Beyond Constant Coefficients; Variable Coefficients etc...: (§4.7)

You may have noticed the theory discussed on 61 → 66 largely applies to the variable coefficient problem,

$$\boxed{a(x)\,y''(x) + b(x)\,y'(x) + c(x)\,y(x) = g(x)} \quad (*)$$

We still have $L[y] = (aD^2 + bD + c)[y] = g$. However, since $Da, Db, Dc \neq 0$ we can no longer factor L as we discussed on 67 → 74. It is still the case that $(*)$ has a <u>unique</u> sol$^{\underline{n}}$ given some initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_1$, however the domain of the sol$^{\underline{n}}$ will not necessarily extend to $\mathbb{R}$. We can only hope for solutions to extend over domains where a, b, c, g are continuous. When a sol$^{\underline{n}}$ of $(*)$ exists it still has the general form

$$\boxed{y = c_1 y_1 + c_2 y_2 + y_p \quad \text{where } L[y_1] = L[y_2] = 0}$$

Variation of parameters also works for $(*)$. The main difference is we need new methods to find y_1 and y_2.

<u>Cauchy Euler Problems:</u>

Let a, b, c be constants. We can solve $\boxed{at^2y'' + bty' + cy = 0}$ (CE) without too much trouble. Assume the sol$^{\underline{n}}$ has the form $y = t^r$ then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$ hence we need

$$at^2 r(r-1)t^{r-2} + btrt^{r-1} + ct^r = 0$$

$$(ar(r-1) + br + c)t^r = 0, \quad \text{assume } t > 0.$$

Our <u>guess</u> of $y = t^r$ will solve (CE) if the constant r satisfies the $\boxed{\underline{\text{characteristic}} \text{ equation } ar(r-1) + br + c = 0}$

(text calls this the indicial eq$^{\underline{n}}$ in §8.5)

$\boxed{\text{E87}}$ Solve $t^2y'' + 6ty' + 6y = 0$. Use calculation above, we need $r(r-1) + 6r + 6 = r^2 + 5r + 6 = (r+3)(r+2) = 0 \therefore r = -3, -2$.
The sol$^{\underline{n}}$ has the form $\underline{y = c_1 t^{-3} + c_2 t^{-2}}$.

E88 Solve $t^2y'' + ty' + y = 0$. Look for sol$^{\underline{n}}$ $y = t^r$. The Char. Eq$^{\underline{n}}$ reads $r(r-1) + r + 1 = r^2 + 1 = 0 \therefore \underline{r = \pm i}$.
We find sol$^{\underline{n}}$'s $y = t^i$ and $y = t^{-i}$, notice these are <u>complex</u>-valued sol$^{\underline{n}}$'s. No problem, we have real sol$^{\underline{n}}$'s $y_1 = \text{Re}(t^i)$ and $y_2 = \text{Im}(t^i)$ hidden inside $y = t^i$.
Observe that:

$$t^i = e^{\ln(t^i)} = e^{i\ln(t)} = \cos(\ln(t)) + i\sin(\ln(t)).$$

Therefore, assuming $t > 0$,

$$y_1 = \text{Re}(t^i) = \cos(\ln(t))$$
$$y_2 = \text{Im}(t^i) = \sin(\ln(t))$$

The general sol$^{\underline{n}}$ is $\boxed{y = c_1\cos(\ln(t)) + c_2\sin(\ln(t))}$

E89 Solve $y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$. Again we search for sol$^{\underline{n}}$'s of the form $y = t^r$ by solving Char. Eq$^{\underline{n}}$ $r(r-1) - r + 1 = 0$ which yields $(r-1)^2 = 0$ and so $r = 1$. Hence $\underline{y_1 = t}$.
is a sol$^{\underline{n}}$. <u>BUT</u>, what is y_2? We know the general sol$^{\underline{n}}$ will have the form $y = c_1t + c_2y_2$.
<u>We are unable to solve this w/o a way to find y_2!</u>
Fortunately, <u>reduction</u> <u>of</u> <u>order</u> gives us a technique to find a 2<u>nd</u> linearly independent sol$^{\underline{n}}$ y_2 if we already have y_1. (we'll prove this formula on 96 but for now let's use it)

$$y_2 = y_1\int\frac{e^{-\int P\,dt}}{y_1^2}\,dt = t\int\frac{e^{\int\frac{1}{t}dt}}{t^2}\,dt = t\int\frac{e^{\ln(t)}}{t^2}\,dt = t\int\frac{dt}{t}$$

