I will select some subset of these problems to collect. The more you work, the more you know. The ordering of topics in these problems is rather lumpy, I've more or less cut and pasted multiple old homeworks and quizzes one after the other.

PP 310 Calculate the Laplace transform of $f(t) = t$ from the definition of the Laplace transform. That is, calculate $\mathcal{L}\{t\}(s) = \int_0^\infty te^{-st}dt$.

PP 311 Calculate $\mathcal{L}\{te^{3t}\}(s)$.

PP 312 Let $f(t) = \sin t$ for $0 \le t \le \pi$ and $f(t) = 0$ for $t > \pi$. Calculate $\mathcal{L}\{f(t)\}(s)$.

PP 313 Calculate $\mathcal{L}\{e^{3t}\sin(6t) - t^3 + e^t\}(s)$

PP 314 Calculate $\mathcal{L}\{t^4 - t^2 - t + \sin(\sqrt{2}t)\}(s)$

PP 315 Calculate $\mathcal{L}\{2t^2e^{-t}\}(s)$

PP 316 Calculate $\mathcal{L}\{t^2e^{3t} + e^{-2t}\sin(2t)\}(s)$

PP 317 Calculate $\mathcal{L}\{\sin(3t)\cos(3t)\}(s)$

PP 318 Calculate $\mathcal{L}\{\cos^3(t)\}(s)$

PP 319 Derive $\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$

PP 320 Derive $\mathcal{L}\{\cosh(bt)\}(s) = \dfrac{s}{s^2 - b^2}$

PP 321 Calculate $\mathcal{L}^{-1}\left\{\dfrac{6}{(s-1)^4}\right\}(t)$.

PP 322 Calculate $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\}(t)$.

PP 323 Calculate $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+2s+10}\right\}(t)$.

PP 324 Calculate $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\}(t)$.

PP 325 Let $F(s) = \dfrac{3s-15}{2s^2-4s+10}$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

PP 326 Let $F(s) = \dfrac{6s^2-13s+2}{s(s-1)(s-6)}$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

PP 327 Let $F(s) = \dfrac{s+11}{(s-1)(s+3)}$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

PP 328 Let $F(s) = \dfrac{7s^3 - 2s^2 - 3s + 6}{s^3(s-2)}$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

PP 329 Given $s^2F(s) + sF(s) - 6F(s) = \dfrac{s^2+4}{s^2+5}$ calculate $f = \mathcal{L}^{-1}\{F\}$.

PP 330 Let $F(s) = \ln\left(\dfrac{s+2}{s-5}\right)$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

PP 331 Let $F(s) = \tan^{-1}\left(\dfrac{1}{s}\right)$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

PP 332 Solve $y'' - 2y' + 5y = 0$ with $y(0) = 2$ and $y'(0) = 4$ via the Laplace transform technique.

PP 333 Solve $y'' + 6y' + 5y = 12e^t$ with $y(0) = -1$ and $y'(0) = 7$ via the Laplace transform technique.

PP 334 Solve $w'' + w = t^2 + 2$ with $w(0) = 1$ and $w'(0) = -1$ via the Laplace transform technique.

PP 335 Solve $y'' - 4y = 4t - 8e^{-2t}$ with $y(0) = 0$ and $y'(0) = 5$ via the Laplace transform technique.

PP 336 Solve $y'' + 3ty' - 6y = 1$ with $y(0) = 0$ and $y'(0) = 0$ via the Laplace transform technique.

PP 337 Solve $y'' + y = t$ with $y(\pi) = 0$ and $y'(\pi) = 0$ via the Laplace transform technique.

PP 338 Let $g(t) = \begin{cases} 0, & 0 < t < 2 \\ t+1, & 2 < t \end{cases}$. Calculate $G(s)$.

PP 339 Let $g(t) = \begin{cases} 0, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 3, & 3 < t \end{cases}$. Calculate $G(s)$.

PP 340 Let $g(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & 3 < t \end{cases}$. Calculate $G(s)$.

PP 341 Let $G(s) = \dfrac{e^{-3s}}{s^2}$. Calculate $g(t)$.

PP 342 Calculate $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s} - 3e^{-4s}}{s+2}\right\}(t)$.

PP 343 Calculate $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s^2+4}\right\}(t)$.

PP 344 Solve $y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi)$ with $y(0) = 0$ and $y'(0) = 0$ via the Laplace transform technique.

