linear algebra.

you may use technology to aid with the matrix calculations in the next three problems. That said, you don't really need it for these in my view

PP 245 Suppose $\frac{dx}{dt} = 5x - 6y - 6z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 6y - 4z$. Find the general real solution via the e-vector method.

PP 246 Suppose $\frac{dx}{dt} = 5x - 5y - 5z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 5y - 3z$. Find the general real solution via the e-vector method.

PP 247 Suppose $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = 3y + z$ and $\frac{dz}{dt} = 3z$. Find the general real solution via the generalized e-vector method.

PP 248 To solve $\frac{d\vec{x}}{dt} = A\vec{x}$ in the case $A = \begin{bmatrix} -3 & 0 & -3 \\ 1 & -2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ by the following calculations:

(a) find the e-values and corresponding e-vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$. (you may use technology)

(b) construct $P = [\vec{u_1}|\vec{u_2}|\vec{u_3}]$ and calculate $P^{-1}AP$. (you may use technology)

(c) note the solution of $AP\vec{y} = \frac{d}{dt}[P\vec{y}] = P\frac{d\vec{y}}{dt}$ is easily found since multiplying by P^{-1} yields $P^{-1}AP\vec{y} = P^{-1}P\frac{d\vec{y}}{dt} = I\frac{d\vec{y}}{dt} = \frac{d\vec{y}}{dt}$. Solve $P^{-1}AP\vec{y} = \frac{d\vec{y}}{dt}$. (this should be really easy, just solve 3 first order problems, one at a time)

(d) $AP\vec{y} = \frac{d}{dt}[P\vec{y}]$ means $\vec{x} = P\vec{y}$ solves $\frac{d\vec{x}}{dt} = A\vec{x}$. Solve the original system by multiplying the solution from (3.) by P.

The method outlined above is more meaningful in a larger discussion involving coordinate change for linear transformations. The coordinates $\vec{y} = P^{-1}\vec{x}$ are **eigencoordinates**. *A matrix is said to be*
diagonalizable *iff there exists some coordinate change matrix P such that $P^{-1}AP = D$ where D is diagonalizable. Not all matrices are diagonalizable. We've seen this. When there are less than n-LI e-vectors then we cannot build the P-matrix as above and it turns out there is no other way to diagonalize a matrix. On the other hand, the generalized e-vectors always exist and cojugating by P made of generalized e-vectors will place* **any** *matrix in Jordan-form (possibly complex).*

PP 249 Suppose A is an 7×7 matrix with complex e-value $\lambda_1 = 3i$ repeated and a real e-value of $\lambda_2 = 1$ repeated three times. You are given a complex vector $\vec{u}_1 = \vec{a}_1 + i\vec{b}_1$ a second LI complex-vector $\vec{u}_2 = \vec{a}_2 + i\vec{b}_2$ such that

$$(A - 3iI)\vec{u}_1 = 0 \qquad (A - 3iI)\vec{u}_2 = \vec{u}_1.$$

We assume $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ are all real vectors. Furthermore, you are given $\vec{u}_3, \vec{u}_4, \vec{u}_5$ LI vectors such that

$$(A - I)\vec{u}_3 = 0, \qquad A\vec{u}_4 = \vec{u}_4, \qquad A\vec{u}_5 = \vec{u}_5 + \vec{u}_4$$

Find the general, manifestly real, solution.

PP 250 Suppose a force $F(x) = 3x^4 + 16x^3 + 6x^2 - 72x$ is the net-force on some mass $m = 1$. Newton's Equation is $\ddot{x} = 3x^4 + 16x^3 + 6x^2 - 72x$.

(a) make the substitution $v = \dot{x}$ and write Newton's equation as a system in normal form for x and v.

(b) find all three critical points for the system in (1.). (the potential should factor nicely)

(c) plot the potential plane and phase plane juxtaposed vertically with the potential at the top and the phase plane at the base. Plot several trajectories and include arrows to indicate the direction of physically feasible solutions.

