MATH 422:	MISSION 1: GROUPS, ACTIONS AND RINGS

Show **your** work carefully. Write on one-side of paper, it is important that the problems be done in order with labels as given here. About 3pts per problem.

Problem 1: Solve $y^3 - 7y + 6 = 0$ via Cardano's formulas (much like E2 of Lecture 1)

Problem 2: Let $Isom(\mathbb{R}^n)$ denote the set of isometries of Euclidean space. In particular, $\varphi \in Isom(\mathbb{R}^n)$ if and only if φ has $\|\varphi(P) - \varphi(Q)\| = \|P - Q\|$ for all $P, Q \in \mathbb{R}^n$.

(a.) If $T_a(x) = x + a$ defines $T_a : \mathbb{R}^n \to \mathbb{R}^n$ then show $T_a \in Isom(\mathbb{R}^n)$ for any $a \in \mathbb{R}^n$.

(b.) If $L_R(x) = Rx$ defines $L_R : \mathbb{R}^n \to \mathbb{R}^n$ where R is an **orthogonal matrix** ($R^T R = I$) then show $L_R \in Isom(\mathbb{R}^n)$.

(c.) If $G = \{T_a \circ L_R \mid a \in \mathbb{R}^n, R \in O(n, \mathbb{R})\}$ then show G is a group with respect to composition of maps and show $G \subseteq Isom(\mathbb{R}^n)$.

(d.) Prove $Isom(\mathbb{R}^n) = G$.

Problem 3: Prove there exists a subgroup of $Isom(\mathbb{R}^n)$ which is isomorphic to $\mathbb{Z}^n$.

Problem 4: Prove there exists a subgroup of $Isom(\mathbb{R}^2)$ which is isomorphic to $\mathbb{Z}_n$.

Problem 5: **Direct Product:** Let H, K be groups and consider $G = H \times K$ with the direct product $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$. Let $H_1 = H \times \{1\}$ and $K_1 = \{1\} \times K$. Prove H_1 and K_1 are normal subgroups of G.

Problem 6: **Semidirect Product:** Let K and H be groups and let $\theta : H \to aut(K)$ be a homomorphism. Write $\theta(h) = \sigma_h$ for all $h \in H$. If $G = K \times H$ is the Cartesian product, define an operation on G as follows:

$$(k_1, h_1)(k_2, h_2) = (k_1 \sigma_{h_1}(k_2), h_1 h_2)$$

for all $(k_1, h_1), (k_2, h_2) \in G$. Let $K_1 = K \times \{1\}$ and $H_1 = \{1\} \times H$. You may assume G forms a group with respect to the given operation.

(a.) $K_1, H_1 \leq G$ and $K_1 \cong K$ and $H_1 \cong H$,

(b.) $G = H_1 K_1$ and $K_1 \trianglelefteq G$ and $H_1 \cap K_1 = \{1\}$.

Problem 7: Let $G = \mathbb{R}^n \rtimes O(n, \mathbb{R})$ have multiplication defined by

$$(a_1, R_1)(a_2, R_2) = (a_1 + R_1 a_2, R_1 R_2).$$

Prove $G \cong Isom(\mathbb{R}^n)$. (this illustrates the claim that $Isom(\mathbb{R}^n)$ is the **semidirect product** of the group of translations with the orthogonal transformations. Note $O(n, \mathbb{R}) = \{R \in \mathbb{R}^{n \times n} \mid R^T R = I\}$.)

Problem 8: Denote the dihedral group $D_4 = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$. Use generalized cycle notation to express the isomorphic image of the subgroup $\langle a \rangle$ within S_{D_4}.

Problem 9: Let $G = \mathrm{GL}_n(\mathbb{R})$ act on $\mathbb{R}^n$ via matrix multiplication. Find the orbit of $v \in \mathbb{R}^n$ under this group action. Break into cases if needed.

Problem 10: (10pts) Consider the set $\mathbb{R}^{m\times n}$ and define $(P,Q)\star A = PAQ^{-1}$ for each $A \in \mathbb{R}^{m\times n}$ and $(P,Q) \in GL_m(\mathbb{R}) \times GL_n(\mathbb{R})$. Prove $\star$ defines a group action and describe how we can decide if two matrices are in the same orbit.