PP 345 Solve $y'' + 5y' + 6y = g(t)$ given $y(0) = 0$ and $y'(0) = 2$ where $g(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 5 \\ 1, & 5 < t \end{cases}$.

PP 346 Solve $y'' - y = u(t-1) - u(t-4)$ given $y(0) = 0$ and $y'(0) = 2$.

PP 347 Calculate $\int_{-\infty}^{\infty} (t^2 - 1)\delta(t)dt$.

PP 348 Calculate $\int_{-\infty}^{\infty} e^{3t}\delta(t)dt$.

PP 349 Calculate $\int_{-\infty}^{\infty} \sin(3t)\delta\left(t - \frac{\pi}{2}\right) dt$.

PP 350 Calculate $\int_{-\infty}^{\infty} e^{-2t}\delta(t+1)dt$.

PP 351 Calculate $\mathcal{L}\{\delta(t-1) - \delta(t-3)\}(s)$.

PP 352 Calculate $\mathcal{L}\{\delta(t-\pi)\sin t\}(s)$.

PP 353 Solve $w'' + w = \delta(t-\pi)$ where $w(0) = 0$ and $w'(0) = 0$.

PP 354 Solve $y'' + y = 4\delta(t-2) + t^2$ given $y(0) = 0$ and $y'(0) = 2$.

PP 355 A hammer hits a spring mass system at time $t = \pi/2$ and thus Newton's Second Law gives

$$\frac{d^2x}{dt^2} + 9x = -3\delta(t - \pi/2)$$

with $x(0) = 1$ and $x'(0) = 0$ since the spring is initially stretched to 1-unit and released from rest. Calculate the equation of motion and explain what happens after the hammer hits the spring at time $t = \pi/2$.

PP 356 Calculate the Laplace transforms of the following functions

(a.) $f(t) = \sin(t)\cos(2t) + \sin^2(3t)$

(b.) $f(t) = e^t u(t-3) + \sin(t)u(t-6)$

PP 357 Calculate the Laplace transforms of the following functions

(a.) $f(t) = \begin{cases} t, & 0 \le t \le 2 \\ \sin(t) & t > 2 \end{cases}$.

(b.) $f(t) = te^{-2t} + t\sin(t)$

PP 358 Compute the inverse Laplace transforms of $F(s) = \dfrac{3s+9}{s^2 - 8s + 7}$

PP 359 Compute the inverse Laplace transform of $F(s) = \dfrac{e^{-2s}}{s(s^2 + 6s + 13)}$

PP 360 Compute the inverse Laplace transform of $F(s) = \dfrac{4s}{s^4 - 1}$

PP 361 Solve the following differential equations with the given initial conditions by the method of Laplace transforms.

- **(a.)** $y'' + y' - 2y = 0$ where $y(0) = 2$ and $y'(0) = 1$
- **(b.)** $y'' - 2y' + y = \delta(t-2)$ where $y(0) = 1$ and $y'(0) = 0$

PP 362 Solve $y'' - 8y' + 7y = u(t-2)$ where $y(0) = 0$ and $y'(0) = 0$ by the method of Laplace transforms.

PP 363 Solve $y'' - 8y' + 7y = u(t-2) + u(t-4)$ where $y(0) = 0$ and $y'(0) = 0$ by the method of Laplace transforms.

PP 364 Solution of IVP with periodic forcing functions.

- **(a.)** find the Laplace transform of the periodic function f where $T = 2a$ and we define $f(t) = 1$ for $0 < t < a$ and $f(t) = 0$ for $a \leq t \leq 2a$. This is the square wave pictured in Problem 25 of Nagel Saff and Snider, page 422 of §7.6. (5th edition, you might need to look around given the current edition)
- **(b.)** solve $y'' + 3y' + 2y = f(t)$ for $t > 0$ given $y(0) = y'(0) = 0$.

PP 365 A spring with stiffness $k = 4$ is attached to a mass $m = 1$ and oscillates in one-dimensional motion such that it has $x(0) = 1$ and $x'(0) = 1$. Is it possible to strike the mass / spring system with a hammer such that the system is motionless after the strike ? Assume an idealized hammer which produces a force $F(t) = J_o\delta(t-a)$, you are free to adjust J_o and a as needed.

PP 366 (Ritger & Rose section 9-6 problem 1a) Use convolution to find the inverse Laplace transform of $\dfrac{1}{s^2(s-a)}$ for $a \neq 0$.

