Don't even think about working these problems out on these pages alone. The solutions should be written neatly on lined or unlined paper with the work clearly labeled. Do not omit scratch work. I need to see all steps. Thanks and enjoy.

Problem 1 [5pts] Solve,

$$(e^x+e^{-x})\frac{dy}{dx} = y^2.$$

.

Problem 2 [5pts] Solve,

$$\frac{dy}{dx} = \frac{y^2-1}{x^2-1}, \qquad y(2) = 2.$$

.

Problem 3 [5pts] Solve,

$$x\frac{dy}{dx}+2y = 3, \qquad y(2) = 2$$

.

Problem 4 [5pts] Given L, R, E, I_o are constants solve the following differential equation:

$$L\frac{dI}{dt}+RI = E, \qquad I(0) = I_o.$$

.

Problem 5 [5pts] Solve,

$$\frac{dy}{dx}+2y = f(x), \;\; y(0) = 0, \;\; \text{with} \qquad f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

Problem 6 [5pts] Solve,

$$(e^x+y)dx+(2+x+ye^y)dy = 0, \qquad y(0) = 1.$$

Problem 7 [5pts] Show the following differential equation is **not** exact:

$$(5x^2y + 6x^3y^2 + 4xy^2)dx + (2x^3 + 3x^4y + 3x^2y)dy = 0.$$

Find integers m, n such that

$$x^my^n(5x^2y + 6x^3y^2 + 4xy^2)dx + x^my^n(2x^3 + 3x^4y + 3x^2y)dy = 0.$$

is an exact equation. Solve the resulting exact differential equation and use the solutions to solve the original given inexact differential equation. (usually multiplication by an integrating factor $\mu = x^my^n$ loses a few exceptional solutions)

Problem 8 [5pt] Solve (this is section 2.5 #8),

$$(2xy)dx+(y^2-3x^2)dy=0$$

.

Problem 9 [5pt] Use the substitution $y=vx^2$ to solve

$$\frac{dy}{dx}=\frac{2y}{x}+\cos(y/x^2).$$

.

Problem 10 [8pts] (this is section 3.4 #24). A rocket with initial mass m_o is launched vertically from the ground at time zero. The rocket thrusts gas at α mass per time with speed β relative to the rocket. Assuming the usual law of gravity (assuming the rocket is near the surface of the earth), Newton's Second Law reads as follows:

$$(m_o-\alpha t)\frac{dv}{dt}-\alpha\beta=-g(m_o-\alpha t)$$

where $v=\frac{dx}{dt}$ is the velocity of the rocket at height x above the ground. Notice that $m_o-\alpha t=m(t)$ is the mass at time t. **Given the rocket launched with initial velocity zero find the equation of motion for** $0\leq t<\frac{m_o}{\alpha}$. *Comment: there are a number of simplifying assumptions made in this model: no friction and* $F_{gravity}=m(t)g$ *for example. If one was to model an intercontinental ballistic missle we would probably need to include those. That is why the text says this is for a "model" rocket*

Problem 11 [8pts] (this is section 3.4 #12). A cannon ball of mass 2 kg is shot vertically with an initial velocity of 200 m/s. The frictional force of air resistance is $|v|/20$ pointed opposite the direction of motion. When will the cannon ball reach it's maximum height above the ground? Assuming the initial height of the cannon ball is zero, find the maximum height it reaches.

Problem 12 [8pts] Hopefully by now you have solved dozens of differential equations. In this problem I wish to pause for a moment and consider the graphical meaning of those calculations. What precisely is a "general solution"? The answer is that a general solution is in fact a whole family of curves. Each member of this family is a solution and we can usually label each by a particular constant. This constant can sometimes be uniquely prescribed by an initial condition (but not always, see #29 of section 2.2). Let's study a few cases:

(a.) $\int f(x)dx=y$ is equivalent to the differential equation $\frac{dy}{dx}=f(x)$. Graph at least 4 members of the general solution to $\frac{dy}{dx}=\cos(x)$. Is the sum of two distinct solutions again a solution? Given an initial condition is the solution uniquely selected from the family of general solutions?

(b.) Solve $xdx+ydy=0$ and graph at least 4 members of the general solution. Is the sum of solutions a solution? Given an initial condition is the solution uniquely selected from the family of general solutions?

