SUPERPOSITION PRINCIPLE:

Given a linear DEq^{n} $L[y]=0$ if we have solutions to corresponding nonhomogeneous problems,

$$L[y_{p_1}] = g_1 \quad \text{and} \quad L[y_{p_2}] = g_2$$

Then we can solve $L[y] = a g_1 + b g_2$ with the sol^{n} $y = a\,y_{p_1} + b\,y_{p_2}$, assuming a, b constants

$\boxed{E85}$ We studied $L = D^2+9$ in $\boxed{E71}$ and $\boxed{E80}$ on pgs. 75 and 80. We found that $L[\frac{1}{8}\sin(x)] = \sin(x)$ in $\boxed{E71}$ then we found that $L[e^{2x}(\frac{3}{40}\sin(x) - \frac{1}{40}\cos(x))] = e^{2x}\sin(x)$.

Given all these previous results solve

$$\boxed{y'' + 9y = 7\sin(x) + 40e^{2x}\sin(x)} \quad (*)$$

Using superposition we find the general sol^{n} to $(*)$

$$\boxed{y = c_1\cos(3x) + c_2\sin(3x) + \underbrace{\tfrac{7}{8}\sin(x)}_{7y_{p_1}} + \underbrace{e^{2x}(3\sin(x) - \cos(x))}_{40\,y_{p_2}}}$$

Where I have made the identifications

$$g_1(x) = \sin(x) \quad \text{with} \quad y_{p_1} = \tfrac{1}{8}\sin(x) \qquad (L[y_{p_1}] = g_1)$$

$$g_2(x) = e^{2x}\sin(x) \quad \text{with} \quad y_{p_2} = \tfrac{1}{40}e^{2x}(3\sin(x) - \cos(x)) \qquad (L[y_{p_2}] = g_2)$$

PHYSICAL INTERPRETATION AND SIGNIFICANCE

- NEWTON'S 2ND LAW: $mx'' = \sum_{\text{all forces}} F_i = F_1 + F_2 + \cdots + F_n$

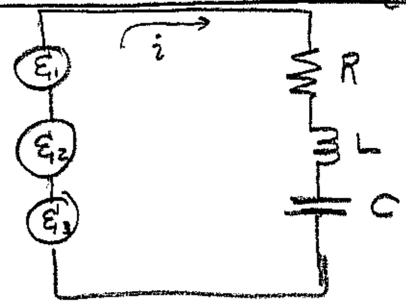
  if n-forces act on m then the equation of motion will have the form

$$x(t) = \underbrace{x_h(t)}_{\substack{\text{response} \\ \text{to initial} \\ \text{conditions}}} + \underbrace{x_{p_1}(t)}_{\substack{\text{consequence} \\ \text{of force} \\ F_1}} + x_{p_2}(t) + \cdots + \underbrace{x_{p_n}(t)}_{\substack{\text{how } F_n \\ \text{influences} \\ \text{motion}}}$$

The net-motion is a sum of all the constituent influences. A <u>nonlinear</u> system need not be the sum of it's parts.

More on Significance of Superposition

- RLC Circuit with Voltage sources $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ has



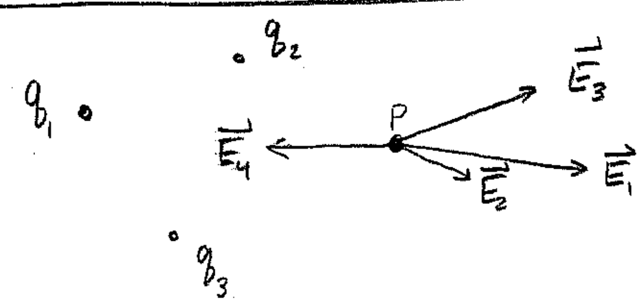
$$iR + L\frac{di}{dt} + \frac{Q}{C} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

In terms of charge, $i = \frac{dQ}{dt}$ we have

$$LQ'' + RQ' + \frac{1}{C}Q = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

Again this is a linear DEq$^{\underline{n}}$ thus the influence of $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ can be calculated separately then combined. Conversely, from an experimental perspective, we can remove $\mathcal{E}_1$ and know what to expect.

- Electric Field due to a collection of Point Charges:



$\vec{\mathscr{r}}_i = \vec{r} - \vec{r}_i$ where q_i is positioned at $\vec{r}_i$ and we use $\vec{r}$ for position of P.

At the point P the net electric field is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 \quad , \text{ where } \quad \vec{E}_i = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathscr{r}}_i}{\mathscr{r}_i^2}$$

If a charged mass Q is placed at P then the equation of motion for the mass m will be governed by:

$$m\frac{d^2\vec{r}}{dt^2} = Q\vec{E} = Q\vec{E}_1 + Q\vec{E}_2 + Q\vec{E}_3 + Q\vec{E}_4$$

If we assume q_1, q_2, q_3, q_4 are immovable then this is a linear system. The net motion of m is a superposition of the motion caused by q_1, q_2, q_3 and q_4. Technically this is outside the scope of this course since we don't treat how to solve systems of DEq$^{\underline{n}}$'s. Anyhow, linear systems and the superposition principle go hand-in-hand.

$\boxed{E86}$ Solve $\boxed{y''+4y = x^3+x^2+2+\tan(2x)}$. From $\boxed{E84}$, $y_1 = -\frac{1}{4}\cos(2x)\ln|\sec 2x+\tan 2x|$ is a particular sol^n for $y''+4y = \tan(2x)$. Find y_{p_2} a particular sol^n for $y''+4y = x^3+x^2+2$ then we'll be able to construct $y_p = y_{p_1}+y_{p_2}$ the particular sol^n for $\boxed{y''+4y = x^3+x^2+2+\tan(2x)}$ —(*). Let's find Y_{p_2} as we already have Y_{p_1}. Notice we can use undet. coeff and there is no over lap as $Y_1 = \cos 2x$ & $Y_2 = \sin 2x$.

$$Y_{p_2} = Ax^3+Bx^2+Cx+D$$

$$Y_{p_2}' = 3Ax^2+2Bx+C$$

$$Y_{p_2}'' = 6A\cancel{x}+2B$$

Now substitute into $y_{p_2}''+4y_{p_2} = x^3+x^2+2$,

$$(6Ax+2B)+4Ax^3+4Bx^2+4Cx+4D = x^3+x^2+2$$

$$4A = 1 \longrightarrow A = 1/4$$

$$4B = 1 \longrightarrow B = 1/4$$

$$6A+4C = 0 \rightarrow C = -\frac{1}{4}(6A) = -\frac{3}{2}\left(\frac{1}{4}\right) = -\frac{3}{8} = C$$

$$2B+4D = 2 \rightarrow D = \frac{1}{4}(2-2B) = \frac{1}{4}\left(\frac{3}{2}\right) = \frac{3}{8} = D$$

Therefore, the general sol^n of (*) (using superposition)

$$\boxed{y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\ln|\sec 2x+\tan 2x| + \frac{1}{4}x^3+\frac{1}{4}x^2-\frac{3}{8}x+\frac{3}{8}}$$