$\boxed{\text{Th}^{m}(3)}$ (Shift Theorem) If the Laplace transform $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > \alpha$ then for $s > \alpha + a$

$$\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a)$$

Proof:

$$\mathcal{L}\{e^{at} f(t)\}(s) = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$
$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$
$$= F(s-a).$$

E105 $\mathcal{L}\{e^{at}\sin bt\}(s) = F(s-a) = \boxed{\dfrac{b}{(s-a)^2 + b^2}}$ for $f(t) = \sin bt$, $F(s) = \dfrac{b}{s^2+b^2}$

E106 $\mathcal{L}\{e^{at}\}(s) = F(s-a) = \boxed{\dfrac{1}{s-a}}$ for $f(t) = 1$, $F(s) = 1/s$

Th^m(4) (Laplace Transform of Derivative). Let f and f' be piecewise continuous with exponential order α then for $s > \alpha$,

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0)$$

Proof:

$$\mathcal{L}\{f'\}(s) = \lim_{N\to\infty} \int_0^N e^{-st} \frac{d}{dt}\big(f(t)\big)\,dt$$
$$= \lim_{N\to\infty} \int_0^N \left[\frac{d}{dt}\left(e^{-st} f(t)\right) - \frac{d}{dt}\left(e^{-st}\right) f(t)\right] dt$$
$$= \lim_{N\to\infty} \left(e^{-sN} f(N) - f(0) + \int_0^N s\, e^{-st} f(t)\,dt\right)$$
$$= \lim_{N\to\infty}\left(e^{-sN} f(N)\right) - f(0) + s \lim_{N\to\infty} \int_0^N e^{-st} f(t)\,dt$$
$$= -f(0) + s\,\mathcal{L}\{f\}(s).$$

Where we noted since for $t > \alpha$ we have $|f(t)| \le Me^{\alpha t}$ thus $|e^{-sN} f(N)| \le e^{-sN} Me^{\alpha N} = Me^{N(\alpha - s)} \longrightarrow 0$ as $N \to \infty$ provided $s > \alpha$.

We can derive similar formulas for higher derivatives,

$$\mathcal{L}\{f''\}(s) = s\,\mathcal{L}\{f'\}(s) - f'(0) \quad \text{using Th}^{m}(4).$$
$$= s\left(s\,\mathcal{L}\{f\}(s) - f(0)\right) - f'(0)$$
$$= s^2\,\mathcal{L}\{f\} - s f(0) - f'(0).$$

Th^{m}(5) Let $f, f', \dots, f^{(n-1)}$ be continous and $f^{(n)}$ piecewise continuous all of them of exponential order α. Then for $s > \alpha$,

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\,\mathcal{L}\{f\}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Proof: follows from calculation similar to the one above Th^{m}(5).

E107 Lets use Th^{m}(4) and Th^{m}(5) to transform the constant coefficient DEq$^{\underline{n}}$ $aY'' + bY' + cY = 0$ where $Y' = \frac{dY}{dt}$

$$\mathcal{L}\{Y''\}(s) = s^2\,\mathcal{L}\{Y\}(s) - sY(0) - Y'(0) = s^2\,\mathcal{Y}(s) - sY(0) - Y'(0)$$
$$\mathcal{L}\{Y'\}(s) = s\,\mathcal{L}\{Y\}(s) - Y(0) = s\,\mathcal{Y}(s) - Y(0)$$
$$\mathcal{L}\{Y\}(s) = \mathcal{Y}(s)$$

It is customary to use lower case Y for $Y(t)$ then uppercase $\mathcal{Y}$ to denote the Laplace transform $\mathcal{L}\{Y\}(s) \equiv \mathcal{Y}(s)$. Taking the Laplace transform of $aY'' + bY' + cY = 0$ yields

$$a\left(s^2\mathcal{Y} - sY(0) - Y'(0)\right) + b\left(s\mathcal{Y} - Y(0)\right) + c\,\mathcal{Y} = 0$$

$$(as^2 + bs + c)\,\mathcal{Y} = asY(0) + aY'(0) + bY(0)$$

Notice we have change a differential eq$^{\underline{n}}$ in t to an algebraic eq$^{\underline{n}}$ in s. We'll come back to this example later.

E108 Let $g(t) = \int_0^t f(u)\,du$ then $g'(t) = f(t)$ by FTC. Thus

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{g'\}(s)$$
$$= s\,\mathcal{L}\{g\}(s) - g(0)$$
$$= s\,G(s) - \cancel{\int_0^0 f(u)\,du} \rightarrow 0$$

$$\therefore \boxed{\frac{1}{s}\,\mathcal{L}\{f\}(s) = \mathcal{L}\left\{\int_0^t f(u)\,du\right\}(s)}$$

Remark: In $\boxed{E107}$ and $\boxed{E108}$ we observe that the Laplace transform changes differentiation w.r.t. t into multiplication by s, and some extra stuff that relates to the initial conditions.