PP 367 Find an integral solution of $y'' + y = g$ via Laplace transforms and convolution. You may assume g is an integrable function of time t.

PP 368 (Ritger & Rose pg. 302 section 9-8) Suppose $L[y] = f$ is a second order linear system. If the possible inputs (we use a complex notation to treat sines and cosines at once) are given by $f(t) = ce^{i\omega t}$ for $c \in \mathbb{C}$ and $\omega \in \mathbb{R}$ then **show** that the output is given by

$$y(t) = H(i\omega)ce^{i\omega t} + y_t(t)$$

where $y_t(t) \to 0$ as $t \to \infty$ (y_t is the *transient solution*). The function $H(i\omega)$ is called the **frequency-response function** for the system. Notice that we can express

$$H(i\omega) = A(\omega)e^{i\phi(\omega)}$$

The factor $A(\omega)$ is the **amplification factor** for the system whereas $\phi(\omega)$ is the **phase lag**. **Find formulas** for $A(\omega) \in (0, \infty)$ and $\phi(\omega)$ in the particular cases:

(a.) $H(s) = \dfrac{1}{s^2 + 5s + 6}$

(b.) $H(s) = \dfrac{1}{s^2+s+1}$

(c.) $H(s) = \dfrac{1}{s^2+s}$

PP 369 Consider $L[y] = (D-2)(D^2+4D+5)[y] = f$ where $D = d/dt$. Find:

(a.) green's function $K(u,t)$ (see my notes for the meaning of this),

(b.) transfer function $H(s)$ and $h(t)$,

(c.) an integral solution of $L[y] = f$ subject $y(0) = y'(0) = y''(0) = 0$ for $f(t) = t^2\cos(t)$. **NOTE: DO NOT DO THIS INTEGRAL, THIS IS WHAT IS MEANT BY "INTEGRAL" SOLUTION, IT IS THE ANSWER REDUCED TO AN INTEGRAL**

PP 370 Calculate the Laplace transforms of the following functions using the table of basic Laplace transforms plus possibly the given Theorems and trigonometry.

(a.) $f(t) = \begin{cases} t, & 0 \le t \le 2 \\ \sin(t) & t > 2 \end{cases}.$

(b.) $f(t) = te^{-2t} + t\sin(t)$

PP 371 Compute the inverse Laplace transforms of,

(a.) $F(s) = \dfrac{3s+9}{s^2-8s+7}$

(b.) $F(s) = \dfrac{e^{-2s}}{s(s^2+6s+13)}$

(c.) $F(s) = \dfrac{4s}{s^4-1}$

PP 372 Solve the following differential equations with the given initial conditions by the method of Laplace transforms.

(a.) $y'' + y' - 2y = 0$ where $y(0) = 2$ and $y'(0) = 1$

(b.) $y'' - 2y' + y = \delta(t-2)$ where $y(0) = 1$ and $y'(0) = 0$

PP 373 Solve the following differential equations with the given initial conditions by the method of Laplace transforms.

(a.) $y'' - 8y' + 7y = u(t-2)$ where $y(0) = 0$ and $y'(0) = 0$

(b.) $y'' - 8y' + 7y = u(t-2) + u(t-4)$ where $y(0) = 0$ and $y'(0) = 0$

PP 374 Solution of IVP with periodic forcing functions.

(a.) find the Laplace transform of the periodic function f where $T = 2a$ and we define $f(t) = 1$ for $0 < t < a$ and $f(t) = 0$ for $a \le t \le 2a$. This is the square wave pictured in Problem 25 of Nagel Saff and Snider, page 422 of §7.6.

(b.) solve $y'' + 3y' + 2y = f(t)$ for $t > 0$ given $y(0) = y'(0) = 0$.

PP 375 Let $f(t) = \begin{cases} \sin(t) & 0 \leq t \leq 2 \\ e^t & t > 2 \end{cases}$. Calculate the Laplace transform of f.

PP 376 Suppose $F(s) = \dfrac{72s}{s^4 - 81}$. Calculate the inverse Laplace transform of $F(s)$.

PP 377 Solve $y'' + 6y' + 13y = u(t-1)$ given $y(0) = 1$ and $y'(0) = 3$

PP 378 Suppose a force $F(x) = 3x^4 + 16x^3 + 6x^2 - 72x$ is the net-force on some mass $m = 1$. Newton's Equation is $\ddot{x} = 3x^4 + 16x^3 + 6x^2 - 72x$.