(d) classify each critical point by examining your plot from (3.)

in this context the phase plane is also called the Poincare plane in honor of the mathematician who did much pioneering work in this realm of qualitative analysis. Incidentally, given any autonomous system $\frac{dx}{dt} = g(x,y)$ and $\frac{dy}{dt} = f(x,y)$ we can study the timeless phase plane equation $\frac{dy}{dx} = \frac{f}{g}$ to indirectly analyze the solutions to the system. Solutions to the phase plane equation are the Cartesian level curves which are parametrized, with parameter t, by the solutions to the system

PP 251 The Volterra-Lotka equations are a nonlinear system of ODEs which model the population interaction between some prey with population x and predator of population y. For example, $\frac{dx}{dt} = x(3-y)$ and $\frac{dy}{dt} = y(x-3)$. This means that when the predator population is over 3 then prey population declines. On the other hand, if the prey population goes beyond 3 then the predator population grows. This competition can lead to a variety of outcomes. Find all the critical points of the system and plot the phase plane via the pplane tool, plot about 20 interesting trajectories. Comment on the stability of the critical points. (you'll need to print this out and attach it to this homework)

PP 252 Show that nontrivial solutions for the cauchy-euler system $t\frac{d\vec{x}}{dt} = A\vec{x}$ of the form $\vec{x}(t) = t^R\vec{u}$ must have R an e-value of A with $\vec{u}$ the corresponding e-vector. Solve $t\frac{d\vec{x}}{dt} = A\vec{x}$ in the case $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$ for $t > 0$.

PP 253 Difference equations can sometimes be written in the form $\vec{x}_{k+1} = B\vec{x}_k$ where $k = 0, 1, 2, \dots$. It is easy to show that if $\vec{x}_o$ is the given **initial state** of the system then the k-th state is found by $\vec{x}_k = B^k\vec{x}_o$. There is a natural connection with this difference equation and the linear differential equations we have studied. Consider this: for small $\triangle t$,

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \Rightarrow \quad \frac{\vec{x}(t+\triangle t) - \vec{x}(t)}{\triangle t} \approxeq A\vec{x}(t) \quad \Rightarrow \quad \vec{x}(t+\triangle t) = \vec{x}(t) + \triangle t A\vec{x}(t)$$

Hence, $\vec{x}(t+\triangle t) = (I + \triangle t A)\vec{x}(t)$. Identify that this approximation resembles the difference equation where $\vec{x}(t) = \vec{x}_k$ and $\vec{x}(t+\triangle t) = \vec{x}_{k+1}$ and $B = I + \triangle t A$.

(a) Suppose $\vec{x}_o = [2, 0]^T$ is the initial state. Calculate the states up to $k = 10$ for $\vec{x}_{k+1} = B\vec{x}_k$ where $B = \begin{bmatrix} 1.1 & -1 \\ 1 & 1.1 \end{bmatrix}$.

(b) Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 0.1 & -1 \\ 1 & 0.1 \end{bmatrix}$ given the initial condition $\vec{x}(0) = [2, 0]^T$.

(c) plot the states from (1.) as dots and the solution from (2.) as a curve on a common xy-plane. Comment on what you see. (what $\triangle t$ did I choose ? How could we make the difference equation more closely replicate the differential equation?)

PP 254 Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Calculate A^2.

PP 255 Let A be as in the previous problem. Suppose $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.

(a.) calculate Av_1 c

(b.) calculate Av_2

(c.) calculate $A[v_1|v_2]$ (here $[v_1|v_2]$ is the 3×2 matrix made from gluing (aka concatenating) the column vectors v_1 and v_2)

(d.) Does $A[v_1|v_2] = [Av_1|Av_2]$?

PP 256 A square matrix X is invertible iff there exists Y such that $XY = YX = I$ where I is the identity matrix. Moreover, linear algebra reveals that X is invertible iff $det(X) \neq 0$. For a 2×2 matrix $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we define $det(X) = ad - bc$. Suppose X is invertible and show $X^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This formula is worth memorizing for future use in two-dimensional problems. Please understand, all I'm asking here is for you to multiply X and my propsed formula for X^{-1} to obtain $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

PP 257 Differentiation of matrices of functions is not hard. Let $X(t) = \begin{bmatrix} e^t & t \\ 1/t & e^{-t} \end{bmatrix}$. Calculate:

(a.) calculate $\frac{dX}{dt}$

(b.) calculate $\frac{dX^{-1}}{dt}$

(c.) simplify $\frac{dX}{dt}X^{-1} + X\frac{dX^{-1}}{dt}$.

