# Additional Techniques for $1^{st}$ order Differential Eq$^{n}$'s (§2.5-2.6) 

Let me summarize the two main techniques we have for solving $1^{st}$ order ODEs,

1. Separation of Variables
2. Exact Equations

The integrating factor method (for solving linear equations §2.3) takes a differential eq$^{n}$ $\frac{dy}{dx} + Py = Q$ and converts it to a separable eq$^{n}$ $\frac{d(\mu y)}{dx} = \mu Q$ simply through a multiplication by $\mu = \exp(\int P dx)$. A natural question to ask:

> Are there other "integrating factors" which convert a $1^{st}$ order ODE to an exact ODE?

The answer is YES. This is the idea of §2.5. Let's formalize the idea of an integrating factor

> **Def$^{n}$** If the DEq$^{n}$ $M dx + N dy = 0$ is not exact however the DEq$^{n}$ $\mu M dx + \mu N dy = 0$ is exact then we say $\mu$ is an integrating factor of $M dx + N dy = 0$.

**E25** Consider $\left(-\frac{2}{x}y - x^2\cos(x)\right)dx + dy = 0$. We identify $M = -2y/x - x^2\cos(x)$ and $N = 1$. Clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ thus this is not an exact DEq$^{n}$. Try multiplying by $\mu = 1/x^2$. We obtain the exact eq$^{n}$ below,

$$\left(\frac{-2}{x^3}y - \cos(x)\right)dx + \frac{1}{x^2}dy = 0 \qquad (*)$$

A moments reflection reveals $(*)$ can be expressed as $dF = 0$ for the function $F(x,y) = y/x^2 - \sin(x)$. Hence sol$^{n}$'s are level curves $F(x,y) = k$ which is $y/x^2 - \sin(x) = k$

$$\therefore \boxed{y = x^2(k + \sin(x))}$$

Notice this is the sol$^{n}$ we found in E

Remark: the previous example demonstrates that the "integrating factor" of § 2.3 also can fit the def$^{\underline{n}}$ we just formalized on the last page. Let me prove it in general for fun,

**E 26** Solve $\boxed{\dfrac{dy}{dx} + Py = Q}$ — (*) Note this is the same as the "Pfaffian" form $dy + (Py - Q)dx = 0$. We defined $\mu = \exp(\int P\,dx)$ and found it had the especially useful identity $\dfrac{d\mu}{dx} = P\mu$. Multiply by $\mu$,

$$\boxed{\mu\, dy + (\mu P y - \mu Q)dx = 0} \quad \text{— (**)}$$

Eq$^{\underline{n}}$ (**) is an exact eq$^{\underline{n}}$ since $\dfrac{\partial}{\partial x}(\mu) = \dfrac{\partial}{\partial y}(\mu P y - \mu Q) = \mu P$. I'll not attempt to solve (**) since we already have a nice clean sol$^{\underline{n}}$ to (*). I merely include this discussion connect § 2.3 and § 2.5. These methods are not mutually exclusive.

**E 27** Some examples have $\mu = x^m y^n$ for $m, n$ constants. Consider $(2y - 6x)dx + (3x - 4x^2y^{-1})dy = 0$. Multiply by $\mu = x^m y^n$

$$x^m y^n (2y - 6x)dx + x^m y^n (3x - 4x^2 y^{-1})dy = 0$$

$$\underbrace{(2x^m y^{n+1} - 6x^{m+1} y^n)}_{P}dx + (\underbrace{3x^{m+1}y^n - 4x^{m+2}y^{n-1}}_{Q})dy = 0$$

We need $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ to insure exactness.

