Name: (please print name here →) ____ .

MATH 334:	MISSION 3: SERIES SOLUTIONS & SYSTEMS OF DEQNS [50PTS]

While complex solutions may be useful as in-between work the solutions requested on this assignment are real solutions. Please write answers in space provided, do not write scratch work on these pages. However, for Problems 69, 70, 71, 72, please write your solution below the problem statement printed out Thanks!

Problem 49 Find the first 4 nonzero terms in the power series solution about $x = 0$ for $z'' - x^2 z = 0$.

$$\boxed{z = c_0\left(1 + \frac{1}{12}x^4 + \cdots\right) + c_1\left(x + \frac{1}{20}x^5 + \cdots\right)}$$

Problem 50 Find the complete power series solution (including a formula for the general coefficient) about $x = 0$ for:

$$y' - 2xy = 0.$$

$$\boxed{y = k\sum_{n=0}^{\infty} \frac{1}{n!}x^{2n}}$$

Problem 51 Find the complete power series solution (including a formula for the general coefficient) about $x = 0$ for:

$$y'' - xy' + 4y = 0.$$

$$\boxed{y = c_0\left(1 - 2x^2 + \frac{1}{3}x^4\right) + c_1\left(x + \sum_{k=1}^{\infty} \frac{(-3)(-1)(1)\cdots(2k-5)}{(2k+1)!}x^{2k+1}\right)}$$

Problem 52 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$y'' - e^{2x}y' + (\cos x)y = 0, \qquad y(0) = -1,\ y'(0) = 1.$$

$$\boxed{y = -1 + x + x^2 + \frac{1}{2}x^3 + \cdots}$$

Problem 53 Find the first four nonzero terms in the power series solution about $x = 0$ for:

$$z'' + xz' + z = x^2 + 2x + 1.$$

$$\boxed{z = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right) + c_1\left(x - \frac{1}{3}x^3 + \cdots\right) + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4 + \cdots}$$

(probably giving up to quadratic order suffices here)

Problem 54 Find the singularities of $x(x^2 + 2x + 2)y'' + (x^2 + 1)y' + 3y = 0$ and determine the largest open interval of convergence for a solution of the form $y = \sum_{n=0} a_n(x + 2)^n$. *Think. Do not try to solve this, I'm asking you about the interval of convergence, I'm* **not** *asking for what a_n are in particular*

$\boxed{\text{largest open I.O.C} = (-2-\sqrt{2}, -2+\sqrt{2})}$

Problem 55 Suppose we define $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$. Show that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

– See other page –

Problem 56 Suppose $\sum_{k=0}^{\infty} (a_{2k}x^{2k} + b_{2k+1}x^{2k+1} = e^x + \cos(x + 2)$. Find explicit formulas for a_{2k} and b_{2k+1} via Σ-notation algebra.

$\boxed{a_{2k} = \frac{1}{(2k)!}\left(1 + (-1)^k \cos(2)\right)}$ & $\boxed{b_{2k+1} = \frac{1}{(2k+1)!}\left(1 - (-1)^k \sin(2)\right)}$

Problem 57 Find a power series solution to the integrals below:

(a.) $\displaystyle\int \frac{x^3 + x^6}{1 - x^3}\, dx = \boxed{C + \frac{x^4}{4} + \sum_{n=1}^{\infty} \frac{2}{3n+4} x^{3n+4}}$

$$= C + \sum_{n=0}^{\infty} \frac{x^{3n+4}}{3n+4} + \sum_{n=0}^{\infty} \frac{x^{3n+7}}{3n+7}$$ (sloppy version)

(b.) $\displaystyle\int x^8 e^{x^3+2}\, dx = \boxed{C + e^2 \sum_{n=0}^{\infty} \frac{x^{3n+9}}{(3n+9)n!}}$

(could factor out 3 of course)

Problem 58 Suppose $\frac{dx}{dt} = x + 4y$ and $\frac{dy}{dt} = x + y$. Find the general real solution via the e-vector method. Also, calculate e^{tA} for $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

$$\vec{r}(t) = c_1 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$$e^{tA} = \left[\begin{array}{c|c} \frac{1}{2}(e^{3t}+e^{-t}) & \frac{1}{4}(e^{3t}-e^{-t}) \\ \hline \frac{1}{4}(e^{3t}-e^{-t}) & \frac{1}{2}(e^{3t}+e^{-t}) \end{array}\right].$$

Problem 59 Suppose $\frac{dx}{dt} = 2x + y$ and $\frac{dy}{dt} = 2y$. Find the general real solution via the generalized e-vector method.

$$\vec{r}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Problem 60 Suppose $\frac{dx}{dt} = 4x - 3y$ and $\frac{dy}{dt} = 3x + 4y$. Find the general real solution via the e-vector method.

$$\vec{r}(t) = c_1 e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}.$$

Problem 61 Suppose $\frac{dx}{dt} = 5x - 6y - 6z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 6y - 4z$. Find the general real solution via the e-vector method.

$$\vec{r}(t) = c_1 e^{t}\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 62 Suppose $\frac{dx}{dt} = 5x - 5y - 5z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 5y - 3z$. Find the general real solution via the e-vector method.

$$\vec{r}(t) = c_1 e^{2t}\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t}\left(\cos t\begin{bmatrix} 5 \\ 8 \\ -5 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + c_3 e^{2t}\left(\sin t\begin{bmatrix} 5 \\ 8 \\ -5 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

Problem 63 Suppose $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = 3y + z$ and $\frac{dz}{dt} = 3z$. Find the general real solution via the generalized e-vector method.

$$\vec{r}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t}\begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}.$$

Problem 64 Suppose A is a 3×3 matrix with nonzero vectors $\vec{u}, \vec{v}, \vec{w}$ such that

$$A\vec{u} = 3\vec{u}, \qquad (A - 3I)\vec{v} = \vec{u}, \qquad A\vec{w} = 0.$$

Write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors.

$$\underline{\vec{x}(t) = c_1 \vec{w} + c_2 e^{3t}\vec{u} + c_3 e^{3t}(\vec{v} + t\vec{u})}.$$

Problem 65 Suppose $(A - \lambda I)\vec{u}_1 = 0$ and $(A - \lambda I)\vec{u}_2 = \vec{u}_1$ where $\lambda = 3 + i\sqrt{2}$ and $\vec{u}_1 = [3+i, 4+2i, 5+3i, 6+4i]^T$ and $\vec{u_2} = [i, 1, 2, 3-i]^T$.

