Don't even think about working these problems out on this page alone. The solutions should be written neatly on lined or unlined paper with the work clearly labeled. Do not omit scratch work. I need to see all steps. Thanks and enjoy.

Problem 16 [50pts] Solve the following constant coefficient linear nonhomogeneous ordinary differential equations by the *Method of Undetermined Coefficients*. If initial conditions are given then find the specific solution which satisfies the given conditions, otherwise find the general solution. For each problem below I expect you to begin by showing how your choice of the form of the particular solution follows from the annihilator method (like E74,E75 and E76 in my notes). Then you should work it out (like E71,E72,E73,E78, E79, E80, E81, E83).

(a.) $y'' - 9y = 54$,

(b.) $y'' - y = x^2 e^x + 5$,

(c.) $y'' + 25y = 20\sin(5x)$

(d.) $y'' - 4y' + 8y = x^3$ subject to initial conditions $y(0) = 2, y'(0) = 4$

(e.) $y'' + y = 8\cos(2x) - 4\sin(x)$ subject to initial conditions $y(\pi/2) = -1, y'(\pi/2) = 0$

Problem 17 [10pts] Solve the following nonhomogeneous linear 2-nd order ODEs via the method of *Variation of Parameters*. In part (a.) I give you two linearly independent solutions to the corresponding homogeneous ODE. In part (b.) I let you find the fundamental solution set.

(a.) $x^2y'' + xy' + (x^2 - 1/4)y = x^{\frac{3}{2}}$ $\qquad y_1 = x^{\frac{-1}{2}}\cos(x)$ and $y_2 = x^{\frac{-1}{2}}\sin(x)$ form a fundamental solution for the given ODE. By the way, the *Method of Frobenius* is a way to derive these solutions. We'll cover that method later in the course.

(b.) $y'' - y = \cosh(x)$,

Problem 18 [5pts] Given that $y_1 = x^4$ is a solution to

$$x^2y'' - 7xy' + 16y = 0$$

find a second linearly independent solution by *Reduction of Order*. In other words, use the mathematics developed on pages 96-97 of my notes to find y_2. (you don't have to prove the formula, you can just use it)

Problem 19 [15pts] The problem is devoted to the concept of superposition. We focus on second order ODEs in this course, but the principle applies generally to all linear problems.

(a.) work out the details of Problem #30 of §4.7,

(b.) complete #48 of §4.5,

(c.) solve $y'' - y = x^2e^x + 5 + 3\cosh(x)$ via the method of superposition you proved in part (a.). Notice you should use your work from Problems 16b and 17b, there's no need to do all that again.

Problem 20 [10pts]Real world problems. Springs and RLC circuits.

(a.) work out the details of Problem #7 of §4.9,

(b.) work out the details of Problem #5 of §4.10,

(c.) work out the details of Problem #3 of §5.7.

Problem 21 [4pts] Prove the following sets of functions are linearly independent on $\mathbb{R}$,

(a.) $\{e^{2x},\ e^{3x}\}$
(b.) $\{x,\ e^x\}$

You can use the *Wronskian* for part (a.) if you wish, however that is not an option for part (b.). I would advise you make a direct argument from the definition of linear independence. (see E53, E54 or E55, it's not that tricky)

Problem 22 [6pts] Springs with damping and RLC circuits share the same underlying mathematics. Essentially both are based on solving the constant coefficient problem $ay'' + by' + cy = 0$ in the special case $a > 0$ and $b, c \geq 0$.

(a.) Find the general form of the solution in the following cases:

$$(I.)\ b^2 - 4ac > 0, \qquad (II.)\ b^2 - 4ac = 0, \qquad (III.)\ b^2 - 4ac < 0$$

(b.) Next, analyze the *asymptotic* behaviour of the general solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$. Show that $y(t)$ either $y(t) \to 0$ or $y(t)$ is bounded as $t \to \infty$. To say the solution is *bounded* means to show there exists $M \in \mathbb{R}$ such that $0 \leq |y(t)| \leq M$ for all $t \in \mathbb{R}$.

(c.) Discuss the physical significance in (i.) a spring of mass $m > 0$, damping constant $b \geq 0$ and spring constant $k > 0$ or (ii.) an RLC circuit with $L, C > 0$ and $R \geq 0$. In particular, what is the significance of the $b = 0$ and $R = 0$ case?

Why did I ask these questions? in the nonhomogeneous case we call the homogeneous solution the *transient* solution because after a long time it usually vanishes. For large time the motion (or current) is dominated by the particular solution which is known as the *steady-state* solution. The steady state solution arises as a response to an external driving force for the spring or a voltage (or current I suppose) source for an RLC circuit.

Problem 23 [10pts][BONUS based on an extension of pg. 101 of my notes]. If we are given a differential equation of the form $P(L)[y] = 0$ for some polynomial P in the linear differential operator L, then the operator $P(L)$ can be factored in the same way as the polynomial; $P(L) = P_1(L)P_2(L)\cdots P_n(L)$ where $P_m(L) = (L-\lambda_m)^{s_m}$. This means the problem of finding solutions for $P(L)[y] = 0$ reduces to the problem of finding eigenfunctions for L, and generalized eigenfunctions for L. (we have to deal with all the same problems we already have faced in the constant coefficient case, it could be that P has a complex zero. That would cause us to use real and imaginary parts of the complex eigenfunction as solutions to our real-valued problem.). Enough of the groundwork, let me get to our problem.

(a.) Suppose $L = f(x)D$ where $D = d/dx$. Find eigenfunctions for this particular type of operator; solve $f(x)D[y] = \lambda y$. Notice $f(x) = 1$ gives constant coefficient case, and $f(x) = x$ gives the Cauchy Euler problem. (*answer is* $y = exp(\lambda \int dx/f(x))$).

(b.) Let $P(x^2D) = (x^2D)^2 + 5x^2D + 6$. Solve the differential equation $P(x^2D)[y] = 0$. Your solution should be a linear combination two eigenfunction solutions.

(c.) Find the general solution of

$$x^2\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + 4x^2\frac{dy}{dx} + 4y = 0.$$

The solution has one eigenfunction solution y_1 just like part (b.). To find y_2 use reduction of order. I think y_2 may satisfy $(x^2D + 2)[y_2] = y_1$ however I have not checked it (if that is true then $(x^2D+2)^2[y_2] = 0$ which means that y_2 is a generalized eigenfunction of order 2 with respect to the operator x^2D.

My point? the operator viewpoint allows us to extend our experience with constant coefficient ODEs to a pleathora of differential equations. So long as the ODE is linear we have nice theory to work with. I do not mean to say this problem describes how to solve all linear ODEs. There is no gaurantee that L can be written as polynomial of particular linear operator. If I could prove that though, this problem would give a very general method of solving linear ODEs. We at least have described how to find ODEs which are a lot like the constant coefficient case. That said, the basic cases are most important because those are the ones that applications produce usually. Beyond that we can use series for approximations so a lack of an exact closed-form elementary function solution is not an insurmountable trouble for applications.