Th^{m}(6) Let $F(s) = \mathcal{L}\{f\}(s)$ and assume $f(t)$ is piecewise continous on $[0, \infty)$ and of exponential order α. Then for $s > \alpha$

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$$

"Proof": We'll see why $n=1$ works,

$$\begin{aligned} \frac{dF}{ds} &= \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right) \\ &= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt \\ &= \int_0^\infty -t e^{-st} f(t)\,dt \\ &= (-1)^1 \int_0^\infty e^{-st} t f(t)\,dt \\ &= (-1)^1 \mathcal{L}\{t f(t)\}(s) \end{aligned}$$

nontrivial step, serious proof that this is allowed can take a page of careful reasoning. However, intuitively it's reasonable

Then next try $n=2$,

$$\begin{aligned} \frac{d^2F}{ds^2} = \frac{d}{ds}\left(\frac{dF}{ds}\right) &= (-1)^1 \frac{d}{ds}\int_0^\infty e^{-st} t f(t)\,dt \\ &= (-1)^1 \int_0^\infty \frac{d}{ds}\left(e^{-st} t f(t)\right) dt \\ &= (-1)^2 \int_0^\infty e^{-st} t^2 f(t)\,dt \\ &= (-1)^2 \mathcal{L}\{t^2 f(t)\}(s). \end{aligned}$$

It's clear we'll find that $\frac{d^nF}{ds^n} = (-1)^n \mathcal{L}\{t^n f(t)\}(s)$ which upon multiplication by $(-1)^n$ is exactly the Th^{m}.

PROPERTIES OF LAPLACE TRANSFORMS SUMMARY (table 7.2)

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$
$$\mathcal{L}\{cf\} = c\,\mathcal{L}\{f\}$$
$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$$
$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0)$$
$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$$
$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$
$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s) \quad \text{where } \mathcal{L}\{f\} \equiv F.$$

E109 Let $f(t) = (1+e^{-t})^2 = 1 + 2e^{-t} + e^{-2t}$ then

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{1\}(s) + 2\mathcal{L}\{e^{-t}\}(s) + \mathcal{L}\{e^{-2t}\}(s)$$
$$= \boxed{\frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2}}$$

E110 Let $f(t) = te^{2t}\cos 5t$

$$\mathcal{L}\{f\}(s) = -\frac{d}{ds}\left(\mathcal{L}\{e^{2t}\cos 5t\}(s)\right) \text{, using Th}^{m}(6)\ n=1$$
$$= -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+25}\right) \text{, table 7.1.}$$
$$= -\frac{(s-2)^2+25-(s-2)[2(s-2)]}{[(s-2)^2+25]^2} \text{, quotient rule.}$$
$$= \boxed{\frac{(s-2)^2-25}{[(s-2)^2+25]^2} = F(s)}$$

E111 Let $f(t) = t\sin^2 t$

$$\mathcal{L}\{f\}(s) = -\frac{d}{ds}\left(\mathcal{L}\{\sin^2 t\}(s)\right)$$
$$= -\frac{d}{ds}\left(\mathcal{L}\{\tfrac{1}{2}(1-\cos 2t)\}(s)\right)$$
trig. identity you should know.
$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$
$$= -\frac{1}{2}\left(\frac{-1}{s^2} - \frac{s^2+4-s(2s)}{(s^2+4)^2}\right)$$
$$= \boxed{\frac{1}{2}\left(\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right) = F(s) = \mathcal{L}\{f\}(s)}$$

E112 Calculate Laplace transform of $f(t) = \sin t \cos 2t$

$$\sin(t)\cos(2t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right)\frac{1}{2}\left(e^{2it} + e^{-2it}\right)$$

$$= \frac{1}{4i}\left(e^{3it} + e^{-it} - e^{it} - e^{-3it}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2i}\left(e^{3it} - e^{-3it}\right) - \frac{1}{2i}\left(e^{it} - e^{-it}\right)\right)$$

$$= \frac{1}{2}\sin(3t) - \frac{1}{2}\sin(t) = \sin(t)\cos(2t)$$

$$\mathcal{L}\{\sin t \cos 2t\}(s) = \mathcal{L}\left\{\frac{1}{2}\sin(3t) - \frac{1}{2}\sin(t)\right\}(s)$$

$$= \boxed{\frac{3}{2(s^2+9)} - \frac{1}{s^2+1}}$$

E113 $$f(t) = \begin{cases} e^t & 0 \le t \le 2 \\ t & t > 2 \end{cases}$$

$$f(t) = e^t[u(t) - u(t-2)] + t\,u(t-2)$$

$$f(t) = e^t u(t) + (t - e^t)u(t-2)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^t u(t)\}(s) + \mathcal{L}\{(t-e^t)u(t-2)\}$$

$$= \mathcal{L}\{e^t\}(s) + e^{-2s}\mathcal{L}\{t+2-e^{t+2}\}(s)$$

$$= \frac{1}{s-1} + e^{-2s}\mathcal{L}\{t + 2 - e^2 e^t\}(s)$$

$$= \boxed{\frac{1}{s-1} + e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s} - \frac{e^2}{s-1}\right)}$$