(c.) Solve $\frac{dy}{dx} + y = 1$ and graph at least 4 members of the general solution. Is the sum of distinct solutions a solution? Given an initial condition is the solution uniquely selected from the family of general solutions?

(d.) Do #29 of section 2.2.

Problem 13 [6pts] Another somewhat subtle issue is the *domain of solutions*. Given a differential equation and an intial condition we can ask what is the largest domain on which the solution is in fact a solution. There are at least two things we must consider: (1.) what is the domain of the function (2.) is the differential equation well-defined on the whole domain indicated from (1.)?

(a.) Complete parts a,b,c of section 2.2 #31,

(b.) Consider the differential equation $\frac{dy}{dx} = \sqrt{1-y^2}$. Find the domain(s) of the solution $y = \sin(x)$ (there are infinitely many disjoint intervals for this particular formula, the key here is that the square-root function is by definition non-negative)

(c.) Consider the differential equation $(\frac{dy}{dx})^2 = 1 - y^2$. Show that $y = \sin(x)$ is a solution independent of the particular value of x; argue that $y = \sin(x)$ is a solution for all of $\mathbb{R}$.

Problem 14 [9pts] **Calculus for complex-valued functions of a real variable**. Given a function $f : I \subseteq \mathbb{R} \to \mathbb{C}$ where $f(t) = u(t) + iv(t)$ for each $t \in I$ we define:

$$\boxed{\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}, \qquad \int f(t)dt = \int u(t)dt + i\int v(t)dt.}$$

One such function we will make much use of in this course is the complex exponential function, it is defined in terms of ordinary exponentials and the sine and cosine function as follows:

$$\boxed{e^{x+iy} = e^x(\cos(y) + i\sin(y))}$$

for each $z = x + iy \in \mathbb{C}$ where I am denoting $Re(z) = x \in \mathbb{R}$ and $Im(z) = y \in \mathbb{R}$. Complete the following tasks:

(a.) Let $x, y, a, b \in \mathbb{R}$ so that $z = x + iy, w = a + ib \in \mathbb{C}$. **Show that** $e^z e^w = e^{z+w}$. You may assume properties of complex numbers, laws of exponents for real exponentials and the adding-angles formulas for sine and cosine.

(b.) Let $\alpha, \beta \in \mathbb{R}$ so that $\lambda = \alpha + i\beta \in \mathbb{C}$. **Show that** $\frac{d}{dt}(e^{\lambda t}) = \lambda e^{\lambda t}$.

(c.) Let $\alpha, \beta \in \mathbb{R}$ so that $\lambda = \alpha + i\beta \in \mathbb{C}$. Given that $\int e^{\lambda t}dt = \frac{1}{\lambda}e^{\lambda t} + C$, **derive the integrals of** $e^{\alpha t}\cos(\beta t)$ **and** $e^{\alpha t}\sin(\beta t)$ **by equating the real and imaginary parts of** $\int e^{\lambda t}dt = \frac{1}{\lambda}e^{\lambda t} + C$. You should appreciate this is easier than the integration by parts solution we taught in calculus.

Problem 15 [16pts] Wow, the end is near, finally. Solve the following constant coefficient linear homogeneous ordinary differential equations.

(a.) $y'' + 5y' + 6y = 0$ subject to initial conditions $y(0) = 0, y'(0) = 3$.

(b.) $y'' - 4y = 0$ subject to initial conditions $y(0) = 1, y'(0) = 2$.

(c.) $y'' + 4y' + 5y = 0$

(d.) $z''' + 3z'' + 3z' + z = 0$

(e.) $(D^2 + 1)(D^2 - 1)(D^2 + 3D + 2)[y] = 0$ where $D = d/dx$

(f.) $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} - 8y = 0$

(g.) $y'' = 0$

(h.) $(D^2 + 4D + 5)^2[y] = 0$ where $D = d/dx$ (see notes for additional comments on the meaning of this notation)

As is often the case in differential equations, this type of problem is both easy and useful to a myriad of common applications such as RLC-circuits, springs with or without friction, angular motion with frictional torques, vibrating beams, wave-guides, standing waves in a pipe or on a string, propagation of heat in a solid, quantum-mechanical harmonic oscillator, travelling waves, vibrations on a drum, shock-absorbers in a car, exponential growth and decay, ... All of these physical examples involve in one way or another the mathematics you just completed.