Name: (please print name here →)

MATH 334:	MISSION 4: INTEGRAL CURVES, ENERGY, LAPLACE TRANSFORM TECHNIQUE

While complex solutions may be useful as in-between work the solutions requested on this assignment are real solutions. Please refer to Mission 1 for formatting rules. Thanks!

Problem 73 Determine if the vector field $\vec{F}$ is conservative on S. If so, find the potential energy function U for which $\vec{F} = -\nabla U$.

(a.) $\vec{F} = \langle x - 7x^6y^7, y^2 - 7x^7y^6 \rangle$ for $S = \mathbb{R}^2$

$$U = \frac{-1}{2}x^2 - \frac{1}{3}y^3 + 7x^7y^7$$

(b.) $\vec{F} = \dfrac{1}{x^2+y^2}\langle x, y\rangle$ for $S = \mathbb{R}^2 - \{(0,0)\}$

$$U = \frac{-1}{2}\ln(x^2+y^2)$$

(c.) $\vec{F} = \dfrac{1}{x^2+y^2}\langle -y, x\rangle$ for $S = \{(x,y) \mid x > 0\}$.

$$U = -\tan^{-1}(y/x)$$

(d.) $\vec{F} = \dfrac{1}{x^2+y^2}\langle -y, x\rangle$ for $S = \mathbb{R}^2 - \{(0,0)\}$

not conservative.

Problem 74 Solve the following exact equations:

(a.) $(x - 7x^6y^7)dx + (y^2 - 7x^7y^6)dy = 0$,

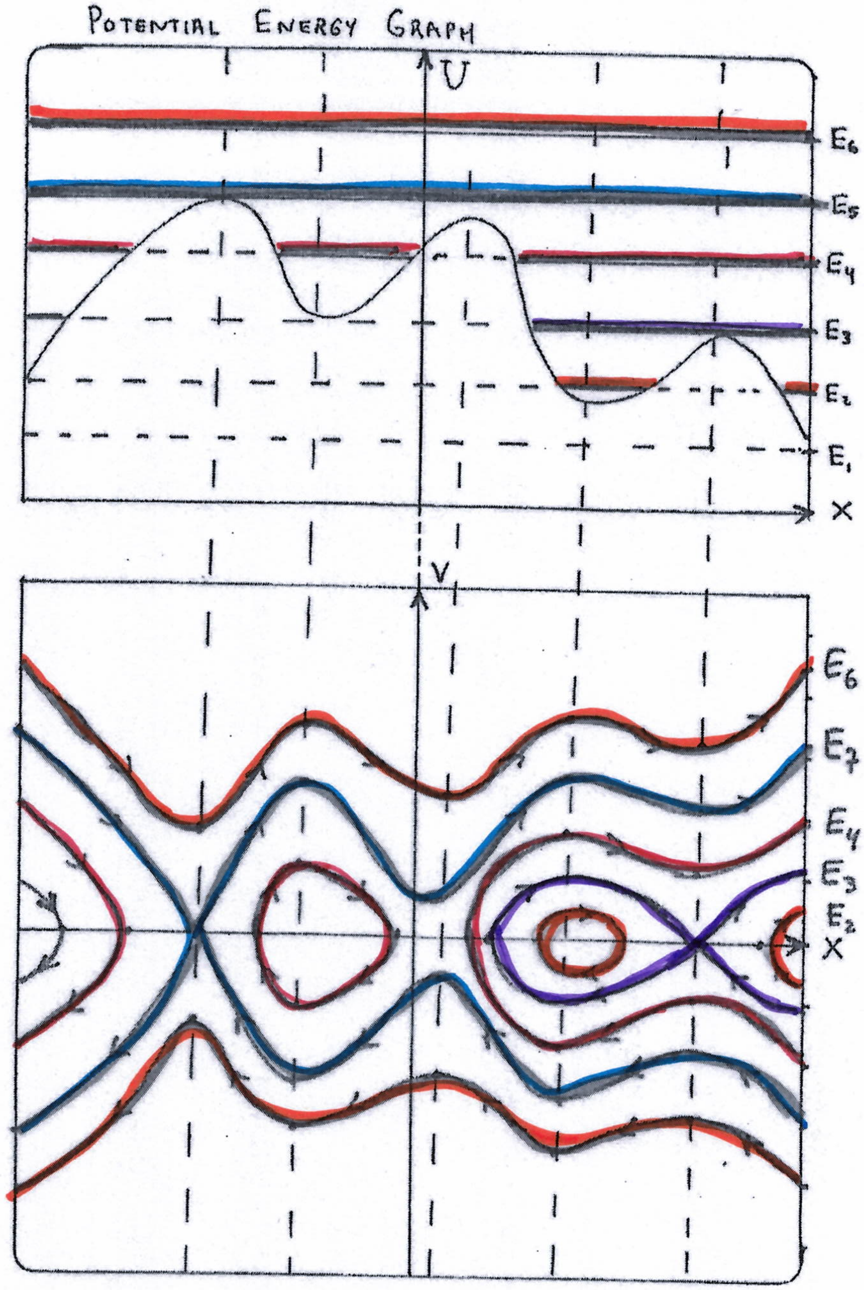
$$\underline{\frac{1}{2}x^2 + \frac{1}{3}y^3 - 7x^7y^7 = C}.$$

(b.) $\dfrac{xdx + ydy}{x^2+y^2} = 0.$

$$\underline{\frac{1}{2}\ln(x^2+y^2) = C}.$$

or simply $\underline{x^2+y^2 = C}$, $C \neq 0$.