(d.) explain the previous part by differentiating $X(t)X^{-1}(t) = I$. Note: the product rule for matrix products is simply $\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}$.

PP 258 If two masses m_1, m_2 are coupled by a spring and then the whole system is attached to springs between to walls (see figure 1 on page 230 of Ritger & Rose for a related picture) then

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3x_2.$$

PP 259 Suppose $k_2 = 0$. Find the equations of motion.

PP 260 Suppose $k_1 = k_3 = 0$. Find the equations of motion.

PP 261 Suppose $k_1 = k_3 = 1$ and $k_2 = 2$ with $m_1 = m_2 = 1$. Find the equations of motion.

PP 262 Solve $x' = 7x + 3y$ and $y' = 3x + 7y$ by the eigenvector method.

PP 263 Use the solution of the previous problem to solve $x' = 7x + 3y + 1$ and $y' = 3x + 7y + 2$ subject the initital condition $x(0) = 1$ and $y(0) = 2$.

PP 264 Solve $x' = -3x - 5y$ and $y' = 3x + y$ with $x(0) = 4$ and $y(0) = 0$ by the eigenvector method.

PP 265 An ice tray has tiny holes between each of its three partititions such that the water can flow from one partition to the next. Let x, y, z denote the height of water in the three water troughs. The holes are designed such that the flow rate is proportional to the height of water above the adjacent trough. For example, supposing x and z are the edge troughs whereas y is in the middle we have $\frac{dx}{dt} = k(y - x)$. For simplicity of discussion suppose $k = 1$. Write the corresponding differential equations to find the water-level in the y and z troughs. If initially there is 3.0 cm of water in the x trough and none in the other two troughs then find the height in all three troughs as a function of time t. Discuss the steady state solution, is it reasonable?

PP 266 Let a, b be constants which are some measure of the trust between two nations. Furthermore, let x be the military expenditure of Bobslovakia and let y be the military expenditure of the Leaf Village. Detailed analysis by strategically gifted ninjas reveal that

$$\frac{dx}{dt} = -x + 2y + a$$

$$\frac{dy}{dt} = 4x - 3y + b$$

Analyze possible outcomes for various initial conditions and values of a, b. Consider drawing an ab-plane to explain your solution(s). Is a stable peace without a run-away arms race possible given the analysis thus far?

PP 267 Suppose $(A - \lambda I)\vec{u}_1 = 0$ and $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 3 + i\sqrt{2}$ and $\vec{u}_1 = [3 + i, 4 + 2i, 5 + 3i, 6 + 4i]^T$ and $\vec{u_2} = [i, 1, 2, 3 - i]^T$.

(a.) find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

(b.) extract four real solutions to write the general real solution (c_1, c_2, c_3, c_4 should be real in this answer)

PP 268 Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Calculate $e^{\theta J}$ where $\theta \in \mathbb{R}$. Express your answer in terms of sine and cosine and relevant matrices.

PP 269 Solve $x' = 2x + y$ and $y' = 2y$ by the method generalized eigenvectors.

PP 270 Introduce variables to reduce

$$y''' + 4y'' + 2y' + 6y = \tan(t)$$

to a system of three first order ODEs in matrix normal form $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$.

PP 271 Introduce variables to reduce

$$y'' + 4ty' + 5y' = 0, \qquad w'' + 9e^{-t}w = 0$$

to a system of four first order ODEs in matrix normal form $\frac{d\vec{x}}{dt} = A\vec{x}$.

PP 272 Linear independence (LI) of vector-valued functions $\{\vec{f}_j : I \subseteq \mathbb{R} \to \mathbb{R}^n \mid j = 1, \dots, k\}$ is defined in the same way as was previously discussed for real-valued functions. In particular, $\{\vec{f}_1, \dots, \vec{f}_k\}$ is LI on $I \subseteq \mathbb{R}$ if $c_1\vec{f}_1(t) + \cdots + c_k\vec{f}_k(t) = 0$ for all $t \in I$ implies $c_1 = 0, \dots, c_k = 0$. We can check LI of n such n-vector-valued functions without any further differentiation; in particular, if $\det[\vec{f}_1(t)|\cdots|\vec{f}_n(t)] \neq 0$ for all $t \in I \subseteq \mathbb{R}$ then $\{\vec{f}_1(t), \dots, \vec{f}_n(t)\}$ is LI on I. Show the following sets of vector-valued functions are LI on $\mathbb{R}$. (notice, my notation is that $(a, b) = [a, b]^T$, in other words, each of the expressions below has lists of column vectors.

