Introduction to my Notes

These notes follow your text mostly. If I can teach you a fraction of the text I'll be happy, it's a great book. Anyway I tend to use some abbreviations here and there so let me list them here for your convenience,

Symbols	Meaning
§	section
$\exists$	there exists
$\nexists$	there does not exist
$a \in B$	a is an element of B
$\mathbb{N}$	Natural #'s 1, 2, 3, ...
$\mathbb{Z}$	Integers ..., -1, 0, 1, 2, ...
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
iff	$\Leftrightarrow$
$\therefore$	therefore
$\#$	change in topic
$\forall$	for all
$\equiv$	definition
def^{n}	definition
Sol^{n}	solution
w.r.t	with respect to
$\notin$	not an element of

Symbols	Meaning
f-la	formula
Th^{m}	theorem
eq^{n}	equation
s.t.	such that
$\mathbb{R}$	real numbers $(-\infty, \infty)$
α	alpha
β	beta
γ	gamma
δ	delta
θ	theta
Ω	omega
ω	omega
ψ	psi
φ	phi

LI = linear independence
ODE = ordinary differential equatio
fnct = function

DIFFERENTIAL EQUATIONS IN PHYSICS

I.

$$\boxed{\text{Schrödinger's Eq}^{n}: \quad \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial \Psi}{\partial t}} \quad \boxed{\text{Eq}^{n}\ 1}$$

is the starting point for non relativistic Quantum Mechanics here the sol^{n} $\Psi(x,t)$ is the wave function. This is a partial differential eq^{n} because there are at least two independent variables x and t. There is a large class of separable sol^{n}'s where

$$\Psi(x,t) = \psi(x)\,\varphi(t)$$

If we substitute into Eq^{n} 1 we get

$$\frac{-\hbar^2}{2m}\psi''\varphi + V\psi\varphi = i\hbar\psi\varphi'$$

Divide by $\psi\varphi$ to get

$$\underbrace{\frac{-\hbar^2}{2m}\frac{\psi''}{\psi} + V}_{\text{function of } x} = \underbrace{i\hbar\frac{\varphi'}{\varphi}}_{\text{function of } t} = \text{constant} \equiv E$$

So we get two ordinary differential eq^{n}'s

$$\boxed{\begin{array}{ll} \dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + V\psi = E\psi & : \text{Time independent Schrödinger Eq}^{n} \\ \hline i\hbar\dfrac{d\varphi}{dt} = E\varphi & : \text{Energy Eigenvalue Eq}^{n} \end{array}}$$

we'll learn how to solve these soon

My Point? At some point it usually ends up being a problem of solving ODEs.

II Maxwell's Eq^n's:

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$: Gauss' Law

$$\nabla \cdot \vec{B} = 0$$: No magnetic monopoles

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$: Faraday's Law

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$: Ampere's Law

Again these are Partial Differential eq^n's since the sol^n's are the electric and magnetic fields $\vec{E}(x,y,z,t)$ and $\vec{B}(x,y,z,t)$. If you study a particular problem these will reduce to several Ordinary Differential Eq^n's.

Example: In vacuum $\rho = J = 0$. Maxwell's Eq^n's can be rewritten as $\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ and $\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$. Each of these is 3 PDE's and in case you're wondering $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\nabla^2 E_x = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad \text{for } \vec{E} = (E_x, E_y, E_z)$$

$$\nabla^2 E_y = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\nabla^2 E_z = \mu_0 \epsilon_0 \frac{\partial^2 E_z}{\partial t^2}$$

Lets pick one and pretend our system is one-dimensional so $\nabla^2 = \partial^2/\partial x^2$ (the real story is more complicated) We should solve

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

The way to solve this is to suppose $E(x,t) = X(x)T(t)$

II) Continued, we're trying to solve $\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$
we <u>guessed</u> $E(x,t) = X(x)\, T(t)$ so substitute into

$$X''(x)\, T(t) = \mu_0 \epsilon_0\, X(x)\, T''(t)$$

Divide by $X(x)\, T(t)$,

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x} = \underbrace{\mu_0 \epsilon_0 \frac{T''(t)}{T(t)}}_{\text{function of } t} = \text{constant} = a$$

So we have to solve a pair of <u>Ordinary Differential Eq^{n}'s</u>

$$\frac{d^2 X}{dx^2} = a X \quad \& \quad \frac{d^2 T}{dt^2} = \frac{a}{\mu_0 \epsilon_0}$$

If $a = -k^2$ we find sol^{n}'s, let $\omega \equiv k/\sqrt{\mu_0 \epsilon_0}$

$$X(x) = A_1 \cos(kx + \varphi_1)$$

$$T(t) = A_2 \cos(\omega t + \varphi_2)$$

Then a little algebra reveals

$$\boxed{E(x,t) = A \cos(kx - \omega t)}$$

This is a wave with speed $V = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = C = 3 \times 10^8 \frac{m}{s}$.

MAXWELL'S EQ^{n}'S $\Rightarrow$ SPEED OF LIGHT IS CONSTANT!