$$2(n+1)x^m y^n - 6n x^{m+1} y^{n-1} = 3(m+1)x^m y^n - 4(m+2)x^{m+1}y^{n-1}$$

Matching powers must have matching coefficients,

$$2(n+1) = 3(m+1)$$
$$-6n = -4(m+2) \Rightarrow n = \tfrac{2}{3}(m+2)$$
$$\Rightarrow 2\left(\tfrac{2}{3}(m+2) + 1\right) = 3(m+1)$$
$$\Rightarrow 4(m+2) + 6 = 9(m+1)$$
$$\Rightarrow 5m = 14 - 9 = 5 \Rightarrow \underline{m = 1} \;\&\; \underline{n = 2}$$

E27 continued

We found multiplication by $\mu = xy^2$ should make the given DEq$^{\underline{n}}$ $\boxed{(2y-6x)dx + (3x - 4x^2y^{-1})dy = 0}$ (*) become exact. Let's solve it. Multiply by $\mu = xy^2$,

$$\underbrace{\boxed{(2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0}}\quad (**)$$

Thinking $\Rightarrow$ $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$ for $\underline{F(x,y) = x^2y^3 - 2x^3y^2}$.

Hence, we find solutions to (**) of the form $\underline{x^2y^3 - 2x^3y^2 = k}$.

What does this tell us about (*) ?

Is the sol$^{\underline{n}}$ to (**) also a sol$^{\underline{n}}$ to (*) ?

Let's differentiate and see, (yes to be more efficient could start at ☺)

$$x^2y^3 - 2x^3y^2 = k$$

$$\Rightarrow 2xy^3 + 3x^2y^2\frac{dy}{dx} - 6x^2y^2 - 4x^3y\frac{dy}{dx} = \frac{d}{dx}(k) = 0$$

$$\Rightarrow 2xy^3\,dx + 3x^2y^2dy - 6x^2y^2dx - 4x^3y\,dy = 0$$

☺ $$\Rightarrow (2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0$$ (this is no surprise)

$$\Rightarrow (2y - 6x)dx + (3x - 4x^2y^{-1})dy = 0$$ : divided by $\mu = xy^2$.

$\Rightarrow$ yes, the sol$^{\underline{n}}$ to (**) is likewise a sol$^{\underline{n}}$ to (*).

The question arises, are <u>all</u> sol$^{\underline{n}}$'s to (*) included in our sol$^{\underline{n}}$'s to (**)? Notice we multiplied by $\mu = xy^2$ which could result in a loss of information in certain <u>special</u> cases. In particular, notice

$$x = 0 \quad \text{and} \quad y = 0$$

(the text likes $x \equiv 0$ here but I'll not bother with the $\equiv$ notation) "identically"

are both possibilities we missed. We have to treat these separately. Substitute $x = 0$ into (*) and notice that $dx = 0$ thus $(2y-2x)\underbrace{dx}_{0} + (\underbrace{3x - 4x^2}_{\text{zero}}y^{-1})dy = 0$

So $\underline{x = 0 \text{ is a sol}^{\underline{n}}}$. However, we find $y = 0$ is not a sol$^{\underline{n}}$ since $\frac{1}{y}$ is undefined. Thus we find sol$^{\underline{n}}$'s $\boxed{x^2y^3 - 2x^3y^2 = k \text{ and } x = 0}$

We've seen that integrating factors can be of the form $\mu = \exp(\int P\,dx)$ or $\mu = x^m y^n$. Are there other possibilities? If so, what general patterns can we look out for? Let's begin with a generic DEq$^{\underline{n}}$ and study the equations that will be needed for an integrating factor to be successful.

$$\boxed{M\,dx + N\,dy = 0} \quad (*)$$

Multiply by $\mu$,

$$\boxed{\mu M\,dx + \mu N\,dy = 0} \quad (**)$$

"Success" is measured by whether or not $(**)$ is exact. We need that:

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N]$$

In general, $\mu, M, N$ are assumed to be functions of $x$ and $y$ thus the product rule yields,

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$\Rightarrow M \frac{\partial \mu}{\partial y} - N \frac{\partial \mu}{\partial x} = \mu \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

The text points out that if $\frac{\partial \mu}{\partial y} = 0$ then

$$\frac{\partial \mu}{\partial x} = \frac{d\mu}{dx} = \frac{\mu(\partial M/\partial y - \partial N/\partial x)}{N} \quad (\star)$$

then we can separate variables ($\mu$ & $x$) and integrate to find $\mu$. I agree. Then the text says this reverses. If $\frac{\partial M/\partial y - \partial N/\partial x}{N}$ does not depend on $y$ then they claim the same eq$^{\underline{n}}$ above results and $\mu(x) = \exp\left[\int \frac{\partial M/\partial y - \partial N/\partial x}{N}\,dx\right]$.