(a.) $\{\, (e^t, e^t),\ (e^t, -e^t)\, \}$

(b.) $\{\, (\cos(t), -\sin(t)),\ (\sin(t), \cos(t))\, \}$,

(c.) $\{\, e^t\vec{u}_1, \ , \ e^t(\vec{u}_2 + t\vec{u}_1)\ , e^t(\vec{u}_3 + t\vec{u}_2 + \frac{t^2}{2}\vec{u}_3)\, \}$ given $\vec{u}_1 = (1, 0, 0), \vec{u}_2 = (0, 1, 1), \vec{u}_3 = (1, 1, 1)$.

PP 273 (Cook 5.1)(problem 13 of section 4.9 in Zill) Solve:

$$2\dot{x} - 5x + \dot{y} = e^t$$

$$2\dot{x} - x + \dot{y} = 5e^t$$

PP 274 (Cook 5.1)(problem 7 of section 7.6 in Zill) Solve:

$$\ddot{x} + x - y = 0,$$

$$\ddot{y} + y - x = 0,$$

subject the initial conditions $x(0) = 0, \dot{x}(0) = -2$ and $y(0) = 0, \dot{y}(0) = 1$. (you could use the technique of section 4.9 or that of 7.6, either method should be a profitable exercise)

PP 275 (matrix multiplication) work problem 6 of Appendix II in Zill (page APP-18)

PP 276 Solve, via the eigenvector technique,

$$\frac{dx}{dt} = 5x - y$$
$$\frac{dy}{dt} = -x + 5y.$$

PP 277 Plot the direction field of the system given in previous Problem using pplane. Plot a few solutions. Can you see the e-vectors' geometric significance? Include a print-out of your investigation.

PP 278 Solve, via the complex eigenvector technique,

$$\frac{dx}{dt} = 4x + 2y$$
$$\frac{dy}{dt} = -x + 2y.$$

PP 279 Plot the direction field of the system given in the previous problem. Plot a few solutions. Can you see the e-vectors' geometric significance? Include a print-out of your investigation.

PP 280 Solve $x' = 7x + 3y + 4z,\ y' = 6x + 2y,\ z' = 5z$ by the eigenvector method.

PP 281 Use technology to find e-values and e-vectors for each of the matrices below. If possible, use the solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$ derived from e-vectors to write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$. If not possible, explain why.

(a.) $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

(b.) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

(c.) $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$

(d.) $A = \begin{bmatrix} -1 & -3 & -9 \\ 0 & 5 & 18 \\ 0 & -2 & -7 \end{bmatrix}$.

(e.) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

PP 282 Find fundamental matrices for each of the systems given in the previous half dozen problems where reasonable.

PP 283 Suppose $\vec{v}$ is an eigevector with eigenvalue λ for the real matrix A. Show A^2 also has e-vector $\vec{v}$. What is the e-value for $\vec{v}$ with respect to A^2.

PP 284 Write down the magic formula for the matrix exponential.

PP 285 Suppose A is a 3×3 matrix with nonzero vectors $\vec{u}, \vec{v}, \vec{w}$ such that

$$A\vec{u} = 3\vec{u}, \qquad (A - 3I)\vec{v} = \vec{u}, \qquad A\vec{w} = 0.$$

Write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors.

PP 286 Suppose $(A - \lambda I)\vec{u}_1 = 0$ and $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 3 + i\sqrt{2}$ and $\vec{u}_1 = [3 + i, 4 + 2i, 5 + 3i, 6 + 4i]^T$ and $\vec{u_2} = [i, 1, 2, 3 - i]^T$.

(a.) find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

(b.) extract four real solutions to write the general real solution (c_1, c_2, c_3, c_4 should be real in this answer)

PP 287 Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and let $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Calculate $e^{\theta J}$ where $\theta \in \mathbb{R}$. Express your answer in terms of sine and cosine and relevant matrices.