PROBLEM 75 Plot the phase plane (or Poincare plot) given the potential energy plot below. For each energy $E_1, E_2, \ldots, E_6$ graph the corresponding trajectories below. Use a couple different colors so your work is easy to follow. Be neat. If no motion is possible then explain why.



← no motion possible. $E_1 < U$ everywhere.

Remark: there is always room for improvement in my sketches here. I hope the main ideas are clear.

(Sorry, my kids stole my good ruler.)

Problem 76 For each vector field below find the parametrization of the integral curve $\vec{\gamma} : \mathbb{R} \to \mathbb{R}^2$ for which $\frac{d\vec{\gamma}}{dt} = \vec{F}(\vec{\gamma}(t))$. In particular, for $\vec{F} = \langle P, Q \rangle$ if we write $\vec{\gamma} = \langle x, y \rangle$ then we wish to solve $\frac{dx}{dt} = P(x,y)$ and $\frac{dy}{dt} = Q(x,y)$.

(a.) $\vec{F} = \langle 3x + y, 3y \rangle$

$$\vec{\gamma}(t) = \left((c_1 + c_2 t)e^{3t},\ c_2 e^{3t} \right)$$

(b.) $\vec{F} = \langle 2x + y, x + 2y \rangle$

$$\vec{\gamma}(t) = \left(-c_1 e^{t} + c_2 e^{3t},\ c_1 e^{t} + c_2 e^{3t} \right)$$

(c.) $\vec{F} = \langle -2y, 2x \rangle$

$$\vec{\gamma}(t) = \left(c_1 \cos(2t) + c_2 \sin(2t),\ -c_1 \sin(2t) + c_2 \cos(2t) \right)$$

Problem 77 For the vector fields of the previous problem, solve the differential equation $\frac{dy}{dx} = \frac{Q}{P}$ and show that the Cartesian solution found here is parametrized by your solution of the previous problem.

(a.) $P = 3x + y$, $Q = 3y$

$$\boxed{\ln|y| - \frac{3x}{y} = c}$$

Remark: (a.) & (b.) actually weren't too bad.

(b.) $P = 2x + y$, $Q = x + 2y$

$$\boxed{\ln|x+y| - 3\ln|x-y| = 2c}$$

(c.) $P = -2y$, $Q = 2x$

$$\underline{x^2 + y^2 = R^2}$$

(where $R^2 = c_1^2 + c_2^2$ if we study how 76c's $\vec{\gamma}(t)$ parametrizes the given circle of radius R)

Problem 78 Calculate $\mathcal{L}\{2t^2e^{-t}\}(s)$ $= \dfrac{2\cdot 2!}{(s+1)^3} = \boxed{\dfrac{4}{(s+1)^3}}$

Problem 79 Calculate $\mathcal{L}\{\sin(2t)\cos(3t)\}(s)$ $= \boxed{\dfrac{1}{2}\left[\dfrac{5}{s^2+25} - \dfrac{1}{s^2+1}\right]}$

Problem 80 Calculate $\mathcal{L}\{te^t\sin(2t+3)\}(s)$ $= \mathcal{L}\left\{\cos(3)te^t\sin(2t) + \sin(3)te^t\cos(2t)\right\}(s)$

$$= -\cos(3)\frac{d}{ds}\left[\frac{2}{(s-1)^2+4}\right] - \sin(3)\frac{d}{ds}\left[\frac{s-1}{(s-1)^2+4}\right]$$

$$= \boxed{\frac{4\cos(3)(s-1) + \sin(3)(s^2-2s+3)}{((s-1)^2+4)^2}}$$

Problem 81 Calculate the inverse Laplace transforms below:

(a.) $\mathcal{L}^{-1}\left\{\dfrac{6}{(s-1)^4}\right\}(t) = \dfrac{6}{3!}e^t t^3 = \boxed{t^3e^t}$

(b.) $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\}(t) = \boxed{\sin(2t)}$

(c.) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+2s+10}\right\}(t) = \mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+1)^2+9}\right\}(t) = \boxed{e^{-t}\cos(3t)}$

(d.) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\}(t) = \dfrac{t^4}{4!} = \boxed{\dfrac{1}{24}t^4}$

Problem 82 Let $F(s) = \dfrac{3s-15}{2s^2-4s+10}$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

$$\underline{f(t) = \frac{3}{2}e^{t}\cos(2t) - 3e^{t}\sin(2t)}.$$

Problem 83 Let $F(s) = \dfrac{6s^2-13s+2}{s(s-1)(s-6)}$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

$$\underline{f(t) = \frac{1}{3} + e^{t} + \frac{14}{3}e^{6t}}.$$

Problem 84 Let $F(s) = \dfrac{7s^3-2s^2-3s+6}{s^3(s-2)}$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

partial fractions.

$$F(s) = \frac{-3}{s^3} + \frac{1}{s} + \frac{6}{s-2} \longrightarrow \boxed{f(t) = \frac{-3t^2}{2} + 1 + 6e^{2t}}$$