(1.) make the substitution $v = \dot{x}$ and write Newton's equation as a system in normal form for x and v.

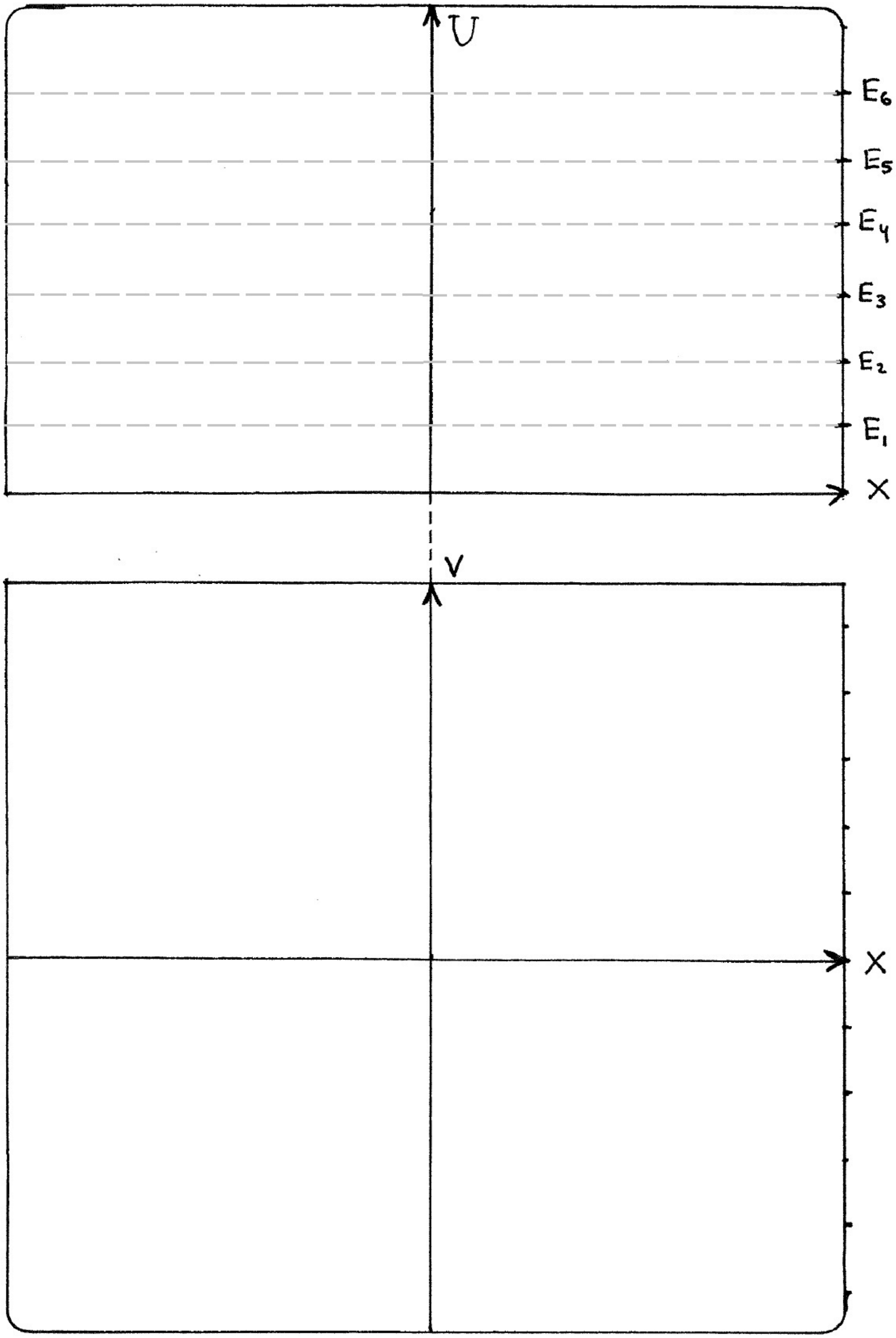
(2.) find all three critical points for the system in (1.). (the potential should factor nicely)

(3.) plot the potential plane and phase plane juxtaposed vertically with the potential at the top and the phase plane at the base. Plot several trajectories and include arrows to indicate the direction of physically feasible solutions.