(a.) find a pair of complex solutions of $\frac{d\vec{x}}{dt} = A\vec{x}$

$$\underline{\vec{z}_1 = e^{(3+i\sqrt{2})t}\vec{u}_1}.$$

$$\underline{\vec{z}_2 = e^{(3+i\sqrt{2})t}(\vec{u}_2 + t\vec{u}_1)}.$$

$$\vec{u}_1 = \begin{bmatrix} 3+i \\ 4+2i \\ 5+3i \\ 6+4i \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}}_{\vec{a}_1} + i\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{\vec{b}_1}$$

$$\vec{u}_2 = \begin{bmatrix} i \\ 1 \\ 2 \\ 3-i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{a}_2} + i\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{\vec{b}_2}$$

(b.) extract four real solutions to write the general real solution (c_1, c_2, c_3, c_4 should be real in this answer)

$$\boxed{\begin{aligned} \vec{x} = {} & c_1 e^{3t}\left(\cos(\sqrt{2}t)\vec{a}_1 - \sin(\sqrt{2}t)\vec{b}_1\right) + c_2 e^{3t}\left(\sin(\sqrt{2}t)\vec{a}_1 + \cos(\sqrt{2}t)\vec{b}_1\right) \\ & + c_3 e^{3t}\left(\cos(\sqrt{2}t)(\vec{a}_2 + t\vec{a}_1) - \sin(\sqrt{2}t)(\vec{b}_2 + t\vec{b}_1)\right) \\ & + c_4 e^{3t}\left(\sin(\sqrt{2}t)(\vec{a}_2 + t\vec{a}_1) + \cos(\sqrt{2}t)(\vec{b}_2 + t\vec{b}_1)\right) \end{aligned}}$$

Problem 66 Consider A is a 3×3 matrix for which there exist nonzero vectors v_1, v_2, v_3 such that:

$$Av_1 = 10v_1, \quad Av_2 = 10v_2, \quad Av_3 = 10v_3 + v_1$$

derive the general solution for $\frac{d\vec{r}}{dt} = A\vec{r}$ with appropriate arguments based on the matrix exponential.

$$\boxed{\vec{r}(t) = c_1 e^{10t}V_1 + c_2 e^{10t}V_2 + c_3 e^{10t}(V_3 + tV_1)}$$

Problem 67 Suppose $A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$. Calculate e^{tA}.
Also, solve $\frac{d\vec{r}}{dt} = A\vec{r}$ given that $\vec{r}(0) = (1,2)$.

$$\underline{e^{tA} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}}. \quad \text{and} \quad \underline{\vec{r}(t) = e^{2t}\begin{bmatrix} 1+10t \\ 2 \end{bmatrix}}.$$

Problem 68 work out problem 15 of section 8.3.2 in Zill. That is, solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$

$$\underline{\vec{x}(t) = c_1\begin{bmatrix} 2e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + \begin{bmatrix} (4t+3)e^t \\ (2t+3)e^t \end{bmatrix}}.$$

Problem 69 Consider the constant coefficient problem $ay'' + by' + cy = 0$ where a, b, c are real constants and $a \neq 0$. Use reduction of order with $x_1 = y$ and $x_2 = y'$ to rewrite the given second order ODE as a system of first order ODEs. Calculate the characteristic equation for your system and comment on how it compares to the usual characteristic equation for the given second order ODE.

$$x_1' = y' = x_2$$
$$x_2' = y'' = \frac{-1}{a}(by' + cy) = \frac{-b}{a}x_2 - \frac{c}{a}x_1$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -c/a & -\lambda - b/a \end{bmatrix}$$
$$= \lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a}$$
$$= \underbrace{(a\lambda^2 + b\lambda + c)\frac{1}{a} = 0}$$

same eq^{n} as the characteristic eq^{n} for
$ay'' + by' + cy = 0$
$\hookrightarrow a\lambda^2 + b\lambda + c = 0$
(just divided by a)

The method outlined below is most meaningful in a larger discussion involving coordinate change for linear transformations. The coordinates $\vec{y} = P^{-1}\vec{x}$ are **eigencoordinates**. *A matrix is said to be* **diagonalizable** *iff there exists some coordinate change matrix P such that $P^{-1}AP = D$ where D is diagonalizable. Not all matrices are diagonalizable. We've seen this.*

Problem 70 To solve $\frac{d\vec{x}}{dt} = A\vec{x}$ in the case $A = \begin{bmatrix} -3 & 0 & -3 \\ 1 & -2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ by the following calculations:

(a) find the e-values and corresponding e-vectors $\vec{u_1}, \vec{u_2}, \vec{u_3}$. (you may use technology)

$$\lambda_1 = -2 = \lambda_2, \quad \lambda_3 = 0$$

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(b) construct $P = [\vec{u_1}|\vec{u_2}|\vec{u_3}]$ and calculate $P^{-1}AP$. (you may use technology)

$$P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) note the solution of $AP\vec{y} = \frac{d}{dt}[P\vec{y}] = P\frac{d\vec{y}}{dt}$ is easily found since multiplying by P^{-1} yields $P^{-1}AP\vec{y} = P^{-1}P\frac{d\vec{y}}{dt} = I\frac{d\vec{y}}{dt} = \frac{d\vec{y}}{dt}$. Solve $P^{-1}AP\vec{y} = \frac{d\vec{y}}{dt}$. (this should be really easy, just solve 3 first order problems, one at a time)

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{-2t} \\ c_3 \end{bmatrix}$$

(d) $AP\vec{y} = \frac{d}{dt}[P\vec{y}]$ means $\vec{x} = P\vec{y}$ solves $\frac{d\vec{x}}{dt} = A\vec{x}$. Solve the original system by multiplying the solution from (3.) by P.

$$\underline{\vec{x} = c_1 e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}.$$

Problem 71 An ice tray has tiny holes between each of its three partititions such that the water can flow from one partition to the next. Let x, y, z denote the height of water in the three water troughs. The holes are designed such that the flow rate is proportional to the height of water above the adjacent trough. For example, supposing x and z are the edge troughs whereas y is in the middle we have $\frac{dx}{dt} = k(y - x)$. For simplicity of discussion suppose $k = 1$. Write the corresponding differential equations to find the water-level in the y and z troughs. If initially there is 3.0 cm of water in the x trough and none in the other two troughs then find the height in all three troughs as a function of time t. Discuss the steady state solution, is it reasonable?

(see solution)

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1.5e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0.5e^{-3t}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Problem 72 Let a, b be constants which are some measure of the trust between two nations. Furthermore, let x be the military expenditure of Bobslovakia and let y be the military expenditure of the Leaf Village. Detailed analysis by strategically gifted ninjas reveal that

$$\frac{dx}{dt} = -x + 2y + a$$

$$\frac{dy}{dt} = 4x - 3y + b$$

Analyze possible outcomes for various initial conditions and values of a, b. Consider drawing an ab-plane to explain your solution(s). Is a stable peace without a run-away arms race possible given the analysis thus far?

$$\underline{\vec{X}(t) = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 3a+2b \\ 4a+b \end{bmatrix}}.$$

If $c_1 \neq 0$ then $\|\vec{X}(t)\| \to \infty$ as $t \to \infty$

Thus $c_1 = 0$ is our only hope, in that case we need b & a related as discussed in sol$^{\underline{n}}$.