Problem 85 Let $F(s) = \tan^{-1}\left(\dfrac{1}{s}\right)$. Calculate $f = \mathcal{L}^{-1}\{F\}$.

$$\boxed{f = \frac{\sin t}{t}}$$

Problem 86 Solve $y'' + 6y' + 5y = 12e^{t}$ with $y(0) = -1$ and $y'(0) = 7$ via the Laplace transform technique.

$$\boxed{y = e^{t} - e^{-t} - e^{-5t}}$$

Problem 87 Let $g(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & 3 < t \end{cases}$. Calculate $G(s)$.

$$\boxed{G(s) = \frac{1}{s^2}\left(e^{-s} - 2e^{-2s} + e^{-3s}\right)}$$

Problem 88 Calculate the Laplace transforms of the following functions

(a.) $f(t) = \sin(t)\cos(2t) + \sin^2(3t)$

$$\boxed{F(s) = \frac{1}{2}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+36}\right]}$$

(b.) $f(t) = e^t u(t-3) + \sin(t)u(t-6)$

$$\boxed{F(s) = e^{-3s}e^{3}\left(\frac{1}{s-1}\right) + e^{-6s}\left(\frac{\cos 6 + s(\sin 6)}{s^2+1}\right)}$$

Problem 89 Solve $y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi)$ with $y(0) = 0$ and $y'(0) = 0$ via the Laplace transform technique.

$$\boxed{\begin{aligned} y = {} & \frac{1}{4}\left(1 - e^{2(t-\pi)} + 2(t-\pi)e^{2(t-\pi)}\right)u(t-\pi) \\ & + \frac{1}{4}\left(1 - e^{2(t-2\pi)} + 2(t-2\pi)e^{2(t-2\pi)}\right)u(t-2\pi) \end{aligned}}$$

Problem 90 Solve $y'' + 5y' + 6y = g(t)$ given $y(0) = 0$ and $y'(0) = 2$ where $g(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 5. \\ 1, & 5 < t \end{cases}$

$$\boxed{\begin{aligned} y = {} & 2e^{-2t} - 2e^{-3t} + \left(\frac{1}{36} + \frac{1}{6}(t-1) - \frac{1}{4}e^{-2(t-1)} + \frac{2}{9}e^{-3(t-1)}\right)u(t-1) \\ & + \left(\frac{-19}{36} - \frac{1}{6}(t-5) + \frac{7}{4}e^{-2(t-5)} - \frac{11}{9}e^{-3(t-5)}\right)u(t-5) \end{aligned}}$$

Problem 91 Let $F(s) = \dfrac{e^{-\pi s}s}{s^2+6s+13}$. Calculate the inverse Laplace transform of $F(s)$.

$$F(s) = \left[\frac{(s+3) - 3}{(s+3)^2 + 4}\right]e^{-\pi s} \xrightarrow{\mathcal{L}^{-1}} \boxed{f = e^{-3(t-\pi)}\left(\cos(2(t-\pi)) - \frac{3}{2}\sin(2(t-\pi))\right)u(t-\pi)}$$

Problem 92 Solve $w'' + w = \delta(t-\pi)$ where $w(0) = 0$ and $w'(0) = 0$.

$$\boxed{w = \sin(t-\pi)\,u(t-\pi)}$$

Problem 93 Solve $y'' + y = 4\delta(t-2) + t^2$ given $y(0) = 0$ and $y'(0) = 2$.

$$\boxed{y = 2\sin t + 4\sin(t-2)\,u(t-2) + t^2 + \cos t - 2}$$

Problem 94 A hammer hits a spring mass system at time $t = \pi/2$ and thus Newton's Second Law gives

$$\frac{d^2x}{dt^2} + 9x = -3\delta(t - \pi/2)$$

with $x(0) = 1$ and $x'(0) = 0$ since the spring is initially stretched to 1-unit and released from rest. Calculate the equation of motion and explain what happens after the hammer hits the spring at time $t = \pi/2$.

$$\boxed{x = \cos(3t) - \cos(3t)\,u(t - \pi/2)}$$

Problem 95 (Ritger & Rose section 9-6 problem 1a) Use convolution to find the inverse Laplace transform of $\dfrac{1}{s^2(s-a)}$ for $a \neq 0$.

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\}(t) = \frac{1}{a^2}\left(e^{at} - 1 - at\right)}$$

Problem 96 Find an integral solution of $y'' + y = g$ via Laplace transforms and convolution. You may assume g is an integrable function of time t.

$$y(t) = y_0 \cos t + y_1 \sin t + \int_0^t \sin(t-u)\, g(u)\, du$$

where $y(0) = y_0$ and $y'(0) = y_1$

MISSION 4 SOLUTION

$\boxed{P73}$ determine if $\vec{F}$ is conservative on S, if so find PE function U for which $\vec{F} = -\nabla U$ on S.

(a.) $\vec{F} = \langle x - 7x^6y^7,\ y^2 - 7x^7y^6 \rangle$ on $S = \mathbb{R}^2$

$\underline{U = -\frac{1}{2}x^2 - \frac{1}{3}y^3 + 7x^7y^7}$ gives $\vec{F} = -\nabla U$

$\therefore$ $\vec{F}$ conservative.