Thus we find $y_2 = t\ln(t)$ and $\underbrace{\boxed{y = c_1t + c_2t\ln(t)}}_{\text{general solution.}}$

> Remark: this is not an isolated occurence. If we get double root of r then the Cauchy-Euler Equation has sol$^{\underline{n}}$ $y = c_1t^r + c_2t^r\ln(t)$ for $t > 0$

Reduction of Order: How to find y_2 from y_1:

Suppose we have a nonzero sol$^{\underline{n}}$ y_1 of $y'' + py' + qy = 0$ where p, q could be functions. Our goal is to find a second linearly independent sol$^{\underline{n}}$ y_2. Since y_1 & y_2 need to be LI if we have $y_2/y_1 = v$ then v cannot be constant. (otherwise y_1, y_2 would be linearly dependent). Consider then,

$$y_2 = v y_1 \quad \text{where} \quad y_1'' + py_1' + qy_1 = 0 \quad (\text{since } y_1 \text{ a sol}^{\underline{n}})$$

Differentiate,

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1''$$

We require $y_2'' + py_2' + qy_2 = 0$,

$$v''y_1 + 2v'y_1' + \underline{vy_1''} + p(v'y_1 + \underline{vy_1'}) + \underline{qvy_1} = 0$$

$$\underbrace{v(y_1'' + py_1' + qy_1)}_{\substack{\text{Zero since} \\ y_1 \text{ is a sol}^{\underline{n}}}} + v''y_1 + 2v'y_1' + pv'y_1 = 0$$

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

Let $w = v'$ then $v'' = w'$ and we find $y_1 \dfrac{dw}{dt} + (2y_1' + py_1)w = 0$

$$\frac{dw}{dt} + \left(\frac{2y_1'}{y_1} + p\right)w = 0$$

We can use integrating factor technique,

$$\mu = \exp\left(\int\left(\frac{2}{y_1}\frac{dy_1}{dt} + p\right)dt\right) = \exp\left(\int \frac{2dy_1}{y_1}\right)\exp\left(\int p\,dt\right)$$

$$= \exp(2\ln|y_1|)\exp\left(\int p\,dt\right)$$

$$= \exp(\ln|y_1|^2)\exp\left(\int p\,dt\right)$$

$$= y_1^2 e^{\int p\,dt}$$

Multiplying by integrating factor & use product rule to find that,

$$\frac{d}{dt}(\mu w) = 0 \implies y_1^2 e^{\int p\,dt} w = k = \text{constant}$$

Continuing derivation of reduction of order formula:

Recall we have $W = \frac{dv}{dt}$ where $y_2 = v y_1$ and y_1 was given whereas y_2 we wish to calculate from y_1 somehow. Furthermore, we just derived that

$$y_1^2 e^{\int P dt} W = k$$

$$\Rightarrow \frac{dv}{dt} = \frac{k}{y_1^2 e^{\int P dt}}$$

$$\Rightarrow v = \int \frac{k e^{-\int P dt}}{y_1^2} dt$$

$$\Rightarrow \frac{y_2}{y_1} = \int \frac{\cancel{k} e^{-\int P dt}}{y_1^2} dt$$

$$\therefore \boxed{y_2 = y_1 \int \frac{\exp(-\int P dt)}{y_1^2} dt}$$

We can choose $k = 1$ for convenience, any other nonzero value would also yield a linearly independent function from y_1.

LOOKING BACK:

The reduction of order formula will derive the 2^{nd} solution in the repeated root case for $ay'' + by' + cy = 0$,

E90 Consider $y'' - 2y' + y = 0$

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \quad \therefore \ \underline{y_1 = e^t}.$$

In this case $P = -2$ thus $-\int P dt = \int 2 dt = 2t$ hence,

$$y_2 = e^t \int \frac{e^{2t}}{(e^t)^2} dt = e^t \int dt = te^t \quad \therefore \ \underline{y_2 = te^t}.$$

- This gives us a way of explaining where the mysterious t in the double root sol$^{\underline{n}}$ came from. You can also derive the t from the matrix exponential applied to the complementary matrix system or Laplace Transforms. The method here is probably the most straight-forward justification.