Remark: I cannot quickly verify the claim of the text that " $\dfrac{\partial M/\partial y - \partial N/\partial x}{N}$ " independent of $y$ implies "

$\dfrac{d\mu}{dx} = \left(\dfrac{\partial M/\partial y - \partial N/\partial x}{N}\right)\mu$ ". This is essentially the claim of Theorem 3 on pg. 70. If you can prove it then I'd be interested. (or reword the text to make me convinced)

**Th$^{\underline{m}}$(3).** If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on $x$, then $\mu(x) = \exp\left[\int \dfrac{\partial M/\partial y - \partial N/\partial x}{N}\,dx\right]$ is an integrating factor for $M\,dx + N\,dy = 0$. Likewise if $(\partial N/\partial x - \partial M/\partial y)/M$ is continuous and depends only on $y$, then $\mu(y) = \exp\left[\int \dfrac{\partial N/\partial x - \partial M/\partial y}{M}\,dy\right]$ is an integrating factor for $M\,dx + N\,dy = 0$

Pf: left for reader. Of course, the method merely guides us to select an appropriate $\int$-factor, so a lack of proof here is not such a big deal.

Remark: §2.5 # 7, 9, 11 are solved for your convenience in the Practice Homework solutions. All three of those illustrate Th$^{\underline{m}}$(3).

$\boxed{E28}$ $\quad xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0 \qquad$ (Ex. 4 in §2.4 of Zill)

this has $M = xy$ and $N = 2x^2 + 3y^2 - 20$. Notice

$$\frac{\partial N/\partial x - \partial M/\partial y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y} \quad \text{is independent of } x$$

Thus, by Th$^{\underline{m}}$(3), we can find an integrating factor of the form,

$$\mu = \exp\left[\int \tfrac{3}{y}\,dy\right] = \exp\left[3\ln|y|\right] = \exp \ln(|y|^3) = |y|^3$$

Assume $y > 0$ for brevity here, multiply by $\mu = y^3$,

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$$

This has sol$^{\underline{n}}$s $\boxed{F(x,y) = \tfrac{1}{2}x^2y^4 + \tfrac{1}{2}y^6 - 5y^4 = k}$

in addition we can see $\boxed{y = 0}$ is also a sol$^{\underline{n}}$. Actually this is covered by the case $k = 0$, but in principle we should always check for sol$^{\underline{n}}$ missed from $\mu = 0$. ? more on this ?

# When are two differential equations the same?

(I) Is $x\,dx + x\,dy = 0$ the same as $dx + dy = 0$?

(II) Is $\frac{dy}{dx} = \frac{1}{g(x,y)}$ same as $g(x,y)dy = dx$?

(III) Is $\frac{dy}{dx} = y$ same as $\frac{dy}{y} = dx$?

Answer: no. these are actually distinct DEq$^{n}$'s in each case above. When we divide (I) by $x$ we may lose the $x=0$ sol$^{n}$. Notice that $dx + dy = 0$ has sol$^{n}$ $F(x,y) = x + y = k$ and clearly $x = 0$ is not included in that family of curves. Likewise in (II) we'll find sol$^{n}$'s that allow $g(x,y) = 0$ for $g(x,y)dy = dx$ whereas a sol$^{n}$ to $\frac{dy}{dx} = \frac{1}{g(x,y)}$ cannot have $g(x,y) = 0$ at any point in the domain of the sol$^{n}$. Finally, for (III) notice that $y = 0$ is clearly a sol$^{n}$ to $\frac{dy}{dx} = y$ whereas $y = 0$ makes $\frac{dy}{y} = dx$ undefined.