(b.) $\vec{F} = \frac{1}{x^2+y^2}\langle x, y\rangle$ on $\mathbb{R}^2 - \{(0,0)\}$ is

conservative since $\underline{U = \frac{-1}{2}\ln(x^2+y^2)}$ gives

$\nabla U = \left\langle \frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right\rangle = -\vec{F}$ for $(x,y) \neq (0,0)$.

(c.) $\vec{F} = \frac{1}{x^2+y^2}\langle -y, x\rangle$ for $S = \{(x,y) \mid x > 0\}$

is conservative since $\nabla \tan^{-1}(y/x) = \left\langle \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right), \frac{1}{1+y^2/x^2}\frac{1}{x} \right\rangle$

yields $\nabla \tan^{-1}(y/x) = \frac{1}{x^2+y^2}\langle -y, x\rangle$

thus $\underline{U = -\tan^{-1}(y/x)}$ is PE function for $\vec{F}$ on S.

(d.) Observe $\vec{r}(t) = \langle \cos t, \sin t\rangle$, $0 \leq t \leq 2\pi$ parametrizes unit circle

$$\int_{\text{unit circle}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\frac{\langle -\sin t, \cos t\rangle}{\cos^2 t + \sin^2 t} \right) \cdot \langle -\sin t, \cos t\rangle\, dt$$

$$= \int_0^{2\pi} dt$$

$$= \underline{2\pi}$$

If $\vec{F} = -\nabla U$ on $\mathbb{R}^2 - \{(0,0)\}$ then $\int_{\text{unit circle}} \vec{F} \cdot d\vec{r} = 0$

Hence $\nexists$ U s.t. $\vec{F} = -\nabla U$, $\vec{F}$ not conservative on S.

$\boxed{P74}$ Solving exact DEq$^{\underline{n}}$

(a.) $(x - 7x^6y^7)dx + (y^2 - 7x^7y^6)dy = 0$

$$\Rightarrow \boxed{\tfrac{1}{2}x^2 + \tfrac{1}{3}y^3 - 7x^7y^7 = C}$$

(note: $\vec{F} = -\nabla U$ with $\vec{F} = \langle P, Q\rangle$
means $d(-U) = Pdx + Qdy$ hence $-U = c$
will solve $Pdx + Qdy = 0$)

(b.) $\dfrac{x\,dx + y\,dy}{x^2+y^2} = 0 \Rightarrow \underline{\tfrac{1}{2}\ln(x^2+y^2) = C}$.
(using 736)

$\boxed{P75}$ see problem sheet, key idea

$$E = \tfrac{1}{2}mv^2 + U(x)$$

thus need $E \geq U(x)$ for possible motions.
Also $v = \frac{dx}{dt}$ so move right in xv-plane with $v > 0$
whereas we move leftward when $v < 0$ in xv-plane.

$\boxed{P76}$ $\vec{F} = \langle 3x+y,\ 3y \rangle$

(a.)

$$\frac{dx}{dt} = 3x + y \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \overbrace{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}}^{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{dy}{dt} = 3y$$

Find $\lambda_1 = \lambda_2 = 3$ and $(A - 3I)e_1 = 0$ and $(A-3I)e_2 = e_1$

thus $\vec{r}(t) = c_1 e^{3t} e_1 + c_2 e^{3t}(e_2 + t e_1)$

$$\boxed{\vec{\gamma}(t) = (c_1 e^{3t} + c_2 t e^{3t},\ c_2 e^{3t})}$$

integral curve to $\vec{F}$

(b.)

$\vec{F} = \langle 2x + y,\ x + 2y \rangle$

$$\frac{dx}{dt} = 2x + 2y \qquad \begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{dy}{dt} = x + 2y$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda - 2)^2 - 1 = (\lambda - 2 + 1)(\lambda - 2 - 1) = (\lambda - 1)(\lambda - 3)$$

Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$

notice $A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $(A - I)v_1 = 0$

notice $A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $(A - 3I)v_2 = 0$

Therefore, $\vec{r}(t) = c_1 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore \boxed{\vec{\gamma}(t) = (-c_1 e^{t} + c_2 e^{3t},\ c_1 e^{t} + c_2 e^{3t})}$$

P76 continued

(c.) $\vec{F} = \langle -2y, 2x \rangle$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 + 4 = 0$$

$\therefore$ $\underline{\lambda = \pm 2i}$.

$$(A - 2iI)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \longrightarrow -2iu - 2v = 0$$

$\hookrightarrow \underline{v = iu}$. set $\underline{u = 1}$.

Choose e-vector $\begin{bmatrix} 1 \\ i \end{bmatrix}$ with $\lambda = 2i$

$$\vec{z} = e^{2it}\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos(2t) + i\sin(2t))\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + i\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

$$\boxed{\vec{\gamma}(t) = (c_1\cos(2t) + c_2\sin(2t),\ -c_1\sin(2t) + c_2\cos(2t))}$$

$\boxed{\text{P77}}$ skip to part (c.)