PP 288 Solve $x' = 2x + y$ and $y' = 2y$ by the method generalized eigenvectors.

PP 289 Show why $\frac{d}{dt}e^{tA} = Ae^{tA}$. Is this enough to show e^{tA} is a fundamental solution matrix? If not, say what else we need to know about the matrix exponential.

PP 290 Show $\vec{x}(t) = e^{tA}\vec{x}_o$ is a solution to $\frac{d\vec{x}}{dt} = A\vec{x}$ with $\vec{x}(0) = \vec{x}_o$. In this sense, the matrix exponential generates the solution of the system of ODEs with coefficient matrix A.

PP 291 (matrix inverse of 2×2) Suppose $X(t) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$. Find $X^{-1}(t)$. (use the nice formula in Example 5.2.7 of Cook)

PP 292 work out problem 15 of section 8.3.2 in Zill. That is, solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$

PP 293 work out problem 21 of section 8.3.2 in Zill. That is, solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} \sec t \\ 0 \end{bmatrix}$

PP 294 Consider the differential equation $y'' - 2y' + y = 0$. I think we can all solve this one. Let $x_1 = y, x_2 = y'$. Let A be the companion matrix which stems from the reduction of order just listed. Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ by translating the fundamental solution set $\{y_1, y_2\} = \{e^t, te^t\}$ into the corresponding fundamental solution set $\{\vec{x}_1, \vec{x}_2\}$. Let $\vec{u}_1 = e^{-t}\vec{x}_1$ and $\vec{u}_2 = e^{-t}\vec{x}_2$. Solve the following equations:

$$(A - I)\vec{u}_1 = \vec{a} \qquad (A - I)\vec{u}_2 = \vec{b}.$$

In other words, find $\vec{a}, \vec{b}$ explicitly. Comment on which of the fundamental solutions to $\{\vec{x}_1, \vec{x}_2\}$ was an eigensolution.

PP 295 Suppose A has n-LI e-vectors and hence we can write the general solution for $\frac{d\vec{x}}{dt} = A\vec{x}$ as a linear combination

$$\vec{x} = c_1 e^{\lambda_1 t} + \cdots + c_n e^{\lambda_n t}$$

Solve $\frac{d\vec{x}}{dt} = A^k\vec{x}$ where $k \in \mathbb{N}$.

PP 296 If $A^T = A$ then we say A is a symmetric matrix. A rather deep theorem of linear algebra states that a symmetric matrix has real eigenvalues and it is possible to select n-LI eigenvectors $\{\vec{u}_1, \dots, \vec{u}_n\}$ for which $A\vec{u}_j = \lambda_j\vec{u}_j$ and $\vec{u}_i \cdot \vec{u}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ for all $i, j \in \mathbb{N}_n$. It follows that $P = [\vec{u}_1|\cdots|\vec{u}_n]$ has $P^TP = I$ which means $P^{-1} = P^T$. This means, if we're studying a system of differential equations $\frac{d\vec{x}}{dt} = A\vec{x}$ with $A^T = A$ we can change coordinates to $\vec{y} = P^T\vec{x}$ and in that new $\vec{y}$-coordinate system the differential equation is simply:

$$\frac{dy_1}{dt} = \lambda_1 y_1, \ \dots, \ \frac{dy_n}{dt} = \lambda_n y_n. \qquad \star.$$

This system is said to be **uncoupled** and it's really the most trivial sort of system you can come across; we can solve each equation in the uncoupled system without knowledge of the remaining variables. Consider $\vec{x} = \langle x, y, z\rangle$ and the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$. Find an orthonormal eigenbasis for A and use it to change coordinates on the given system. Verify the claim $\star$ in the context of A. Use the notation $\vec{y} = \langle \bar{x}, \bar{y}, \bar{z}\rangle$, so $y_1 = \bar{x}$ etc..

PP 297 Consider the solution-set of $4xy + 4xz + 4yz = 1$. Change to the barred-coordinates $\bar{x}, \bar{y}, \bar{z}$ you discovered in the previous problem. Which Quadric surface is this?

PP 298 The Cayley Hamilton Theorem states that a matrix will solve its own characteristic equation. For example, if $P(x) = x^3 + I$ then $P(A) = A^3 + I = 0$. For this A, calculate e^{tA} in terms of A. Recall, as you should know, $e^{tA} = I + tA + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$.