MISSION 3 SOLUTION

P49 $y'' - x^2 y = 0$ find 1st 4 nonzero terms in power series solution about $x = 0$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$
$$y'' = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots$$

Thus,

$$0 = y'' - x^2 y = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots - x^2(c_0 + c_1 x + c_2 x^2 + \cdots)$$
$$= 2c_2 + 6c_3 x + x^2(12c_4 - c_0) + x^3(20c_5 - c_1) + \cdots$$

We find from equating coefficients,

$$\left.\begin{array}{llll} 1: & 0 = 2c_2 & \therefore & c_2 = 0 \\ x: & 0 = 6c_3 & \therefore & c_3 = 0 \\ x^2: & 0 = 12c_4 - c_0 & \therefore & c_4 = c_0/12 \\ x^3: & 0 = 20c_5 - c_1 & \therefore & c_5 = c_1/20 \end{array}\right\} \text{this gives 4-terms,}$$

We find

$$y = c_0 + c_1 x + \frac{c_0}{12}x^4 + \frac{c_1}{20}x^5 + \cdots$$

$$\boxed{y = c_0\left(1 + \frac{1}{12}x^4 + \cdots\right) + c_1\left(x + \frac{1}{20}x^5 + \cdots\right)}$$

P50 Find complete power series solution of $y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy \Rightarrow \int \frac{dy}{y} = \int 2x\,dx$$

$$\Rightarrow \ln|y| = x^2 + c_1$$

$$\Rightarrow |y| = \exp(x^2 + c_1) = \exp(c_1)\exp(x^2)$$

$$\Rightarrow y = k e^{x^2}$$

$$\Rightarrow \boxed{y = k \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}} - *$$

Alternatively,

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\therefore \quad y' - 2xy = \sum_{n=1}^{\infty} n c_n x^{n-1} - 2x \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 2 c_n x^{n+1}$$

$$= \underset{(n-1=j)}{\sum_{j=0}^{\infty} (j+1) c_{j+1} x^j} - \underset{(j=n+1)}{\sum_{j=1}^{\infty} 2 c_{j-1} x^j}$$

$$= c_1 + \sum_{j=1}^{\infty} \left[(j+1) c_{j+1} - 2c_{j-1}\right] x^j = 0$$

Thus $c_1 = 0$ and,

$(j+1)c_{j+1} - 2c_{j-1} = 0$ for $j = 1, 2, 3, \ldots$ Setting $j-1 = n$ we have $j+1 = n+2$ and $c_{n+2} = \dfrac{2c_n}{n+2}$ for $n = 0, 1, 2, \ldots$

Notice $c_1 = 0 \Rightarrow c_{2k+1} = 0$. If we went on, we could also derive $c_{2k} = \dfrac{1}{k!}$ which brings us back to $*$.

P51 $y'' - xy' + 4y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \ , \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \ , \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Thus,

$$y'' - xy' + 4y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 4c_n x^n$$

$$= \sum_{j=0}^{\infty} (j+2)(j+1)c_{j+2} x^j - \sum_{j=1}^{\infty} j c_j x^j + \sum_{j=0}^{\infty} 4c_j x^j$$

$$= 2c_2 + 4c_0 + \sum_{j=1}^{\infty} \left[(j+2)(j+1)c_{j+2} - j c_j + 4c_j \right] x^j$$

$$= 2c_2 + 4c_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)c_{n+2} + (4-n)c_n \right] x^n$$

We find $2c_2 + 4c_0 = 0$ and $(n+1)(n+2)c_{n+2} + (4-n)c_n = 0$ for $n = 1, 2, 3, \dots$ thus $c_{n+2} = \dfrac{(n-4)c_n}{(n+2)(n+1)}$ for $n = 1, 2, 3, \dots$

Note $\underline{c_2 = -2c_0}$. Moreover, $c_4 = \dfrac{(2-4)c_2}{4\cdot 3} = \dfrac{-2}{4\cdot 3}(-2c_0) = \dfrac{c_0}{3}$

and $c_6 = c_{4+2} = \dfrac{(4-4)c_4}{6\cdot 5} = 0$. Then $c_8 = 0 = c_{10} = c_{12} \cdots$

Now study $n = 2k+1$ for $k = 0, 1, 2, \dots$, $n+2 = 2k+3$,

$\underline{k=0}$ $\quad c_3 = \dfrac{-3c_1}{3\cdot 2} = \dfrac{-c_1}{2} = \dfrac{-3c_1}{3!}$

$\underline{k=1}$ $\quad c_5 = \dfrac{-c_3}{5\cdot 4} = \dfrac{-(-c_1)}{5\cdot 4\cdot 2} = \dfrac{(-3)(-1)c_1}{5!}$

$\underline{k=2}$ $\quad c_7 = \dfrac{c_5}{7\cdot 6} = \dfrac{c_1}{7\cdot 6\cdot 5\cdot 4\cdot 2} = \dfrac{(-3)(-1)(1)c_1}{7!}$

$\underline{k=3}$ $\quad c_9 = \dfrac{3c_7}{9\cdot 8} = \dfrac{3c_1}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 2} = \dfrac{(-3)(-1)(1)(3)c_1}{9!}$

$c_{2k+1} = \dfrac{(-3)(-1)\cdots(2k-5)c_1}{(2k+1)!}$

pattern

$$\boxed{y = c_0\left(1 - 2x^2 + \tfrac{1}{3}x^4\right) + c_1\left(x + \sum_{k=1}^{\infty} \frac{(-3)(-1)(1)\cdots(2k-5)}{(2k+1)!} x^{2k+1}\right)}$$

P52 $\quad y'' - e^{2x}y' + \cos(x)y = 0$

given $\quad y(0) = -1 \quad$ and $\quad y'(0) = 1$

$$y = -1 + x + c_2x^2 + c_3x^3 + \cdots$$

$$y' = 1 + 2c_2x + 3c_3x^2 + \cdots$$

$$y'' = 2c_2 + 6c_3x + \cdots$$

Then,

$$y'' - e^{2x}y' + \cos x\, y = (2c_2 + 6c_3x + \cdots) - (1 + 2x + \cdots)(1 + 2c_2x + \cdots) + $$

$$+ (1 - \tfrac{1}{2}x^2 + \cdots)(-1 + x + c_2x^2 + \cdots)$$

$$= 2c_2 - 1 - 1 + x(6c_3 - 2 - 2c_2 + 1) + \cdots = 0$$

$$2c_2 - 2 = 0 \Rightarrow \underline{c_2 = 1}.$$

$$6c_3 - 1 - 2c_2 = 0 \Rightarrow c_3 = \frac{1 + 2c_2}{6} = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \boxed{y = -1 + x + x^2 + \tfrac{1}{2}x^3 + \cdots}$$

$\boxed{P53}$ $y'' + xy' + y = x^2 + 2x + 1$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$y'' = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots$$

Thus,

$$y'' + xy' + y = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots +$$

$$x(c_1 + 2c_2 x + \cdots) +$$

$$c_0 + c_1 x + c_2 x^2 + \cdots = x^2 + 2x + 1$$

$$= (2c_2 + c_0) + x(6c_3 + c_1 + c_1) + x^2(12c_4 + 2c_2 + c_2) + \cdots$$

Then, equating coefficients of $1, x, x^2$ yields,

$1 \rfloor \quad 2c_2 + c_0 = 1 \quad \therefore \quad c_2 = \dfrac{1 - c_0}{2} = \dfrac{1}{2} - \dfrac{1}{2}c_0$

$x \rfloor \quad 6c_3 + 2c_1 = 2 \quad \therefore \quad c_3 = \dfrac{2 - 2c_1}{6} = \dfrac{1}{3} - \dfrac{1}{3}c_1$