(c.) $P = -2y$, $Q = 2x$ find Cartesian form of $\int$-curve of $\vec{F} = \langle -2y, 2x \rangle$

$$\frac{dy}{dx} = \frac{Q}{P} = \frac{2x}{-2y} \longrightarrow 2x\,dx + 2y\,dy = 0$$

$$d(x^2 + y^2) = 0$$

$$\therefore \boxed{x^2 + y^2 = R^2}$$

Now lets check 76c parametrizes $x^2 + y^2 = R^2$,

we have $x = c_1\cos 2t + c_2\sin 2t$ & $y = -c_1\sin 2t + c_2\cos 2t$

Let us set $\theta = 2t$ for convenience,

$$x^2 + y^2 = (c_1\cos\theta + c_2\sin\theta)^2 + (-c_1\sin\theta + c_2\cos\theta)^2$$

$$= c_1^2\cos^2\theta + 2c_1c_2\cos\theta\sin\theta + c_2^2\sin^2\theta +$$

$$+ c_1^2\sin^2\theta - 2c_1c_2\sin\theta\cos\theta + c_2^2\cos^2\theta$$

$$= c_1^2(\cos^2\theta + \sin^2\theta) + c_2^2(\sin^2\theta + \cos^2\theta)$$

$$= \underline{c_1^2 + c_2^2 = R^2}$$

(this relates 76c & 77c's choice of constants)

P77 continued

(a.) Solve $\frac{dy}{dx} = \frac{Q}{P}$ for $Q = 3y$ & $P = 3x + y$

$$\frac{dy}{dx} = \frac{3y}{3x+y}$$

Let $v = y/x \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

Then $\frac{dy}{dx} = x\frac{dv}{dx} + v = \frac{3vx}{3x+vx} = \frac{3v}{v+3}$

$$x\frac{dv}{dx} = \frac{3v}{v+3} - v = \frac{3v - v(v+3)}{v+3} = \frac{-v^2}{v+3}$$

$$\frac{(v+3)\,dv}{v^2} = \frac{-dx}{x}$$

$$\int\left(\frac{1}{v} + \frac{3}{v^2}\right)dv = \int\frac{-dx}{x}$$

$$\ln|v| - \frac{3}{v} = -\ln|x| + C$$

$$\ln|y/x| - \frac{3}{y/x} = -\ln|x| + C$$

$$\ln|y| - \cancel{\ln|x|} - \frac{3x}{y} = -\cancel{\ln|x|} + C$$

$$\boxed{\ln|y| - \frac{3x}{y} = C}\ *$$

Consider, $x = (c_1 + c_2 t)e^{3t}$, $y = c_2 e^{3t}$

then

$$\begin{aligned}\ln|y| - \frac{3x}{y} &= \ln|c_2 e^{3t}| - \frac{3(c_1+c_2t)e^{3t}}{c_2 e^{3t}} \\ &= \ln c_2 + 3t - \frac{3(c_1+c_2t)}{c_2} \\ &= \ln c_2 + 3t - \frac{3c_1}{c_2} - 3t \\ &= \underline{\ln c_2 - 3c_1/c_2}.\end{aligned}$$

$*$ parametrized by P76 a solution.

P77 continued

(6.) $\frac{dy}{dx} = \frac{Q}{P}$ where $Q = x+2y$, $P = 2x+y$

$$\frac{dy}{dx} = \frac{x+2y}{2x+y}$$

$$V = \frac{y}{x} \hookrightarrow y = xV \quad \therefore \quad \frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$V + x\frac{dV}{dx} = \frac{x+2xV}{2x+xV} = \frac{1+2V}{2+V}$$

$$x\frac{dV}{dx} = \frac{1+2V}{2+V} - V = \frac{1+2V-V(2+V)}{2+V} = \frac{1-V^2}{2+V}$$

$$\left(\frac{2+V}{1-V^2}\right)dV = \frac{dx}{x}$$

$$\frac{2+V}{1-V^2} = \frac{A}{1+V} + \frac{B}{1-V} \Rightarrow 2+V = A(1-V) + B(1+V)$$

$\underline{V=1|}$ $3 = 2B \therefore \underline{B = 3/2}$

$\underline{V=-1|}$ $1 = 2A \therefore \underline{A = 1/2}$.

$$\frac{1}{2}\int\left(\frac{1}{V+1} + \frac{3}{1-V}\right)dV = \int\frac{dx}{x}$$

$$\frac{1}{2}\left(\ln|V+1| - 3\ln|1-V|\right) = \ln|x| + C$$

$$\boxed{\ln\left|\frac{y}{x}+1\right| - 3\ln\left|1-\frac{y}{x}\right| = \ln(x^2) + 2C} *$$

(76b) $\hookrightarrow$ $\frac{y}{x} = \frac{c_1e^t + c_2e^{3t}}{-c_1e^t + c_2e^{3t}} = \frac{c_1 + c_2e^{2t}}{-c_1 + c_2e^{2t}}$ $\left(\begin{array}{l} x = c_2e^{3t} - c_1e^t \\ y = c_1e^t + c_2e^{3t}\end{array}\right)$

$$\frac{y}{x} + 1 = \frac{c_1 + c_2e^{2t} - c_1 + c_2e^{2t}}{-c_1 + c_2e^{2t}} = \frac{2c_2e^{2t}}{-c_1 + c_2e^{2t}}$$

$$1 - \frac{y}{x} = \frac{-c_1 + c_2e^{2t} - (c_1 + c_2e^{2t})}{-c_1 + c_2e^{2t}} = \frac{-2c_1}{-c_1 + c_2e^{2t}}$$