Moral of Story: when we modify a given DEq$^{n}$ we should keep in mind that we may be adding or deleting solutions. Multiplication by an integrating factor is a good example, when $\mu = 0$ for some sol$^{n}$ then that sol$^{n}$ might be missing from the sol$^{n}$ of the exact eq$^{n}$ $\mu M\,dx + \mu N\,dy = 0$.
It is interesting this did not happen in §2.3 since our integrating factor $\mu = \exp\left[\int P\,dx\right]$ was nonzero because the exponential is never zero.

## Solutions Involving Substitutions:

The idea is simple: take the given DEq$^{\underline{n}}$ and change dependent and/or independent variables such that the transformed DEq$^{\underline{n}}$ is solvable. Then transform back to the original variables to solve given DEq$^{\underline{n}}$.
The general implementation is tricky: obvious question: which variables should we use to substitute? The text gives several classes of problems. For example,

(I) $\dfrac{dy}{dx} = G(y/x)$ is called homogeneous, Use substitution $v = y/x$.

(II) $\dfrac{dy}{dx} = G(ax+by)$ is apparently nameless, Use substitution $z = ax + by$

(III) $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$ is a Bernoulli eq$^{\underline{n}}$, Use substitution $v = y^{1-n}$

(IV) $\underbrace{(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0}$

equation with linear coefficients.

If $a_1b_2 \neq a_2b_1$ then use substitution $\begin{cases} x = u + h \\ y = v + k \end{cases}$
such that the DEq$^{\underline{n}}$ becomes
$(a_1u + b_1v)\,du + (a_2u + b_2v)\,dv$
then this becomes an eq$^{\underline{n}}$ of the form $\dfrac{dv}{du} = G(v/u)$
which is homogeneous. Then can back-track to given DEq$^{\underline{n}}$.

The text goes over these since they are commonly found in many applications. However, the text makes no effort to explain why these substitutions are natural. Of course they <u>work</u> but this is less than satisfying in my view. I'll work a few problems then we'll return to the question of motivation from the lens of <u>symmetry</u>

$\boxed{E29}$ Solve $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$. Note $\dfrac{dy}{dx} = \dfrac{1 + y/x}{1 - y/x} = G(y/x)$
where $G(v) = \dfrac{1+v}{1-v}$. Let $v = \dfrac{y}{x}$ then $y = xv$ hence

$$\frac{dy}{dx} = v + \frac{dv}{dx} \Rightarrow v + \frac{dv}{dx} = \frac{1+v}{1-v} \Rightarrow \frac{dv}{dx} = \frac{1+v}{1-v} - v$$

continued ↷

we transformed the DEq$^{n}$ $\frac{dy}{dx} = \frac{x+y}{x-y}$ into

$$\frac{dv}{dx} = \frac{1+v}{1-v} - v \Rightarrow \frac{1+v-v+v^2}{1-v} = \frac{1+v^2}{1-v} = \frac{dv}{dx}$$

$$\Rightarrow \left(\frac{1-v}{1+v^2}\right)dv = dx$$

$$\Rightarrow \int\left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = dx$$

$$\Rightarrow \tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = x + C$$

$$\Rightarrow \boxed{\tan^{-1}\left(\frac{y}{x}\right) - \ln\sqrt{1+(y/x)^2} = x + C}$$

$\boxed{E30}$ Consider $(x+y+3)dx + (2x-y)dy = 0$. Following text we seek $h, k$ such that $u = x+h$ and $v = y+k$ and,

$$x+y+3 = u+v$$
$$2x-y = 2u-v$$

This means we want

$$x+y+3 = x+h+y+k \Rightarrow 3 = h+k$$
$$2x-y = 2x+2h-y-k \Rightarrow 0 = 2h-k$$

$$k = 2h \qquad 3 = h+2h$$

Thus, $h=1$ and $k=2$. We try the change of variables,

$$u = x+1 \Rightarrow (x+y+3)dx = (u+v)du$$
$$v = y+2 \Rightarrow (2x-y)dy = (2u-v)dv$$