$x^2 \rfloor \quad 12c_4 + 3c_2 = 1$

$$\therefore \quad c_4 = \frac{1 - 3c_2}{12} = \frac{1}{12} - \frac{1}{4}\left(\frac{1 - c_0}{2}\right)$$

$$c_4 = \frac{-1}{24} + \frac{1}{8}c_0$$

$$y = c_0 + c_1 x + \left(\frac{1}{2} - \frac{1}{2}c_0\right)x^2 + \left(\frac{1}{3} - \frac{1}{3}c_1\right)x^3 + \left(\frac{-1}{24} + \frac{1}{8}c_0\right)x^4 + \cdots$$

$$\boxed{y = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots\right) + c_1\left(x - \frac{1}{3}x^3 + \cdots\right) + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4 + \cdots}$$

PS4 Find singularities of

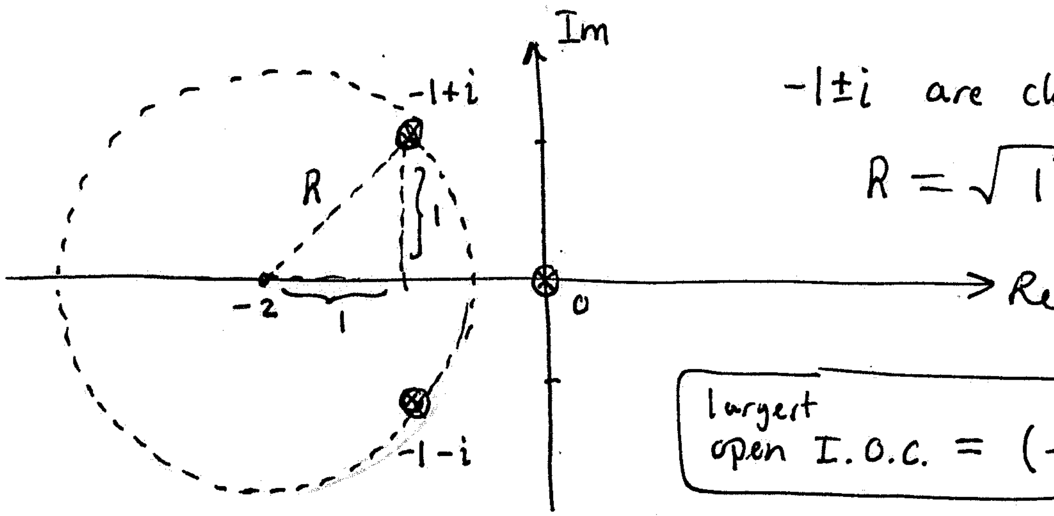
$$x(x^2+2x+2)y'' + (x^2+1)y' + 3y = 0$$

and determine largest open I.O.C. for $y = \sum_{n=0}^{\infty} a_n (x+2)^n$

singularities from $x(x^2+2x+2) = 0$

$\underline{x = 0}$. or $x^2+2x+2 = (x+1)^2+1 = 0$

$\therefore \underline{x = -1 \pm i}$.



$-1 \pm i$ are closer to -2 than 0.

$R = \sqrt{1^2+1^2} = \sqrt{2}$

$\boxed{\text{largest open I.O.C.} = (-2-\sqrt{2}, -2+\sqrt{2})}$

PS5

Suppose we define $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$. Show that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{j=0}^{\infty} \frac{(i\theta)^{2j+1}}{(2j+1)!} \qquad \because (i)^{2j+1} = i(i)^{2j} = i(i^2)^j = i(-1)^j$$

$$= \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}}_{\cos\theta} + i \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j \theta^{2j+1}}{(2j+1)!}}_{\sin\theta}$$

Hence $\underline{e^{i\theta} = \cos\theta + i\sin\theta}$.

PSS continued

Alternatively, & much more popularly,

$$e^{i\theta} = 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \cdots$$

$$= 1 + i\theta - \frac{1}{2}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{i}{5!}\theta^5 + \cdots$$

$$= \underbrace{\left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 - \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots\right)}_{\sin\theta}$$

(from Calculus II where we derived these expansions and possibly proved their convergence via Taylor's estimate ...)

PROBLEM 56

Suppose $\sum_{k=0}^{\infty}\left(a_{2k}x^{2k} + b_{2k+1}x^{2k+1}\right) = e^x + \cos(x+2)$. Find explicit formulas for a_{2k} and b_{2k+1} via Σ-notation algebra.

$$\cos(x+2) = \cos x \cos 2 - \sin x \sin 2$$

$$= \cos 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} - \sin(2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

Hence, noting $e^x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ (broke into even/odd)

$$\sum_{k=0}^{\infty} a_{2k}x^{2k} + \sum_{k=0}^{\infty} b_{2k+1}x^{2k+1} =$$

$$= \sum_{k=0}^{\infty}\left(\cos(2)\frac{(-1)^k}{(2k)!} + \frac{1}{(2k)!}\right)x^{2k} + \sum_{k=0}^{\infty}\left(-\sin(2)\frac{(-1)^k}{(2k+1)!} + \frac{1}{(2k+1)!}\right)x^{2k+1}$$

Therefore, as even & odd do not overlap,

$$\boxed{a_{2k} = \frac{1}{(2k)!}\left(1 + (-1)^k\cos(2)\right)} \ \& \ \boxed{b_{2k+1} = \frac{1}{(2k+1)!}\left(1 - (-1)^k\sin 2\right)}$$

PROBLEM 57

[redacted]) Find a power series solution to the integrals below:

(a.) $\int \frac{x^3+x^6}{1-x^3}\,dx$

(b.) $\int x^8 e^{x^3+2}\,dx$

(a.) Note that $\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}$ for $|x^3|<1$.

Hence,

$$\int \frac{x^3+x^6}{1-x^3}\,dx = \int \left(x^3 \sum_{n=0}^{\infty} x^{3n} + x^6 \sum_{n=0}^{\infty} x^{3n} \right) dx$$

$$= \int \left(\sum_{n=0}^{\infty} x^{3n+3} + \sum_{n=0}^{\infty} x^{3n+6} \right) dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} \frac{x^{3n+4}}{3n+4} + \sum_{n=0}^{\infty} \frac{x^{3n+7}}{3n+7}}$$ sloppy answer

$$3j = 3n+3$$

$$3j+4 = 3n+7.$$

$$= C + \frac{x^4}{4} + \sum_{n=1}^{\infty} \frac{x^{3n+4}}{3n+4} + \sum_{j=1}^{\infty} \frac{x^{3j+4}}{3j+4}$$

$$= \boxed{C + \frac{x^4}{4} + \sum_{n=1}^{\infty} \frac{2}{3n+4} x^{3n+4}}$$ $\Leftarrow$ better answer (earned bonus)

(b.) $$\int x^8 e^{x^3+2}\,dx = e^2 \int x^8 \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}\,dx$$

$$= e^2 \int \sum_{n=0}^{\infty} \frac{x^{3n+8}}{n!}\,dx$$

$$= \boxed{e^2 \sum_{n=0}^{\infty} \frac{x^{3n+9}}{(3n+9)n!} + C}$$

PS8

Problem 43 Suppose $\frac{dx}{dt} = x + 4y$ and $\frac{dy}{dt} = x + y$. Find the general real solution via the e-vector method.

$$\left.\begin{aligned} \frac{dx}{dt} &= x + 4y \\ \frac{dy}{dt} &= x + y \end{aligned}\right\} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} = (\lambda-1)^2 - 4 = (\lambda - 1 - 2)(\lambda - 1 + 2)$$

Hence $\lambda_1 = 3$ or $\lambda_2 = -1$.