too ugly I'll try another path

P77 continued

return to *

$$\ln\left|\frac{1}{x}(y+x)\right| - 3\ln\left|\frac{1}{x}(x-y)\right| = \ln(x^2) + 2C$$

$$\ln|x+y| - 3\ln|x-y| + \ln\left|\frac{1}{x}\right| - 3\ln\left|\frac{1}{x}\right| = \ln(x^2) + 2C$$

$$\ln|x+y| - 3\ln|x-y| = \ln(x^2) + \ln x - 3\ln x + 2C$$

$$\boxed{\ln|x+y| - 3\ln|x-y| = 2C}$$ (better)

$$x+y = c_2e^{3t} - c_1e^{t} + c_1e^{t} + c_2e^{3t} = 2c_2e^{3t}$$

$$x-y = c_2e^{3t} - c_1e^{t} - c_1e^{t} - c_2e^{3t} = -2c_1e^{t}$$

$$\ln|2c_2e^{3t}| - 3\ln|-2c_1e^{t}| = \ln|2c_2| + 3t - 3\ln|-2c_1| - 3t$$

$$= \underbrace{\underline{\ln|2c_2| - 3\ln|2c_1|}}_{\substack{2C \\ \text{in terms of } c_1 \,\&\, c_2}}.$$

P78 on sheet

P79 $\sin(2t)\cos(3t) = \frac{1}{2i}\left(e^{2it} - e^{-2it}\right)\frac{1}{2}\left(e^{3it} + e^{-3it}\right)$

$$= \frac{1}{4i}\left(e^{5it} - e^{-5it} - e^{it} + e^{-it}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2i}\left(e^{5it} - e^{-5it}\right) - \frac{1}{2i}\left(e^{it} - e^{-it}\right)\right)$$

$$= \frac{1}{2}\sin(5t) - \frac{1}{2}\sin(t)$$

$$\therefore \mathcal{L}\{\sin(2t)\cos(3t)\}(s) = \frac{1}{2}\mathcal{L}\{\sin(5t)\}(s) - \frac{1}{2}\mathcal{L}\{\sin(t)\}$$

$$= \underline{\frac{1}{2}\left(\frac{5}{s^2+25}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right)}.$$

P80 on sheet & P81

P82 $F(s) = \frac{3s-15}{2s^2-4s+10} = \frac{3s-15}{2(s^2-2s+5)} = \frac{3s-15}{2((s-1)^2+4)}$

$$F(s) = \frac{3(s-1) - 12}{2((s-1)^2+4)} \qquad \frac{-12}{2} = \frac{-6(2)}{2} = -3(2)$$

$$\mathcal{L}^{-1}\{F(s)\}(t) = \underline{\frac{3}{2}e^t\cos(2t) - 3e^t\sin(2t)}.$$

P83 $\frac{6s^2-13s+2}{s(s-1)(s-6)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-6}$

$$6s^2-13s+2 = A(s-1)(s-6) + Bs(s-6) + Cs(s-1) \quad \begin{array}{l} \xrightarrow{s=0} 2 = 6A \\ \xrightarrow{s=1} -5 = -5B \\ \xrightarrow{s=6} 140 = 30C \end{array}$$

$$F(s) = \frac{1}{3s} + \frac{1}{s-1} + \frac{14}{3(s-6)}$$

$$\xrightarrow{\mathcal{L}^{-1}} \underline{f(t) = \frac{1}{3} + e^t + \frac{14}{3}e^{6t}}.$$

$\boxed{P84}$ on sheet

$\boxed{P85}$ $F(s) = \tan^{-1}\left(\frac{1}{s}\right)$

Note, $(\tan^{-1}(x))' = \dfrac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$$\Rightarrow \quad \underline{\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}}.$$

($\tan^{-1}(0) = 0$, no constant needed)

$$F(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{s}\right)^{2n+1}$$

$$f = \mathcal{L}^{-1}\{F\} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\mathcal{L}^{-1}\left\{\frac{1}{s^{2n+1}}\right\}(t)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{t^{2n+1-1}}{(2n+1-1)!}$$

$$= \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}t^{2n+1}\right)\frac{1}{t}$$

$$= \boxed{\frac{\sin(t)}{t}}$$

neat.

$\boxed{P86}$ $y'' + 6y' + 5y = 12e^{t}$, $y(0) = -1$, $y'(0) = 7$

$$s^2 \bar{Y} + s - 7 + 6(s\bar{Y} + 1) + 5\bar{Y} = \frac{12}{s-1}$$

$$(s^2 + 6s + 5)\bar{Y} = -s + 1 + \frac{12}{s-1}$$

$$\bar{Y} = \frac{1-s}{s^2+6s+5} + \frac{12}{(s-1)(s^2+6s+5)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+5}$$

$$(1-s)(s-1) + 12 = A(s+1)(s+5) + B(s-1)(s+5) + C(s-1)(s+1)$$

$\underline{s=1}$ $12 = 12A \Rightarrow \underline{A = 1}$

$\underline{s=-1}$ $8 = B(-2)(4) \Rightarrow \underline{B = -1}$

$\underline{s=-5}$ $6(-6) + 12 = C(-6)(-4) \Rightarrow \underline{C = -1}$.