ANOTHER APPROACH TO CAUCHY EULER PROBLEM:

Consider $at^2 y'' + bty' + cy = 0$

Try the change of variables $t = e^x$

That is $x = \ln(t)$. Notice

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx} = \underline{\frac{1}{t}\frac{dy}{dx} = \frac{dy}{dt}} \quad .(i)$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{1}{t}\frac{dy}{dx}\right] = \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \underline{\frac{d^2y}{dt^2} = \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}} \quad .(ii)$$

Substitute (i) & (ii) into the Cauchy Euler Problem,

$$at^2\left(\frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}\right) + bt\left(\frac{1}{t}\frac{dy}{dx}\right) + c\,y(x) = 0$$

$$\Rightarrow \underline{a\frac{d^2y}{dx^2} + (b-a)\frac{dy}{dx} + c\,y(x) = 0}$$

We'll find solutions of the form

I.) $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \xrightarrow{x = \ln(t)} y = c_1 t^{\lambda_1} + c_2 t^{\lambda_2}$

II.) $y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x} \xrightarrow{x = \ln(t)} y = c_1 t^{\lambda} + c_2 t^{\lambda}\ln(t)$

III.) $y(x) = c_1 e^{\alpha x}\sin\beta x + c_2 e^{\alpha x}\cos\beta x \xrightarrow{x = \ln(t)} y = c_1 t^{\alpha}\sin(\beta\ln(t)) +$
$+ c_2 t^{\alpha}\cos(\beta\ln(t))$

- Why is the Cauchy-Euler Problem so similar to the constant coefficient case?

- See §8.5 for the text's treatment of the Cauchy Euler Problem.

On Linear Differential Equations and Eigenfunctions

Let T be a linear operator on smooth functions on $I \subseteq \mathbb{R}$ means $T : C^\infty(I) \to C^\infty(I)$ and it satisfies

(1.) $T(cf) = cT(f)$, for all $f \in C^\infty(I)$ and $c \in \mathbb{R}$

(2.) $T(f+g) = T(f) + T(g)$, for all $f, g \in C^\infty(I)$.

$\boxed{E91}$ A constant coefficient ODE has the following form:

$$\underbrace{\frac{d^n}{dx^n}(y) + P_n \frac{d^{n-1}}{dx^{n-1}}(y) + \cdots + P_2 \frac{d}{dx}(y) + P_1 y}_{P(D)[y]} = 0$$

$$P(D)[y] = 0$$

Where $P(D) = D^n + P_n D^{n-1} + \cdots + P_2 D + P_1$. The nice feature of this smooth linear operator is that it factored completely into

$$P(D) = L_1 \circ L_2 \circ \cdots \circ L_k$$

where $k \leq n$ and $L_1, L_2, \ldots, L_k$ are linear operators. In particular $L_j = (D + \lambda_j)^{m_j}$ or $L_j = \left[(D - \alpha_j)^2 + \beta_j^2\right]^{m_j}$ for each $j = 1, 2, \ldots, k$. Fortunately if $L_j(f) = 0$ for any particular j then $P(D)(f) = 0$. This result depended strongly on the fact we can commute the factors in $P(D)$,

$$P(D) = L_1 \circ L_2 \circ \cdots \circ L_j \circ \cdots \circ L_k = L_1 \circ L_2 \circ \cdots \circ L_k \circ L_j$$

Hence $L_j(f) = 0 \Rightarrow P(D)(f) = L_1 \circ L_2 \circ \cdots \circ L_k \circ L_j(f)$

$$= L_1 \circ L_2 \circ \cdots \circ L_k(0) = 0.$$

Def$^{\underline{n}}$/ f is an eigenfunction with eigenvalue λ of the linear operator T if $T(f) = \lambda f$. Equivalently $(T - \lambda)(f) = 0$. If $(T - \lambda)^2(f) = 0$ and $(T - \lambda)(f) \neq 0$ then f is a generalized eigenfunction of order two.