$\lambda_1 = 3$

$$(A - 3I)\vec{u}_1 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow u - 2v = 0$$

Let $v = 1$, $u = 2v = 2$. $\therefore \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\Rightarrow \underline{\vec{x}_1(t) = e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}.$$

$\lambda_2 = -1$

$$(A + I)\vec{u}_2 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow u + 2v = 0$$

Let $v = 1$, $u = -2v = -2$. $\therefore \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\Rightarrow \vec{x}_2(t) = e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Therefore,

$$\boxed{\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

announcement said this was constant

Problem 47 Suppose X is a fundamental matrix for $\frac{d\vec{x}}{dt} = A\vec{x}$. Suppose B is a square matrix with $det(B) \neq 0$. Show that XB is a fundamental matrix for $\frac{d\vec{x}}{dt} = A\vec{x}$.

since X is fund. matrix it is sol$^{\underline{n}}$ matrix.

1) $\frac{d}{dt}(XB) = \frac{dX}{dt}B + X\cancelto{0}{\frac{dB}{dt}} = \frac{dX}{dt}B = AXB$

thus $(XB)' = A(XB)$ hence XB is a sol$^{\underline{n}}$ matrix.

2.) $\det(XB) = \underbrace{\det(X)}_{\neq 0}\underbrace{\det(B)}_{\neq 0} \neq 0$ $\therefore$ XB is nonsingular. Hence XB is a fundamental matrix for $\vec{x}' = A\vec{x}$.

PS8 continued

Calculate e^{tA} for $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$.

Trick: by general sol$^{\underline{n}}$ Th$^{\underline{m}}$ can write

$$e^{At} = X(t)B \quad \text{for some constant } B.$$

But, $e^{0} = I = X(0)B \Rightarrow B = X^{-1}(0)$.

Hence, $\underline{e^{At} = X(t)X^{-1}(0)}$ (a neat, but usually useless formula for e^{At})

$$e^{At} = \begin{bmatrix} 2e^{3t} & -2e^{-t} \\ e^{3t} & e^{-t} \end{bmatrix}\begin{bmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{bmatrix}$$

(plugging $t=0$ into $X^{-1}(t)$ in Problem 46)

$$= \boxed{\begin{bmatrix} \frac{1}{2}(e^{3t}+e^{-t}) & \frac{1}{4}(e^{3t}-e^{-t}) \\ \frac{1}{4}(e^{3t}-e^{-t}) & \frac{1}{2}(e^{3t}+e^{-t}) \end{bmatrix}}$$

As a quick check, $e^{A(0)} = \begin{bmatrix} \frac{1}{2}(1+1) & \frac{1}{4}(1-1) \\ \frac{1}{4}(1-1) & \frac{1}{2}(1+1) \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$.

P59

Suppose $\frac{dx}{dt} = 2x + y$ and $\frac{dy}{dt} = 2y$. Find the general real solution via the generalized e-vector method.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} \rightsquigarrow \det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix}$$

$$= (\lambda-2)^2$$

$$= 0 \Rightarrow \underline{\lambda_1 = \lambda_2 = 2}.$$

Find e-vector for $\lambda_1 = 2$

$$(A-2I)\vec{u}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} v = 0 \\ u \text{ free} \end{matrix} \longrightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}.$$

We find one sol$^{\underline{n}}$, $\vec{x}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We need another!

Consult the matrix exp. / gen. e-vector technique.

$$(A-2I)\vec{u}_2 = \vec{u}_1 \leftarrow \text{find } \vec{u}_2 \text{ to make}$$

$(A-2I)^2\vec{u}_2 = 0$

and $\vec{u}_1, \vec{u}_2$ is a chain

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$v = 1$, u-free $\Rightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ nice choice.

Use the fact e^{At} is sol$^{\underline{n}}$ matrix & the magic f-la to find

$$\vec{X}_2(t) = e^{At}\vec{u}_2 = e^{2t}\left(\vec{u}_2 + t(A-2I)\vec{u}_2 + \frac{t^2}{2!}(A-2I)^2\vec{u}_2 + \cdots\right)$$

$$= e^{2t}(\vec{u}_2 + t\,\vec{u}_1)$$

$$= e^{2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

(ok to skip some of these details if you understand them.)

Therefore, $\boxed{\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ 1 \end{bmatrix}}$

P60

Suppose $\frac{dx}{dt} = 4x - 3y$ and $\frac{dy}{dt} = 3x + 4y$. Find the general real solution via the e-vector method.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix} = (\lambda - 4)^2 + 9 = 0$$

$$\Rightarrow \underline{\lambda = 4 \pm 3i}.$$

<u>Consider $\lambda = 4 + 3i$</u>

$$(A - (4+3i)I)\vec{u}_1 = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow -3iu - 3v = 0$$

$\hookrightarrow \underline{v = -iu}$.

Let $u = 1$ then $v = -i$ hence $\vec{u}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

We find <u>complex</u> sol$^{\underline{n}}$

$$\begin{aligned}
\vec{x}(t) &= e^{(4+3i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&= e^{4t}\left(\cos(3t) + i\sin(3t)\right)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\
&= \underbrace{e^{4t}\left(\cos(3t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)}_{\vec{x}_1(t) = Re(\vec{x}(t))} + i\underbrace{e^{4t}\left(\cos 3t\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin 3t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}_{\vec{x}_2(t) = Im(\vec{x}(t))}
\end{aligned}$$

We've shown a complex sol$^{\underline{n}}$ to $\vec{x}' = A\vec{x}$ contains <u>two</u> real sol$^{\underline{n}}$'s, the real & imaginary parts indicated above.

Thus,

$$\boxed{\vec{x}(t) = c_1 e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \sin 3t \\ -\cos 3t \end{bmatrix}}$$

P61

[illegible]

Suppose $\frac{dx}{dt} = 5x - 6y - 6z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 6y - 4z$. Find the general real solution via the e-vector method.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \underbrace{\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{pmatrix}$$

$$= (5-\lambda)\left[(\lambda+4)(\lambda-4)+12\right] + 6\left[\lambda+4-6\right] - 6\left[6+3(\lambda-4)\right]$$

$$= (5-\lambda)(\lambda^2-4) + 6(\lambda-2) - 6(3\lambda-6)$$

$$= (\lambda-2)\left[(5-\lambda)(\lambda+2)+6-18\right]$$

$$= (\lambda-2)\left[-\lambda^2+3\lambda-2\right]$$

$$= -(\lambda-2)(\lambda-1)(\lambda-2) \quad \to \quad \lambda_1 = 1, \quad \lambda_2 = 2 = \lambda_3$$

$\lambda_1 = 1)$

$$(A - I)\vec{u}_1 = 0 \Rightarrow \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u - w = 0$$
$$v + \tfrac{w}{3} = 0$$

Choose $w = 3$, $u = w = 3$, $v = -1$.

Hence $\vec{u}_1 = [3, -1, 3]^T$.