$$\bar{Y} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{1}{s+5}$$

$$\boxed{y = e^{t} - e^{-t} - e^{-5t}}$$

$\boxed{P87}$

$$g(t) = \begin{cases} 0 & t < 1 \\ t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \\ 0 & 3 < t \end{cases}$$

$$g(t) = (t-1)(u(t-1) - u(t-2)) + (3-t)(u(t-2) - u(t-3))$$

$$g(t) = (t-1)u(t-1) + (3-t-(t-1))u(t-2) + (t-3)u(t-3)$$

$$g(t) = (t-1)u(t-1) + (4-2t)u(t-2) + (t-3)u(t-3)$$

$$\mathcal{L}\{g(t)\}(s) = e^{-s}\mathcal{L}\{t+1-1\}(s) + e^{-2s}\mathcal{L}\{4-2(t+2)\}(s) + e^{-3s}\mathcal{L}\{t+3-3\}(s)$$

$$= e^{-s}\mathcal{L}\{t\}(s) + e^{-2s}\mathcal{L}\{-2t\}(s) + e^{-3s}\mathcal{L}\{t\}(s)$$

$$= \underline{e^{-s}\frac{1}{s^2} - 2e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}}.$$

Problem 60 Calculate the Laplace transforms of the following functions

(a.) $f(t) = \underbrace{\sin(t)\cos(2t)} + \overbrace{\sin^2(3t)}^{\frac{1}{2}(1-\cos 6t)}$

$$\frac{1}{2i}\left(e^{it}-e^{-it}\right)\frac{1}{2}\left(e^{2it}+e^{-2it}\right) = \frac{1}{4i}\left(e^{3it}-e^{-3it}-e^{it}+e^{-it}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2i}\left(e^{3it}-e^{-3it}\right)\right) - \frac{1}{2}\left(\frac{1}{2i}\left(e^{it}-e^{-it}\right)\right)$$

$$= \frac{1}{2}\sin 3t - \frac{1}{2}\sin t$$

Thus,

$$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\left\{\frac{1}{2}\sin 3t - \frac{1}{2}\sin t + \frac{1}{2}(1-\cos(6t))\right\}(s)$$

$$= \boxed{\frac{1}{2}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+36}\right]}$$

(b.) $f(t) = e^t u(t-3) + \sin(t)u(t-6)$

$$\mathcal{L}\{f(t)\} = e^{-3s}\mathcal{L}\{e^{t+3}\}(s) + e^{-6s}\mathcal{L}\{\sin(t+6)\}(s)$$

$$= e^{-3s}\mathcal{L}\{e^{3}e^{t}\}(s) + e^{-6s}\mathcal{L}\{\cos 6\sin t + \sin 6\cos t\}(s)$$

$$= \boxed{\underbrace{e^{-3s}e^{3}}_{e^{-3(s-1)}}\left(\frac{1}{s-1}\right) + e^{-6s}\left(\frac{\cos 6 + s(\sin 6)}{s^2+1}\right)}$$

(if you like.)

$\boxed{P89}$ $y'' + 4y' + 4y = u(t-\pi) - u(t-2\pi)$, $y(0) = y'(0) = 0$

$$s^2 Y - 4sY + 4Y = \frac{1}{s}e^{-\pi s} - \frac{1}{s}e^{-2\pi s}$$

$$\underbrace{(s^2 - 4s + 4)}_{(s-2)^2} Y = \frac{1}{s}\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$Y = \frac{1}{s(s-2)^2}\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$\frac{1}{s(s-2)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$1 = A(s-2)^2 + Bs(s-2) + Cs$$

$\underline{s=0|}$ $1 = 4A$ $\therefore$ $A = 1/4$

$\underline{s=2|}$ $1 = 2C$ $\therefore$ $C = 1/2$

$\underline{s=1|}$ $1 = A - B + C \hookrightarrow B = A + C - 1 = \frac{1}{4} + \frac{1}{2} - 1 = \frac{-1}{4}$

$$Y = \Bigg(\underbrace{\frac{1}{4s} - \frac{1}{4(s-2)} + \frac{1}{2(s-2)^2}}_{G(s)}\Bigg)\left(e^{-\pi s} - e^{-2\pi s}\right)$$

$$Y(s) = G(s)e^{-\pi s} - G(s)e^{-2\pi s}$$

$$\boxed{\begin{aligned} y = {} & \left(\frac{1}{4} - \frac{1}{4}e^{2(t-\pi)} + \frac{1}{2}(t-\pi)e^{2(t-\pi)}\right)u(t-\pi) + \\ & + \left(\frac{1}{4} - \frac{1}{4}e^{2(t-2\pi)} + \frac{1}{2}(t-2\pi)e^{2(t-2\pi)}\right)u(t-2\pi) \end{aligned}}$$