PP 299 Solve $x' = -x - 4y$ and $y' = 8x + 11y$ using matrix methods.

PP 300 Solve $x' = -7x - 6y$ and $y' = 15x + 11y$ using matrix methods.

PP 301 Suppose $A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$. Calculate e^{tA}.
Also, solve $\frac{d\vec{r}}{dt} = A\vec{r}$ given that $\vec{r}(0) = (1, 2)$.

PP 302 Consider A is a 3×3 matrix for which there exist nonzero vectors v_1, v_2, v_3 such that:

$$Av_1 = 10v_1, \quad Av_2 = 10v_2, \quad Av_3 = 10v_3 + v_1$$

derive the general solution for $\frac{d\vec{r}}{dt} = A\vec{r}$ with appropriate arguments based on the matrix exponential.

PP 303 Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ let $B = I + A$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also set $M = \begin{bmatrix} 8 & 5 & 9 \\ 6 & 3 & 0 \\ 7 & 0 & 0 \end{bmatrix}$

(a.) Calculate AB and calculate BA. (this doesn't usually happen)

(b.) We say $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$. Calculate Me_1 and Me_2 then check that $M \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = [Me_1|Me_2]$. *This ought to illustrate the column-by-column multiplication rule in the sense that* $M[e_1|e_2] = [Me_1|Me_2]$. *Recall this was important for us as we analyzed how the solution matrix gives us a matrix where each column is itself a solution*

PP 304 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 0 \end{bmatrix}$. Calculate the following items for A,

(a.) show the eigenvalues of A are $\lambda_1 = -3$, $\lambda_2 = -1$ and ,$\lambda_3 = 6$

(b.) find eigenvectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ with eigenvalues $\lambda_1 = -3$, $\lambda_2 = -1$ and $,\lambda_3 = 6$ respective. Normalize the eigenvectors so that each has length one.

(c.) show $\vec{u}_i \bullet \vec{u}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

(d.) Let $P = [\vec{u}_1|\vec{u}_2|\vec{u}_3]$ and show that $P^T P = I$ (this shows that $P^{-1} = P^T$ and that P is what is known as an **orthgonal matrix**)

(e.) Calculate $P^T AP$. You should get something really pretty.

Remark: the problem above illustrates the real spectral theorem which implies that a symmetric matrix has an orthonormal eigenbasis and eigenvalues which are all real

PP 305 Find the general solution of $\frac{d\vec{r}}{dt} = A\vec{r}$ where A is was given in the previous problem.

PP 306 Let $x' = 2x - 3y$ and $y' = 3x + 2y$. Find the general real solution via the techinque of eigenvectors and/or generalized eigenvectors. In addition, set-up the solution to $x' = 2x - 3y + f_1$ and $y' = 3x + 2y + f_2$ via the method of variation of parameters for systems. *hint: I think this one requires calculation of a complex eigenvector*

PP 307 Let $x' = 3x - 18y$ and $y' = 2x - 9y$. Find the general real solution via the techinque of eigenvectors and/or generalized eigenvectors. Then solve the initial value problem for the given system of DEqns with initial data $x(0) = 1$ and $y(0) = 0$. *hint: I believe this problem will require you to find one eigenvector and one generalized eigenvector, both with the same eigevalue*

PP 308 Let I be the 2×2 identity matrix and let

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Prove that $e^{tK} = \cosh(t)I + \sinh(t)K$. Is $\frac{d\vec{r}}{dt} = K\vec{r}$ a system of differential equations obtained by reduction of order ? If so, do the solutions you found in e^{tK} coincide logically with those you find by directly solving the corresponding 2-nd order problem ?

PP 309 Suppose A is a 4×4 matrix with nonzero real vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ and $\vec{u}_4$ for which:

$$A\vec{u_1} = 3\vec{u}_1, \quad (A - 3I)\vec{u_2} = \vec{u_3}, \quad (A - 3I)\vec{u}_3 = \vec{u_1}, \qquad A\vec{u}_4 = 0$$

Find the general solution to $\frac{d\vec{r}}{dt} = A\vec{r}$. Do not assume it fits a pattern. You need to THINK.