(II) $\boxed{\text{Newton's Law: } m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{net}}$ (assuming $\frac{dm}{dt}=0$)

In this case we have one independent variable, the time t. Again to solve particular problem it boils down to solving ODEs.

Example: Gravity near earth's surface.

$$m\frac{d^2\vec{r}}{dt^2} = -mg\hat{k} \longrightarrow \left.\begin{array}{l} mx'' = 0 \\ my'' = 0 \\ mz'' = -mg \end{array}\right\} \text{three ordinary diff. eq}^{\underline{n}}\text{s.}$$

QUESTION: where will we use ordinary differential eq$^{\underline{n}}$s in the real world?

ANSWER: MOST EVERYWHERE THAT AN EQUATION CAN MODEL AN ACTIVITY. MY EXAMPLES ARE FROM FOUNDATIONAL PHYSICS, BUT THERE ARE MANY MANY MORE EXAMPLES IN APPLICATIONS OF PHYSICS, ENGINEERING, BIOLOGY, ETC... SO ORDINARY DIFFERENTIAL EQUATIONS ARE BASIC KNOWLEDGE YOU SHOULD ADD TO YOUR ARSENAL OF MATHEMATICAL TOOLS.

Remark: Systems of Differential Equations naturally suggest the use of eigenvectors. Since linear algebra builds up the theory underlying eigenvectors we treat systems of ODE's in math 321.

BASIC TERMINOLOGY

Let us suppose that x is the independent variable and y is the dependent variable. That means we can view y as a function of x. An ordinary differential equation (O.D.E.) is an eq^{n} of the form,

$$\boxed{F\left(y^{(n)}, y^{(n-1)}, \ldots, y^{(2)}, y', y, x\right) = 0 \qquad n^{th} \text{ order differential eq}^{n}}$$

Where $y^{(n)} = \frac{d^n y}{dx^n}$ and $y' = \frac{dy}{dx}$. This highest derivative that appears gives the order of the DEq^{n}. It is called ordinary because all the derivatives are just w.r.t. x. When a differential eq^{n} has partial derivatives (like $\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t}$ for example) then it is called a partial differential eq^{n} (PDE). We'll learn how to solve a few basic PDEs at the end of this course.

Def^{n}/ A n^{th} order linear differential eq^{n} has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x) y = F(x)$$

EO

$\frac{dy}{dx} = y^2$ (non-linear // solved by seperating variables

$\frac{dy}{dx} + y = \cos(x)$ (linear // solve with method of undeter. coefficients,

$y'' + 3y' + 3y = \tan(t)$ (linear // variation of parameters will solve it)

$\left(\frac{dy}{dx}\right)^3 + \left(\frac{dy}{dx}\right)^2 = \sin(x)$ (non-linear // if you can solve this I'll be impressed.)

Def^{n}/ A solution to a differential eq^{n} $F(y^{(n)}, \ldots, x) = 0$ is a function f such that

$$F\left(f^{(n)}(x), f^{(n-1)}(x), \ldots, f'(x), f(x), x\right) = 0$$

- Given a n^{th} order ODE $F(y^{(n)},\dots,y',x)=0$ we will find there is not a unique sol^{n} just on the basis of the DEq^{n} itself. Roughly speaking solving a n^{th} order DEq^{n} amounts to taking n-integrals. For each integral we get an integration constant, so as those constants are arbitrary we see we get infinitely many sol^{n}'s. As a common abuse of terminology I will often refer to this infinite family of solutions as the "general sol^{n}".

- Now, if DEq^{n}'s are to describe the real world this is discouraging, it would seem we can never uniquely describe a system with an ODE. The sol^{n} to this is simple, physical systems are in practice modelled by <u>DEq^{n}'s</u> + <u>INTIAL CONDITIONS</u>.

Def^{n}/ A sol^{n} to an <u>intial value problem</u> (IVP) on an interval I containing x_0 is a sol^{n} to a DEq^{n} $F(y^{(n)},\dots,y',x)=0$ that <u>also</u> satisfies n-extra intial conditions

$$y(x_0)=y_0\ ,\ \frac{dy}{dx}(x_0)=y_1\ ,\dots,\ \frac{d^{n-1}y}{dx^{n-1}}(x_0)=y_{n-1}$$

where $y_0, y_1,\dots,y_{n-1}$ are given constants.

- Usually an intial value problem has a unique sol^{n}, but not always (you have a hwk. problem to illustrate this)

Th^{m}(1) Given an IVP $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$. If f and $\frac{\partial f}{\partial y}$ are continous in a rectangle that contains the point (x_0,y_0) then the IVP has a unique sol^{n} in some interval $(x_0-\delta, x_0+\delta)$ for $\delta>0$.

- The sol^{n} may or may not extend to the whole rectangle. This is a local theorem, it just tells us about sol^{n}'s <u>near</u> the point x_0.