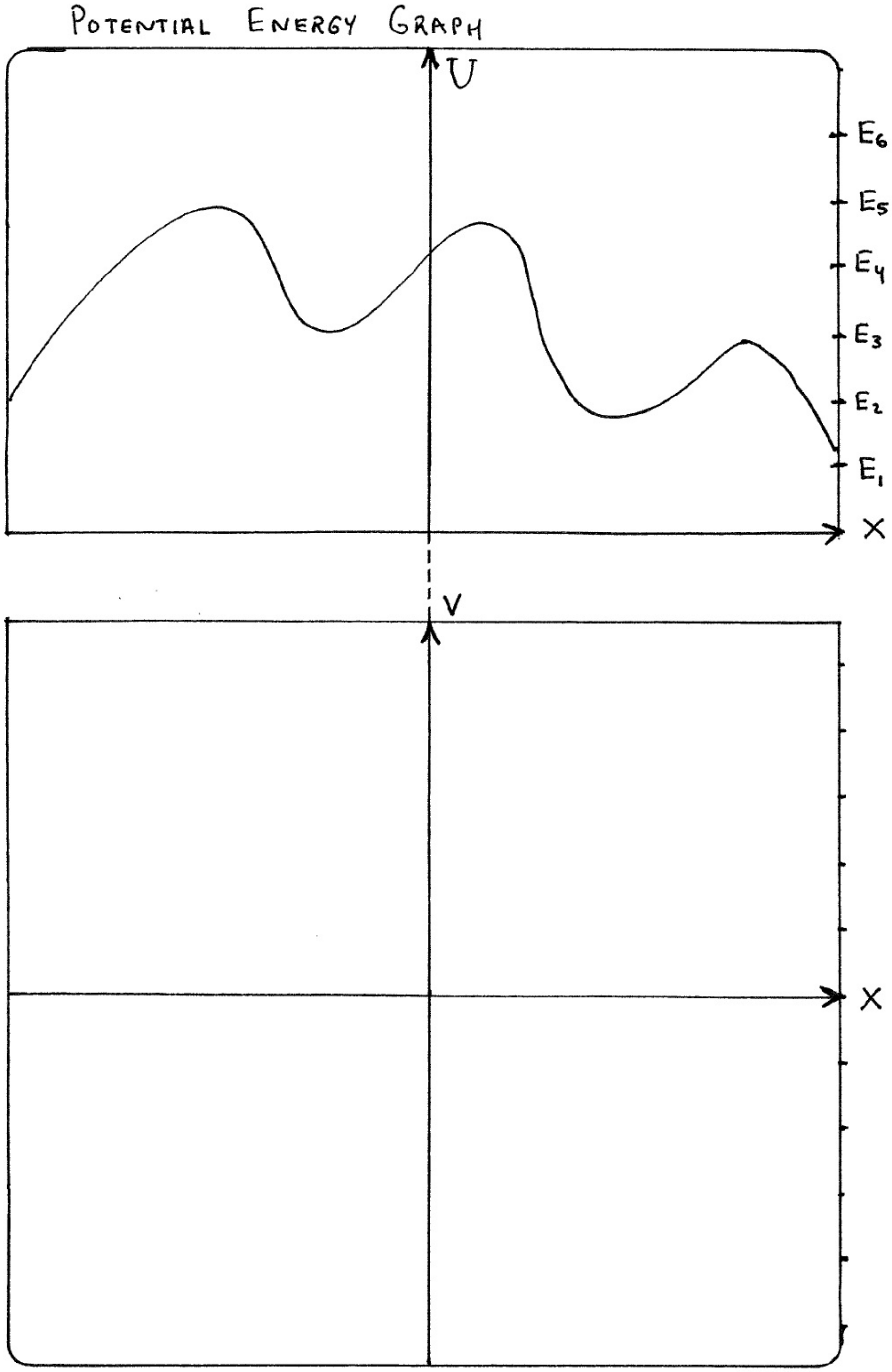
(4.) classify each critical point by examining your plot from (3.)

in this context the phase plane is also called the Poincare plane in honor of the mathematician who did much pioneering work in this realm of qualitative analysis. Incidentally, given any autonomous system $\frac{dx}{dt} = g(x, y)$ *and* $\frac{dy}{dt} = f(x, y)$ *we can study the timeless phase plane equation* $\frac{dy}{dx} = \frac{f}{g}$ *to indirectly analyze the solutions to the system. Solutions to the phase plane equation are the Cartesian level curves which are parametrized, with parameter* t*, by the solutions to the system*

