Матн 334:

Mission 5: Series Solution Techniques & PDEs [50pts]

While complex solutions may be useful as in-between work the solutions requested on this assignment are real solutions. Please refer to Mission 3 for formatting rules. 50 points to earn here, 5 points for completely correct formatting. Thanks!

**Problem 92** (1pt) Find the first 4 nonzero terms in the power series solution about x=0 for  $z''-x^2z=0$ .

**Problem 93** (2pt) Find the complete power series solution (including a formula for the general coefficient) about x = 0 for:

$$y' - 2xy = 0.$$

**Problem 94** (2pt) Find the complete power series solution (including a formula for the general coefficient) about x = 0 for:

$$y'' - xy' + 4y = 0.$$

**Problem 95** (1pt) Find the first four nonzero terms in the power series solution about x = 0 for:

$$y'' - e^{2x}y' + (\cos x)y = 0,$$
  $y(0) = -1, y'(0) = 1.$ 

**Problem 96** (1pt) Find the first four nonzero terms in the power series solution about x = 0 for:

$$z'' + xz' + z = x^2 + 2x + 1.$$

- **Problem 97** (1pt) Find the singularities of  $x(x^2+2x+2)y''+(x^2+1)y'+3y=0$  and determine the largest open interval of convergence for a solution of the form  $y=\sum_{n=0}a_n(x+2)^n$ . Think. Do not try to solve this, I'm asking you about the interval of convergence, I'm **not** asking for what  $a_n$  are in particular
- **Problem 98** (1pt) Suppose we define  $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ . Show that  $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ .

**Problem 99** (1pt) Suppose  $\sum_{k=0}^{\infty} (a_{2k}x^{2k} + b_{2k+1}x^{2k+1}) = e^x + \cos(x+2)$ . Find explicit formulas for  $a_{2k}$  and  $b_{2k+1}$  via  $\Sigma$ -notation algebra.

**Problem 100** (2pt) Find a power series solution to the integrals below:

(a.) 
$$\int \frac{x^3 + x^6}{1 - x^3} dx$$

**(b.)** 
$$\int x^8 e^{x^3+2} \, dx$$

**Problem 101** (3pt) Solve via method of Frobenius:  $2x^2y'' + 3xy' - (1+x)y = 0$ .

**Problem 102** (3pt) Solve via method of Frobenius:  $3x^2y'' + (5x + 3x^3)y' + (3x^2 - 1)y = 0$ .

**Problem 103** (3pt) Solve via method of Frobenius:  $(2x^2 - x^3)y'' + (7x - 6x^2)y' + (3 - 6x)y = 0$ .

**Problem 104** (3pts) Find all nontrivial solutions to the following Boundary Value Problems (BVPs) on the given interval I. For each case, determine what values of K permit a family of nontrivial solutions. Use  $n \in \mathbb{N}$  to index the solutions as appropriate.

(a.) 
$$y'' + Ky = 0$$
 with  $y(0) = y(\pi) = 0$ ,  $I = (0, \pi)$ 

**(b.)** 
$$y'' + Ky = 0$$
 with  $y'(0) = y'(1) = 0$ ,  $I = (0, 1)$ 

(c.) 
$$y'' + Ky = 0$$
 with  $y'(0) = y(2) = 0$ ,  $I = (0, 2)$ 

**Problem 105** (3pt) Find the Fourier series which represents  $f(x) = x^2$  on  $-\pi < x < \pi$ . Use your result to calculate the sum of the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  and  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ 

**Problem 106** (1pt) Find the Fourier sine series which represents  $f(x) = x - x^2$  for 0 < x < 1.

**Problem 107** (1pt) Find the Fourier cosine series which represents  $f(x) = \pi - x$  for  $0 < x < \pi$ .

Problem 108 (1pts) Consider separation of variables for the PDE given below

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

given by the proposed product  $u(r,\theta) = R(r)T(\theta)$ . Show this proposal yields

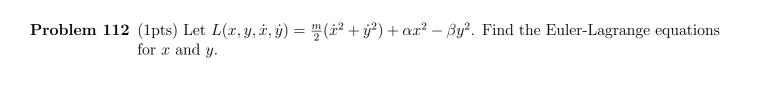
$$T''(\theta) + \lambda T(\theta) = 0$$

where  $\lambda$  is a constant and find related ODE for R which also involves  $\lambda$ .

**Problem 109** (2pts) Solve the heat equation  $\frac{\partial u}{\partial t} = 3 \frac{\partial^2 u}{\partial x^2}$  on  $0 < x < \pi$  for t > 0 given the boundary conditions  $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$  for t > 0 and initial condition u(x,0) = x for  $0 < x < \pi$ .

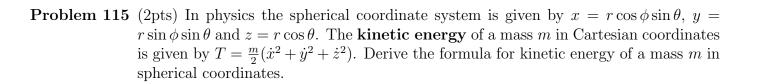
**Problem 110** (2pts) Solve the wave equation  $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$  for 0 < x < 1, t > 0 given the boundary conditions u(0,t) = u(1,t) = 0 for t > 0 and u(x,0) = x(1-x) and  $\frac{\partial u}{\partial t}(x,0) = \sin(7\pi x)$  for 0 < x < 1.

**Problem 111** (2pts) Solve the Laplace equation  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  for  $0 \le x, y \le \pi$  given the boundary conditions  $u(0,y) = u(\pi,y) = 0$  for  $0 \le y \le \pi$  and u(x,0) = f(x) and  $u(x,\pi) = 0$  for  $0 \le x \le \pi$ . You can expect the solution to be formed in terms of appropriate integrals involving f based on Fourier analytic argumentation.



**Problem 113** (1pts) In physics, polar coordinates are sometimes given by  $x = s \cos \phi$  and  $y = s \sin \phi$ . The **kinetic energy** of a mass m undergoing two-dimensional motion is given by  $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$ . Derive the formula for kinetic energy of a mass m in polar coordinates.

**Problem 114** (1pts) Find the equations of motion for the polar coordinates of a free particle of mass m. Identify any conserved quantities.



**Problem 116** (2pts) Find the equations of motion for the spherical coordinates of a free particle. Identify any conserved quantities.

**Problem 117** (2pts) Suppose that the net-force on a given mass m is a conservative force which depends only on the distance from the origin. In this case,  $F = -\frac{dU}{dr}$  and we can express the Lagrangian L = T - U using the formula you derived in the previous problem and the arbitrary potential U = U(r). Derive the Euler Lagrange equations for r,  $\theta$  and  $\phi$ .