$\boxed{E92}$ Let $\lambda \in \mathbb{R}$. The solutions to the constant coefficient ODEs are eigenfunctions or generalized eigenfunctions of $D = d/dx$,

$$(D-\lambda)(e^{\lambda x}) = \lambda e^{\lambda x} - \lambda e^{\lambda x} = 0. \quad \therefore\ f_1(x) = e^{\lambda x} \text{ eigenfnct.}$$
$$(D-\lambda)^2(xe^{\lambda x}) = 0 \quad \therefore\ f_2(x) = xe^{\lambda x} \text{ generalized eigenfnct. order 2.}$$
$$(D-\lambda)^3(x^2e^{\lambda x}) = 0 \quad \therefore\ f_3(x) = x^2e^{\lambda x} \text{ generalized eigenfnct. order 3.}$$

When $\lambda \in \mathbb{C}$ then we find $e^{\lambda x}, xe^{\lambda x}, \dots$ are generalized eigenfunctions of D which are complex-valued. The $\text{Re}(x^s e^{\lambda x})$ and $\text{Im}(x^s e^{\lambda x})$ are not eigenfunctions of $(D-\lambda)^{s+1}$ however they are linearly independent. Eigenfunctions with distinct eigenvalues are LI and we can show generalized eigenfunctions $e^{\lambda x}, xe^{\lambda x}, x^2e^{\lambda x}$ etc... form a LI set of functions.

Remark: for notational simplicity assume the roots of a constant coefficient ODE $L[y]=0$ are all real $\lambda_1, \lambda_2, \dots, \lambda_k$ with multiplicities $m_1, m_2, \dots, m_k$ then

$$L[y] = \underbrace{(D-\lambda_1)^{m_1}(D-\lambda_2)^{m_2}\cdots(D-\lambda_k)^{m_k}}_{\substack{\text{polynomial in } D. \\ \text{made of commuting factors}}}[y]$$

① D commutes with D
② D commutes with λ
③ λ_1 commutes with λ_2

$$(D-\lambda_1)(D-\lambda_2)(f) = (D-\lambda_1)(f'-\lambda_2 f) = f''-\lambda_2 f'-\lambda_1 f'+\lambda_1\lambda_2 f$$
$$(D-\lambda_2)(D-\lambda_1)(f) = (D-\lambda_2)(f'-\lambda_1 f) = f''-\lambda_1 f'-\lambda_2 f'+\lambda_2\lambda_1 f$$

Clearly $(D-\lambda_1)(D-\lambda_2)(f) = (D-\lambda_2)(D-\lambda_1)(f)$ for all f thus the operators commute.

OPERATORS AND THE CAUCHY – EULER PROBLEM

I'll cut straight to the point here

$$ax^2y'' + bxy' + cy = 0$$

$$\Big(\underbrace{a(xD)^2 + b(xD) + c}_{\text{polynomial in } xD}\Big)[y] = 0$$

$\Rightarrow$ can factor into smaller polynomials in xD.

$\Rightarrow$ can break down problem to finding eigenfunctions of xD instead.

Notice $y = x^r$ is an eigenfunction of xD since $xD(x^r) = x r x^{r-1} = r x^r$

$\Rightarrow$ $y = x^r$ is eigenfunction of xD with eigenvalued r.

How do we factor $(xD)^2 + b(xD) + c$? It's not as simple as the constant coefficient case. Let's work backwards to see how it goes,

$$\begin{aligned}(xD+\alpha)(xD+\beta)[f] &= (xD+\alpha)[xf' + \beta f] \\ &= xD(xf') + xD(\beta f) + \alpha x f' + \alpha\beta f \\ &= xf' + x^2 f'' + \beta x f' + \alpha x f' + \alpha\beta f \\ &= \left(x^2D^2 + (\alpha+\beta+1)xD + \alpha\beta\right)[f]\end{aligned}$$

$\boxed{\text{E93}}$ Factor $L = x^2D^2 + 4xD + 2$. Identify $\alpha = 1, \beta = 2$ thus $L = (xD+1)(xD+2)$. We can solve

$$x^2y'' + 4xy' + 2y = 0 \Rightarrow (xD+1)(xD+2)[y] = 0$$

eigenfunction $y_1 = x^{-1}$ eigenfunction $y_2 = x^{-2}$

Remark: Don't worry I'm majorly digressing for the last 4 pages.

$$\therefore \quad \boxed{y = c_1/x + c_2/x^2}$$