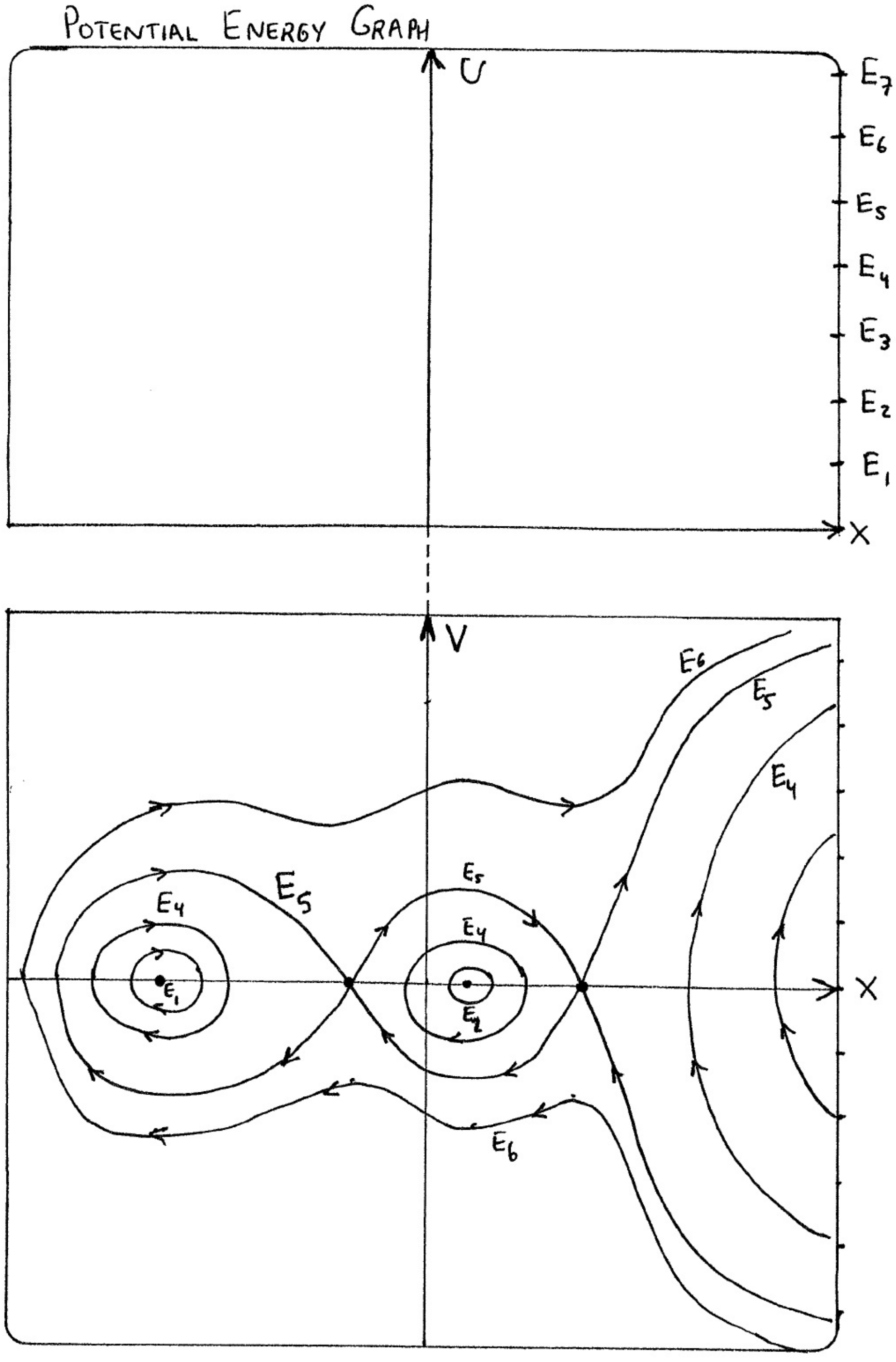
POTENTIAL ENERGY GRAPH



PP 379 Plot the phase plane (or Poincare plot) given the potential energy plot below. For each energy $E_1, E_2, \ldots, E_6$ graph the corresponding trajectories below. Use a couple different colors so your work is easy to follow. Be neat. If no motion is possible then explain why.



PP 380 You are given a not so great phase plane (or Poincare plot) of the motion of a particle with various energies as listed. Plot the potential energy responsible for such motion.



Sorry, sketch not quite true to math ☺