P90 $\underline{y'' + 5y' + 6y = g(t)}_{*}$, $y(0) = 0$, $y'(0) = 2$

$$g(t) = t\,(u(t-1) - u(t-5)) + 1 \cdot u(t-5)$$

$$g(t) = t\,u(t-1) + (1-t)\,u(t-5)$$

Take Laplace transform of $*$,

$$s^2 Y - 2 + 5s\,Y + 6Y = e^{-s}\mathcal{L}\{t+1\}(s) + e^{-5s}\mathcal{L}\{1-(t+5)\}(s)$$

$$(s^2 + 5s + 6)\,Y = 2 + e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + e^{-5s}\left(\frac{-4}{s} - \frac{1}{s^2}\right)$$

$$Y = \frac{2}{s^2+5s+6} + \frac{1}{s^2+5s+6}\left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} + \frac{1}{s^2+5s+6}\left(\frac{-4}{s} - \frac{1}{s^2}\right)e^{-5s}$$

Partial fractions algebra,

$$Y = \frac{2}{s+2} - \frac{2}{s+3} + \left(\frac{1}{36s} + \frac{1}{6s^2} - \frac{1}{4(s+2)} + \frac{2}{9(s+3)}\right)e^{-s}$$
$$+ \left(\frac{-19}{36s} - \frac{1}{6s^2} + \frac{7}{4(s+2)} - \frac{11}{9(s+3)}\right)e^{-5s}$$

$$\boxed{\begin{aligned} y = 2e^{-2t} - 2e^{-3t} &+ \left(\frac{1}{36} + \frac{1}{6}(t-1) - \frac{1}{4}e^{-2(t-1)} + \frac{2}{9}e^{-3(t-1)}\right)u(t-1) + \\ &+ \left(\frac{-19}{36} - \frac{1}{6}(t-5) + \frac{7}{4}e^{-2(t-5)} - \frac{11}{9}e^{-3(t-5)}\right)u(t-5) \end{aligned}}$$

P91 See problem sheet.

P92 $\underline{w'' + w = \delta(t-\pi)\,,\; w(0) = w'(0) = 0}$

$$s^2 W + W = e^{-\pi s}$$

$$W = \left(\frac{1}{s^2+1}\right)e^{-\pi s}$$

$$\underline{w = \sin(t-\pi)\,u(t-\pi)}$$

$\boxed{P93}$ Solve $y'' + y = 4\delta(t-2) + t^2$, $y(0) = 0$, $y'(0) = 2$

$$s^2 Y - 2 + Y = 4e^{-2s} + \frac{2}{s^3}$$

$$(s^2+1) Y = 2 + 4e^{-2s} + \frac{2}{s^3}$$

$$Y = \frac{2}{s^2+1} + \frac{4}{s^2+1} e^{-2s} + \frac{2}{s^3(s^2+1)}$$

algebra.

$$Y = \frac{2}{s^2+1} + \frac{4}{s^2+1} e^{-2s} + \frac{2}{s^3} + \frac{2s}{s^2+1} - \frac{2}{s}$$

$$\boxed{y = 2\sin t + 4\sin(t-2)\,u(t-2) + t^2 + \cos t - 2}$$

$\boxed{P94}$ hammer hits a spring mass system at $t = \pi/2$ gives

$$\frac{d^2x}{dt^2} + 9x = -3\delta(t - \pi/2)$$

with $x(0) = 1$ and $x'(0) = 0$. Find the eq^{n} of motion

$$s^2 X - s + 9X = -3e^{-\pi s/2}$$

$$X = \frac{s}{s^2+9} - \frac{3}{s^2+9} e^{-\pi s/2}$$

$$x = \cos(3t) - \sin(3(t - \pi/2))\, u(t - \pi/2)$$

$$x = \cos(3t) - (\sin(3t)\cos(-3\pi/2) - \cos(3t)\sin(3\pi/2))\, u(t - \pi/2)$$

$$\underline{x = \cos(3t) - (\cos 3t)\, u(t - \pi/2)}$$

$$\underline{x(t) = 0 \quad \text{for} \quad t > \pi/2}$$

(hammer stops motion at $t = \pi/2$)

PROBLEM 95

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\}(t) = t * e^{at}$$
$$= \int_0^t (t-u)e^{au}\,du$$
$$= t\int_0^t e^{au}\,du - \int_0^t ue^{au}\,du$$
$$= t\left[\frac{1}{a}e^{au}\Big|_0^t\right] - \frac{1}{a}ue^{au}\Big|_0^t + \int_0^t \frac{1}{a}e^{au}\,du$$
$$= \frac{t}{a}\left[e^{at}-1\right] - \frac{1}{a}\left[te^{at}\right] + \frac{1}{a^2}e^{au}\Big|_0^t$$
$$= -t/a + \frac{1}{a^2}\left[e^{at}-1\right]$$
$$= \frac{-1}{a^2} + \frac{1}{a^2}e^{at} - \frac{t}{a}$$
$$= \boxed{\frac{e^{at}-1-at}{a^2}}$$

PROBLEM 96 Find $\int$-sol$^{\underline{n}}$ of $y''+y=g$. Let $y(0)=y_0$, $y'(0)=y_1$

$$s^2Y - sy_0 - y_1 + Y = \mathcal{L}\{g\} = G$$
$$Y = \frac{sy_0+y_1}{s^2+1} + \left(\frac{1}{s^2+1}\right)G(s)$$
$$y = y_0\cos t + y_1\sin t + (\sin t) * g(t)$$
$$\Rightarrow \boxed{y(t) = y_0\cos t + y_1\sin t + \int_0^t \sin(t-u)\,g(u)\,du}$$