We are now faced with solving,

$$(u+v)du + (2u-v)dv = 0$$

$$\frac{dv}{du} = \frac{u+v}{2u-v} = \frac{1+v/u}{2-v/u} \quad \text{(homogeneous)}$$

Let's use $w = v/u$ thus $uw = v$ and $w + \frac{dw}{du} = \frac{dv}{du}$ hence,

$$w + \frac{dw}{du} = \frac{1+w}{2-w} \Rightarrow \frac{dw}{du} = \frac{1+w}{2-w} - w = \frac{1+w-2w+w^2}{2-w}$$

Thus $\left(\frac{2-w}{w^2-w+1}\right)dw = du$. This integration requires a little thought, complete square to find $\frac{2-w}{(w-1/2)^2+3/4} = \frac{-(w-1/2)+7/4}{(w-1/2)^2+3/4}$ hence,

$$\int \frac{-(w-1/2)+7/4}{\underbrace{(w-1/2)^2+3/4}}dw = -\frac{1}{2}\ln\left((w-1/2)^2+3/4\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(w-1/2)\right) + C$$

$$w - 1/2 = \frac{\sqrt{3}}{2}\tan\theta$$

E30 Continued:

We've calculated that

$$u = \frac{-1}{2}\ln\left((w - 1/2)^2 + 3/4\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(w - 1/2)\right) + c$$

Recall that $w = v/u$ and $u = x+1$ and $v = y+2$.
We substitute back in $x, y$ we find the $sol^n$

$$\boxed{x+1 = \frac{-1}{2}\ln\left[\left(\frac{y+2}{x+1} - \frac{1}{2}\right)^2 + \frac{3}{4}\right] + \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(\frac{y+2}{x+1} - \frac{1}{2}\right)\right] + C}$$

I think it's fair to say that it is less than obvious that the above is a $sol^n$ to $\frac{dy}{dx} = \frac{x+y+3}{y-2x}$.

E31 Solve the Bernoulli $DEq^n$ for $x > 0$

$$\frac{dy}{dx} + \frac{y}{x} = x^2y^2$$

we substitute $v = y^{1-2} = \frac{1}{y}$

note $\frac{dv}{dx} = \frac{d}{dx}\left[\frac{1}{y}\right] = \frac{-1}{y^2}\frac{dy}{dx}$

We have $y = 1/v$ and $dy/dx = -y^2\, dv/dx = \frac{-1}{v^2}\frac{dv}{dx}$

$$\frac{-1}{v^2}\frac{dv}{dx} + \frac{1}{xv} = \frac{x^2}{v^2}$$

$$\frac{dv}{dx} = \left(\frac{x^2}{v^2} - \frac{1}{xv}\right)(-v^2) = \left(\frac{v}{x} - x^2\right)$$

$$\Rightarrow \frac{dv}{dx} - \frac{1}{x}v = -x^2$$ (linear $eq^n$ can use $\int$-factor method)

$$\mu = \exp\left(\int -\frac{1}{x}\,dx\right) = \exp(-\ln|x|) = \frac{1}{|x|} = \frac{1}{x}$$ (assume $x > 0$)

$$\underbrace{\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v}_{\frac{d}{dx}\left(\frac{1}{x}v\right)} = -x$$

$$\frac{d}{dx}\left(\frac{1}{x}v\right) = -x \Rightarrow \frac{v}{x} = -\frac{x^2}{2} + C \Rightarrow \underline{v = cx - \frac{x^3}{2}}.$$

Therefore, $\boxed{y = \frac{1}{cx - x^3/2}}$