E114 $\mathcal{L}\{te^{-t}-3\} = \mathcal{L}\{te^{-t}\} - \mathcal{L}\{3\}$
$$= \boxed{\frac{1}{(s+1)^2} - \frac{3}{s}}$$

E115 $\mathcal{L}\{13e^{2t}\sin(t+\pi)\} = \mathcal{L}\{13e^{2t}(\sin t\cos\pi + \sin\pi\cos t)\}$
$$= \mathcal{L}\{-13e^{2t}\sin t\}$$
$$= \boxed{-13\left(\frac{1}{(s-2)^2+1}\right)}$$

E116 $\mathcal{L}\{te^{-2t}\sin(3t)\} = -\frac{d}{ds}\left(\mathcal{L}\{e^{-2t}\sin 3t\}(s)\right)$ using Table 7.2
$$= -\frac{d}{ds}\left(\frac{3}{(s+2)^2+9}\right)$$
$$= +3\left[(s+2)^2+9\right]^{-2}2(s+2)$$
$$= \boxed{\frac{6(s+2)}{[(s+2)^2+9]^2}}$$

E117
$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ t^2 & 1 \le t \le 2 \\ \sin t & 2 < t \end{cases} = [u(t)-u(t-1)] + [u(t-1)-u(t-2)]t^2 + u(t-2)\sin t$$

$$f(t) = \underbrace{u(t)}_{\text{I}} + \underbrace{[t^2-1]u(t-1)}_{\text{II}} + \underbrace{[\sin t - t^2]u(t-2)}_{\text{III}}$$

I $\mathcal{L}\{u(t)\} = \frac{1}{s}$

II Use $\mathcal{L}\{g(t)u(t-1)\}(s) = e^{-s}\mathcal{L}\{g(t+1)\}(s)$. Identify that $g(t) = t^2-1 \therefore g(t+1) = (t+1)^2-1 = t^2+2t+1-1$

$$\mathcal{L}\{g(t+1)\}(s) = \mathcal{L}\{t^2+2t\}(s) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$\therefore \mathcal{L}\{(t^2-1)u(t-1)\}(s) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right)$$

continued ↗

III Again use $\mathcal{L}\{g(t)\,u(t-2)\}(s) = e^{-2s}\,\mathcal{L}\{g(t+2)\}(s)$
identify that $g(t) = \sin t - t^2$ thus

$$g(t+2) = \sin(t+2) - (t+2)^2$$
$$= \sin t\cos 2 + \sin 2\cos t - t^2 + 2t - 4$$

$$\mathcal{L}\{g(t+2)\}(s) = \frac{\cos(2)}{s^2+1} + \frac{s\sin(2)}{s^2+1} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{4}{s}$$

$$\therefore\ \mathcal{L}\{(\sin t - t^2)u(t-2)\}(s) = e^{-2s}\left(\frac{\cos(2)}{s^2+1} + \sin(2)\frac{s}{s^2+1} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{4}{s}\right)$$

Thus putting it together,

$$\boxed{F(s) = \frac{1}{s} + e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) + e^{-2s}\left(\frac{\cos(2)}{s^2+1} + \sin(2)\frac{s}{s^2+1} - \frac{2}{s^3} + \frac{2}{s^2} - \frac{4}{s}\right)}$$

E118

$$\sin(t)\cos^2(t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right)\left[\frac{1}{2}\left(e^{it} + e^{-it}\right)\right]^2$$
$$= \frac{1}{8i}\left(e^{it} - e^{-it}\right)\left(e^{2it} + 2 + e^{-2it}\right)$$
$$= \frac{1}{8i}\left(e^{3it} + 2e^{it} + e^{-it} - e^{it} - 2e^{-it} - e^{-3it}\right)$$
$$= \frac{1}{8i}\left(e^{3it} - e^{-3it}\right) + \frac{1}{8i}\left(e^{it} - e^{-it}\right)$$
$$= \frac{1}{4}\frac{1}{2i}\left(e^{3it} - e^{-3it}\right) + \frac{1}{4}\frac{1}{2i}\left(e^{it} - e^{-it}\right)$$
$$= \frac{1}{4}\sin(3t) + \frac{1}{4}\sin(t).$$

$$\mathcal{L}\{\sin t\cos^2 t\} = \mathcal{L}\left\{\frac{1}{4}\sin(3t) + \frac{1}{4}\sin t\right\}$$
$$= \boxed{\frac{1}{4}\left(\frac{3}{s^2+9} + \frac{1}{s^2+1}\right)}$$