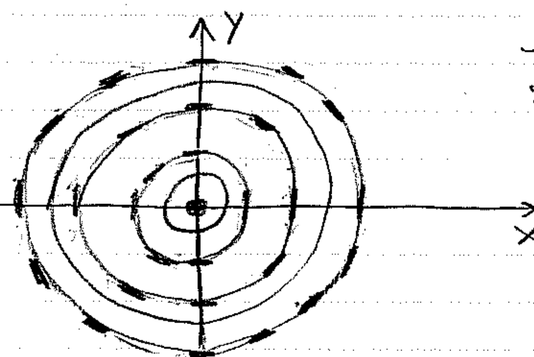
Remark: Sometimes we will given an implicit formula to describe a function. For some examples rather than possesing an explicit f-la for $f(x)$ we'll instead say the sol^{n} is the locus of points satisfying some relation. For example, $x^2+y^2=1$ is a sol^{n} of $\frac{dy}{dx} = \frac{-x}{y}$. At the level of functions we have two solutions $y = f_1(x) = \sqrt{1-x^2}$ and $y = f_2(x) = -\sqrt{1-x^2}$.

Comment: Slope fields and Euler's Method are not req^{d} topics. I include a comment about them for breadth. It is likely you would cover them in depth in a numerical methods course.

SLOPE FIELDS:

One method of visualizing a DEq^{n} is to draw it's slope field, (Chapter 12 of text is devoted to graphics)

E1

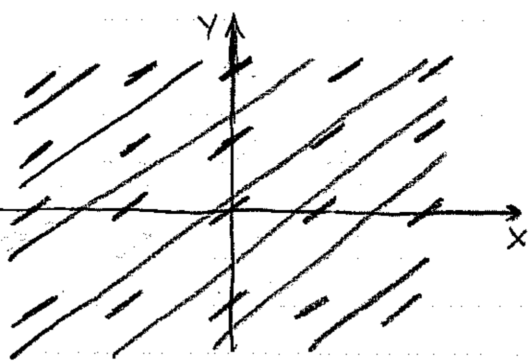


the little dashes — indicate the slope that tangents to the sol^{n} curves must have. Here I've tried to sketch:

$$\frac{dY}{dx} = \frac{-x}{Y}$$

and a few of the sol^{n}'s which happen to be circles.

E2



here I've sketched the slope field for $\frac{dY}{dx} = 1$. Sol^{n}'s have slope one every where, that means they're lines of slope one,

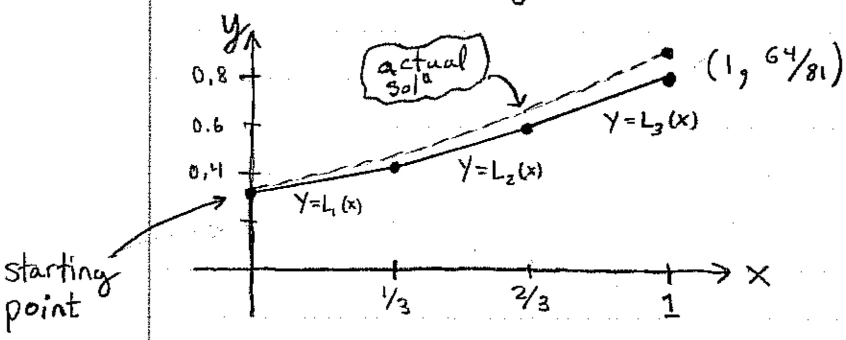
$$Y = x + b$$

Remark: I cannot draw all the sol^{n}'s, they literally fill the (xy)-plane.* We can see that DEq^{n}'s do not usually give unique sol^{n}'s. *(ok, to be careful we should throw out $(0,0)$ in E1)

Euler's Method

- Closely related to the idea of the *slope field*.
- In short, Euler's Method generates approximate sol^n's to a DEq^n by replacing the real sol^n with a piecewise linear version of the sol^n.

$\boxed{E3}$ $\frac{dy}{dx} = y$ given the intial data $y(0) = 1/3$ find $y(1)$ using Euler's method with 3 - steps.



Eulers method with 3 Steps says $y(1) \cong \frac{64}{81}$

$$L_1(x) = 1/3 + (1/3)(x - 0) \quad \Rightarrow L_1(1/3) = 1/3 + 1/9 = 4/9 = y_1$$
$$L_2(x) = 4/9 + (4/9)(x - 1/3) \quad \Rightarrow L_2(2/3) = 4/9 + 4/27 = 16/27 = y_2$$
$$L_3(x) = 16/27 + (16/27)(x - 2/3) \quad \Rightarrow L_3(1) = 16/27 + 16/81 = 64/81 = y_3$$

Remark: Eulers method & Slope fields require alot of brute force arthimatic, as such they are best implemented by technology. We will focus on those methods which do not require numerical assistance. You should be aware that many problems defy closed form sol^n's so we are driven to *solve numerically*.

Separation of Variables (exact sol^n of problem above)

$\boxed{E4}$

$$\frac{dy}{dx} = y \Rightarrow \int \frac{dy}{y} = \int dx \Rightarrow \ln|y| = x + \tilde{C} \Rightarrow y = ce^x$$

$$y(0) = c = 1/3 \Rightarrow y = \frac{1}{3}e^x$$ (exact sol^n to $\boxed{E3}$ above)

- lets compare the approximation to the real deal,

$$y(1) = \frac{1}{3}e^1 = 0.9061 \quad \text{compared to} \quad \frac{64}{81} = 0.79$$