$\lambda_2 = 2)$

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \Rightarrow -u + 2v + 2w = 0$$
$$u = 2v + 2w$$

Hence, $\vec{u}_2 = \begin{bmatrix} 2v+2w \\ v \\ w \end{bmatrix} = v\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ use $\underline{\vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$ & $\underline{\vec{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}$

Thus,
$$\boxed{\vec{x}(t) = c_1 e^{t}\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}$$

no "extra" t !

P62

Suppose $\frac{dx}{dt} = 5x - 5y - 5z$, $\frac{dy}{dt} = -x + 4y + 2z$ and $\frac{dz}{dt} = 3x - 5y - 3z$. Find the general real solution via the e-vector method.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \underbrace{\begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -5 & -5 \\ -1 & 4-\lambda & 2 \\ 3 & -5 & -3-\lambda \end{bmatrix}$$

$$= (5-\lambda)\left[(\lambda-4)(\lambda+3)+10\right] + 5\left[\lambda+3-6\right] - 5\left[5-3(4-\lambda)\right]$$

$$= -(\lambda-5)\left[\lambda^2-\lambda-2\right] + \underbrace{5[\lambda-3] + 35 - 15\lambda}_{-10\lambda+20}$$

$$= -(\lambda-5)(\lambda-2)(\lambda+1) - 10(\lambda-2)$$

$$= -(\lambda-2)\left[(\lambda-5)(\lambda+1)+10\right]$$

$$= -(\lambda-2)((\lambda-2)^2+1) \Rightarrow \underline{\lambda_1 = 2} \text{ and } \underline{\lambda_2 = 2 \pm i}.$$

$\lambda = 2$ | $(A - 2I)\vec{u}_1 = \begin{bmatrix} 3 & -5 & -5 \\ -1 & 1 & 2 \\ 3 & -5 & -5 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$

$$\begin{aligned} 3u - 5v - 5w &= 0 \\ -3u + 3v + 6w &= 0 \\ \hline -2v + w &= 0 \end{aligned} \Rightarrow \underline{w = 2v}$$

Let $v = 1$ then $w = 2$ and $u = v + 2w = 1 + 4 = 5$. Thus $\vec{u}_1 = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$.

$\lambda = 2 + i$ |

$$(A - (2+i))\vec{u}_2 = \begin{bmatrix} 3-i & -5 & -5 \\ -1 & 2-i & 2 \\ 3 & -5 & -5-i \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \rightarrow \begin{pmatrix} (3-i)u - 5v - 5w = 0 \\ 3u - 5v - (5+i)w = 0 \end{pmatrix}$$

$$-iu - iw = 0 \quad \therefore \quad \underline{w = -u}.$$

Let $u = 1$ then $w = -1$ and we can use any eqn with v to fix the value of v;

$$3u - 5v - (5+i)w = 0$$
$$5v = 3u - (5+i)w = 3 + 5 + i = 8 + i$$
$$\Rightarrow v = \tfrac{8}{5} + \tfrac{i}{5}$$

We obtain $\vec{u}_2 = [1, \frac{8+i}{5}, -1]^T$

But, I prefer $\underline{\vec{u}_2 = [5, 8+i, -5]^T}$. It follows that

$$\boxed{\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{2t}\left(\cos t\begin{bmatrix} 5 \\ 8 \\ -5 \end{bmatrix} - \sin t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + c_3 e^{2t}\left(\sin t\begin{bmatrix} 5 \\ 8 \\ -5 \end{bmatrix} + \cos t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)}$$

P63

Suppose $\frac{dx}{dt} = 3x+y$, $\frac{dy}{dt} = 3y+z$ and $\frac{dz}{dt} = 3z$. Find the general real solution via the generalized e-vector method.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \underbrace{\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\det(A - \lambda I) = (\lambda - 3)^3 = 0 \implies \underline{\lambda_1 = \lambda_2 = \lambda_3 = 3}.$$

Moreover, by inspection, (standard basis <u>is</u> generalized e-basis for A in Jordan form!)

$$(A - 3I)e_1 = 0$$
$$(A - 3I)e_2 = e_1$$
$$(A - 3I)e_3 = e_2$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus,

$$\vec{x}_1(t) = e^{At}e_1 = e^{3t}e_1$$

$$\vec{x}_2(t) = e^{At}e_2 = e^{3t}\left(e_2 + t(A-3I)e_2 + \frac{t^2}{2}\underbrace{(A-3I)^2e_2}_{0} + \cdots\right) = e^{3t}(e_2 + te_1)$$

$$\vec{x}_3(t) = e^{At}e_3 = e^{3t}\left(e_3 + t(A-3I)e_3 + \frac{t^2}{2}(A-3I)^2e_3 + \cdots\right)$$
$$= e^{3t}\left(e_3 + te_2 + \frac{t^2}{2}(A-3I)e_2\right)$$
$$= e^{3t}\left(e_3 + te_2 + \frac{t^2}{2}e_1\right)$$

Thus,
$$\boxed{\vec{x}(t) = c_1e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3e^{3t}\begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}}$$

Btw, $\text{col}_j(A) = Ae_j$ thus $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are columns 1, 2 and 3 of e^{At}. In fact,

$$e^{At} = \begin{bmatrix} e^{3t} & te^{3t} & \frac{1}{2}t^2e^{3t} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}$$

Almost as easy as exponentiating a diagonal matrix.

P64

Suppose A is a 3×3 matrix with nonzero vectors $\vec{u}, \vec{v}, \vec{w}$ such that

$$A\vec{u} = 3\vec{u}, \qquad (A - 3I)\vec{v} = \vec{u}, \qquad A\vec{w} = 0. \longrightarrow \vec{w} \text{ is e-vector with e-value zero.}$$

Write the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ in terms of the given vectors.

$e^{At}\vec{w} = \vec{w}$.

$$(A - 3I)\vec{u} = 0 \Rightarrow e^{At}\vec{u} = e^{3t}\vec{u}$$

$$(A - 3I)\vec{v} = \vec{u} \Rightarrow e^{At}\vec{v} = e^{3t}(I + t(A-3I) + \cdots)\vec{v} = e^{3t}(\vec{v} + t\vec{u})$$

$$\therefore \boxed{\vec{x}(t) = c_1\vec{w} + c_2 e^{3t}\vec{u} + c_3 e^{3t}(\vec{v} + t\vec{u})}$$

P65

$$\vec{u}_1 = \underbrace{\begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}}_{\vec{a}_1} + i \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{\vec{b}_1} \quad \& \quad \vec{u}_2 = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}_{\vec{a}_2} + i \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{\vec{b}_2} \qquad (*)$$

(a.) $$e^{At}\vec{u}_1 = \underline{e^{(3+i\sqrt{2})t}\vec{u}_1 = \vec{z}_1}.$$

$$e^{At}\vec{u}_2 = e^{\lambda t}\left(I + t(A - \lambda I) + \frac{t^2}{2}(A - \lambda I)^2 + \cdots\right)\vec{u}_2$$

$$= e^{\lambda t}\left(\vec{u}_2 + t(A - \lambda I)\vec{u}_2 + 0 \ldots\right)$$

$$= \underline{e^{(3+i\sqrt{2})t}(\vec{u}_2 + t\vec{u}_1) = \vec{z}_2}.$$

$\vec{z}_1, \vec{z}_2$ are complex sol^{n}s to $\vec{x}' = A\vec{x}$.

(b.)
$$\boxed{\begin{aligned} \vec{x} = {} & c_1 e^{3t}\left(\cos\sqrt{2}t\,\vec{a}_1 - \sin\sqrt{2}t\,\vec{b}_1\right) + c_2 e^{3t}\left(\sin\sqrt{2}t\,\vec{a}_1 + \cos\sqrt{2}t\,\vec{b}_1\right) \\ & + c_3 e^{3t}\left((\vec{a}_2 + t\vec{a}_1)\cos\sqrt{2}t - (\vec{b}_2 + t\vec{b}_1)\sin\sqrt{2}t\right) \\ & + c_4 e^{3t}\left((\vec{a}_2 + t\vec{a}_1)\sin\sqrt{2}t + (\vec{b}_2 + t\vec{b}_1)\cos\sqrt{2}t\right) \end{aligned}}$$

where $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$ are as given in $(*)$

$\boxed{P66}$ Given $A \in \mathbb{R}^{3\times 3}$ with nonzero V_1, V_2, V_3 such that

$$A V_1 = 10 V_1 \Rightarrow (A - 10I)V_1 = 0$$

$$A V_2 = 10 V_2 \Rightarrow (A - 10I)V_2 = 0$$

$$A V_3 = 10 V_3 + V_1 \Rightarrow (A - 10I)V_3 = V_1$$

We see V_1 & V_2 are eigenvectors and for V_3,

$$e^{tA} V_3 = e^{10t}\left(I + t(A - 10I) + \frac{t^2}{2}(A - 10I)^2 + \cdots\right)V_3$$

$$= e^{10t}\left(V_3 + t(A - 10I)V_3 + \frac{t^2}{2}(A - 10I)V_2 + \cdots\right)$$

$$= e^{10t}\left(V_3 + t V_1\right)$$

Thus,

$$\frac{d\vec{r}}{dt} = A\vec{r}$$

has solution

$$\boxed{\vec{r}(t) = c_1 e^{10t} V_1 + c_2 e^{10t} V_2 + c_3 e^{10t}(V_3 + t V_1)}$$

$\boxed{P67}$ $A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$

Calculate e^{tA} and solve $\frac{d\vec{r}}{dt} = A\vec{r}$ given that $\vec{r}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{bmatrix} = (\lambda-2)^2 = 0 \quad \therefore \quad \underline{\lambda = 2 \text{ twice}}.$$

$$(A - 2I)\vec{u}_1 = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}.$$

$$(A - 2I)\vec{u}_2 = \vec{u}_1 \Rightarrow \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow 5v = 1 \quad \therefore \underline{v = 1/5}$$

u free, I'll choose $\underline{u = 0}$.

Hence $\vec{u}_2 = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$ is generalized e-vector of order 2 subject to $(A - 2I)\vec{u}_2 = \vec{u}_1$ hence $(A - 2I)^2\vec{u}_2 = (A - 2I)\vec{u}_1 = 0$.

$$e^{tA}\vec{u}_2 = e^{2t}\left[I + t(A - 2I) + \cdots\right]\vec{u}_2 = \underline{e^{2t}(\vec{u}_2 + t\vec{u}_1)}.$$

Likewise $e^{tA}\vec{u}_1 = e^{2t}\vec{u}_1$ is solution and thus

$$X(t) = \left[e^{tA}\vec{u}_1 \mid e^{tA}\vec{u}_2\right] = e^{tA}\left[\vec{u}_1 \mid \vec{u}_2\right] = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t}/5 \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t}/5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix}^{-1}$$

$$= e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} 1 & 5t \\ 0 & 1 \end{bmatrix} \quad \therefore \quad \boxed{e^{tA} = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}}$$

Solution to $\frac{d\vec{r}}{dt} = A\vec{r}$ with $\vec{r}(0) = \vec{r}_o$ is given by $\vec{r}(t) = e^{tA}\vec{r}_o$

$$\vec{r}(t) = \begin{bmatrix} e^{2t} & 5te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \boxed{\vec{r}(t) = \begin{bmatrix} (1+10t)e^{2t} \\ 2e^{2t} \end{bmatrix}}$$

$\boxed{P68}$

work out problem 15 of section 8.3.2 in Zill. That is, solve $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$ and $\vec{f}(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix} = \lambda(\lambda-3) + 2 = \lambda^2 - 3\lambda + 2 \cancel{=}$$

$$0 = (\lambda-1)(\lambda-2)$$

$$(A - I)\vec{u}_1 = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \underline{\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}.$$

$$(A - 2I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \underline{\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

$$\Phi(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \longrightarrow \Phi^{-1} = \frac{1}{e^{3t}}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} = \underline{\begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}}$$

$$\vec{X} = \Phi \int \Phi^{-1}\vec{f}\, dt$$

$$= \Phi \int \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}\begin{bmatrix} e^t \\ -e^t \end{bmatrix} dt$$

$$= \Phi \int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt$$

$$= \Phi \begin{bmatrix} 2t + c_1 \\ 3e^{-t} + c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 2t + c_1 \\ 3e^{-t} + c_2 \end{bmatrix}$$

$$= \underline{c_1\begin{bmatrix} 2e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + \begin{bmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{bmatrix}}. //$$

$\boxed{P69}$ (solved on problem sheet)

P70

To solve $\frac{d\vec{x}}{dt} = A\vec{x}$ in the case $A = \begin{bmatrix} -3 & 0 & -3 \\ 1 & -2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ by the following calculations:

1. find the e-values and corresponding e-vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. (you may use technology)
2. construct $P = [\vec{u}_1|\vec{u}_2|\vec{u}_3]$ and calculate $P^{-1}AP$. (you may use technology)
3. note the solution of $AP\vec{y} = \frac{d}{dt}[P\vec{y}] = P\frac{d\vec{y}}{dt}$ is easily found since multiplying by P^{-1} yields $P^{-1}AP\vec{y} = P^{-1}P\frac{d\vec{y}}{dt} = I\frac{d\vec{y}}{dt} = \frac{d\vec{y}}{dt}$. Solve $P^{-1}AP\vec{y} = \frac{d\vec{y}}{dt}$. (this should be really easy, just solve 3 first order problems, one at a time)
4. $AP\vec{y} = \frac{d}{dt}[P\vec{y}]$ means $\vec{x} = P\vec{y}$ solves $\frac{d\vec{x}}{dt} = A\vec{x}$. Solve the original system by multiplying the solution from (3.) by P.

The method outlined above is more meaningful in a larger discussion involving coordinate change for linear transformations. The coordinates $\vec{y} = P^{-1}\vec{x}$ are **eigencoordinates**. *A matrix is said to be* **diagonalizable** *iff there exists some coordinate change matrix P such that $P^{-1}AP = D$ where D is diagonalizable. Not all matrices are diagonalizable. We've seen this. When there are less than n-LI e-vectors then we cannot build the P-matrix as above and it turns out there is no other way to diagonalize a matrix. On the other hand, the generalized e-vectors always exist and cojugating by P made of generalized e-vectors will place* any *matrix in Jordan-form (possibly complex).*

1.) Technology $\Rightarrow$ $\lambda_1 = -2,\ \lambda_2 = -2,\ \lambda_3 = 0$ with $\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

2.) $$P^{-1} = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 2 & -3 \\ -1 & 0 & -1 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

using early part of course.

3.) $P^{-1}AP\vec{y} = \frac{d\vec{y}}{dt}$ $\Rightarrow$ $-2y_1 = \frac{dy_1}{dt} \Rightarrow y_1 = c_1 e^{-2t}$ $(\lambda = -2)$

$\vec{y} = [y_1, y_2, y_3]^T$ $\quad -2y_2 = \frac{dy_2}{dt} \Rightarrow y_2 = c_2 e^{-2t}$ $(\lambda = -2)$

$0 = \frac{dy_3}{dt} \Rightarrow y_3 = c_3$ $(\lambda = 0)$

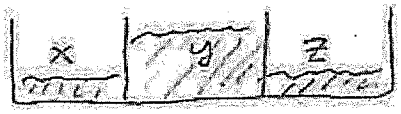
4.) $$\vec{x} = P\vec{y} = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{-2t} \\ c_2e^{-2t} \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} -3c_2e^{-2t} - c_3 \\ c_1e^{-2t} + c_3 \\ c_2e^{-2t} + c_3 \end{bmatrix}$$ $\leftarrow$ also good, just trying to point out the fact this is our usual sol$^{\text{n}}$

$$= \boxed{c_1e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2e^{-2t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

P71

An ice tray has tiny holes between each of its three partititions such that the water can flow from one partition to the next. Let x, y, z denote the height of water in the three water troughs. The holes are designed such that the flow rate is proportional to the height of water above the adjacent trough. For example, supposing x and z are the edge troughs whereas y is in the middle we have $\frac{dx}{dt} = k(y-x)$. For simplicity of discussion suppose $k=1$. Write the corresponding differential equations to find the water-level in the y and z troughs. If initially there is 3.0 cm of water in the x trough and none in the other two troughs then find the height in all three troughs as a function of time t. Discuss the steady state solution, is it reasonable?



Yes! $\vec{x}(t) \longrightarrow \begin{bmatrix}1\\1\\1\end{bmatrix}$ as $t \to \infty$ it levels out in the end. (by the work below)

$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = (x-y)+(z-y)$$
$$\frac{dz}{dt} = y - z$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}' = \begin{bmatrix}-1 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\det\begin{bmatrix}-1-\lambda & 1 & 0\\ 1 & -2-\lambda & 1\\ 0 & 1 & -1-\lambda\end{bmatrix} = -(\lambda+1)\left[(\lambda+2)(\lambda+1)-1\right] - 1\left[-(\lambda+1)\right]$$
$$= -(\lambda+1)\left[(\lambda+1)(\lambda+2) - 1 - 1\right]$$
$$= -(\lambda+1)\left[\lambda^2 + 3\lambda\right]$$
$$= -\lambda(\lambda+1)(\lambda+3) \quad \therefore \lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = -3$$

$\lambda_1 = 0$

$$A\vec{u}_1 = \begin{bmatrix}-1 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \begin{matrix}\longrightarrow v = u\\ \longrightarrow v = w\end{matrix} \longrightarrow \vec{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\lambda_2 = -1$

$$(A+I)\vec{u}_2 = \begin{bmatrix}0 & 1 & 0\\ 1 & -1 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \begin{matrix}\longrightarrow v = 0\\ \longrightarrow w = -u\end{matrix} \longrightarrow \vec{u}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$\lambda_3 = -3$

$$(A+3I)\vec{u}_3 = \begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \begin{matrix}\longrightarrow u = -v/2\\ \longrightarrow w = -v/2\end{matrix} \longrightarrow \vec{u}_3 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$

$$\vec{x}(0) = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix} \longrightarrow \text{rref}\left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\ 1 & 0 & -2 & 0\\ 1 & -1 & 1 & 0\end{array}\right] = \left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 1.5\\ 0 & 0 & 1 & 0.5\end{array}\right]$$

$$\boxed{\vec{x}(t) = \begin{bmatrix}1\\1\\1\end{bmatrix} + 1.5e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix} + 0.5e^{-3t}\begin{bmatrix}1\\-2\\1\end{bmatrix}}$$

P72

Let a, b be constants which are some measure of the trust between two nations. Furthermore, let x be the military expenditure of Bobslovakia and let y be the military expenditure of the Leaf Village. Detailed analysis by strategically gifted ninjas reveal that

$$\frac{dx}{dt} = -x + 2y + a$$

$$\frac{dy}{dt} = 4x - 3y + b$$

Analyze possible outcomes for various initial conditions and values of a, b. Consider drawing an ab-plane to explain your solution(s). Is a stable peace without a run-away arms race possible given the analysis thus far?

(the analysis I omit would ask a, b reasonable? I derive math possibilities but I do not apply common sense to end of this with.)

$$A = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}, \quad \det\begin{bmatrix} -1-\lambda & 2 \\ 4 & -3-\lambda \end{bmatrix} = (\lambda+1)(\lambda+3) - 8 = \lambda^2 + 4\lambda - 5 = (\lambda-1)(\lambda+5)$$

$$\therefore \underline{\lambda_1 = 1}, \quad \underline{\lambda_2 = -5}.$$

$$(A - I)\vec{u}_1 = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow u = v \longrightarrow \underline{\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

$$(A + 5I)\vec{u}_2 = \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow v = -2u \longrightarrow \underline{\vec{u}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}.$$

$$\vec{x}_p' = A\vec{x}_p + \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \vec{x}_p = -A^{-1}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{-1}{3-8}\begin{bmatrix} -3 & -2 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3a - 2b \\ -4a - b \end{bmatrix}$$

Thus, $\underline{\vec{x}(t) = c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 3a + 2b \\ 4a + b \end{bmatrix}}$.

If $c_1 \neq 0$ then $\vec{x}(t) \longrightarrow \infty \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as $t \to \infty \Rightarrow$ run away arms spending for both x & y.

Thus, $c_1 = 0$ is our only hope, how can that occur?

$$\vec{x}(0) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 3a + 2b \\ 4a + b \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 3a + 2b \\ 4a + b \end{bmatrix} = \begin{bmatrix} c_2 \\ -2c_2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = 5\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}^{-1}\left(\begin{bmatrix} c_2 - x_0 \\ -2c_2 - y_0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} c_2 - x_0 \\ -2c_2 - y_0 \end{bmatrix} = \begin{bmatrix} x_0 - c_2 - 4c_2 - 2y_0 \\ 4c_2 - 4x_0 + 6c_2 + 3y_0 \end{bmatrix}$$

Thus $a = x_0 - 2y_0 - 5c_2$ and $b = 3y_0 - 4x_0 + 10c_2$

Eliminating $c_2 = \left(\frac{x_0 - 2y_0 - a}{5}\right)\frac{2}{2} = \frac{b - 3y_0 + 4x_0}{10}$

$$\Rightarrow b = 2x_0 - 4y_0 - 2a + 3y_0 - 4x_0$$

$$\underline{b = -2x_0 - y_0 - 2a}.$$

(I leave further analysis to reader; it appears many sol^{n}'s exist.)