Problem 11: In E8 of Lecture 2 we found there were 48 symmetries of a cube. Only some of those symmetries correspond to motions which are physically possible for a rigid cube. For example, a symmetry which rips off a face and flips it over whilst leaving all the other faces intact is not an allowed **motion** of the cube. Find the number of motions for the cube.

Problem 12: Illustrate the Class Equation for A_4 by describing the conjugacy classes and counting the number of permutations in each class.

Problem 13: Show a group of order 40 cannot be simple. *Use Sylow Theorem(s).*

Problem 14: The **inner automorphisms** of a group G is the set of conjugations; we denote $\text{Inn}(G) = \{i_g : G \to G \mid i_g(x) = g^{-1}xg \text{ for all } x \in G\}$.

(a.) Prove $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$.

(b.) Prove $G/Z(G) \cong \text{Inn}(G)$.

(c.) How many elements of order two are inside $\text{Inn}(D_4)$?

Problem 15: Suppose G is a group with normal subgroups H, K such that $H \leq K$.

(a.) Prove $K/H \trianglelefteq G/H$.

(b.) Define $\Psi : G/H \to G/K$ by $\Psi(gH) = gK$ for each $gH \in G/H$. Show Ψ is well-defined and Ψ is a homomorphism with $\text{Ker}(\Psi) = K/H$

(c.) Prove $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong G/K$.

Problem 16: Suppose $H, K \leq G$ and suppose H is normal. Prove $HK \leq G$ and $HK = KH$.

Problem 17: Suppose $K \trianglelefteq G$. Show G is solvable if and only if both G/K and K are solvable.

Problem 18: Let $R = \mathbb{Z}_4[i] = \{a + ib \mid a, b \in \mathbb{Z}_4,\ i^2 = -1\}$ where addition and multiplication are defined in the expected manner. Find the group of units for R.

Problem 19: Show $R = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} \;\middle|\; x, y \in \mathbb{Z} \right\}$ forms a subring of $\mathbb{Z}^{2\times 2}$.

Problem 20: Once more let $R = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} \;\middle|\; x, y \in \mathbb{Z} \right\}$. Define $\varphi : R \to \mathbb{Z}$ by $\varphi(X) = X_{11} - X_{12}$ for each $X \in R$ and complete the following:

(a.) Show φ is a ring homomorphism.

(b.) Prove φ is a surjection.

(c.) Describe $\text{Ker}(\varphi)$.

(d.) Find a known ring to which $R/\text{Ker}(\varphi)$ is isomorphic.

(e.) Is $\text{Ker}(\varphi)$ a prime ideal?

(f.) Is $\text{Ker}(\varphi)$ a maximal ideal?

Problem 21: Suppose a commutative ring R has an element a such that $a^n = 0$ for some positive integer n. Also suppose u is a unit in R. Prove $u - a$ is a unit in R.
Hint: think about the geometric series.

Problem 22: Suppose S, T are subrings of a ring R. Prove or disprove the following:

(a.) $S \cap T$ is a subring of R

(b.) $S + T = \{s + t \mid s \in S, t \in T\}$ is a subring of R

Problem 23: Prove a surjective homomorphism from a field F onto a ring $R \neq \{0\}$ must be an isomorphism.

Problem 24: Let R be a commutative ring with $1 \in R$. If I is a prime ideal of R then prove $I[x]$ is a prime ideal of $R[x]$.

Problem 25: Let F be a field and $f(x), g(x) \in F[x]$. If only constant polynomials divide both $f(x)$ and $g(x)$ then prove there exist $a(x), b(x) \in F[x]$ for which $a(x)f(x) + b(x)g(x) = 1$. *Your solution should be based on our theorem that $F[x]$ is a PID.*

Problem 26: Let R be a commutative ring with unity. Write $a \mid b$ if there exists $r \in R$ for which $b = ra$. Show the following:

(a.) $Rab \subseteq Ra \cap Rb$,

(b.) If $Ra + Rb = R$ then $Rab = Ra \cap Rb$,

(c.) $u \in R$ is a unit if and only if $Ru = R$,

(d.) Rp is a prime ideal if and only if $p \mid ab$ implies $p \mid a$ or $p \mid b$

(e.) if R is an integral domain then $Ra = Rb$ if and only if $a = ub$ for some unit $u \in R$.

Problem 27: Let R be a ring with unity and define the the **nilradical** of R by $N(R) = \{a \in R \mid a \text{ is nilpotent}\}$.

(a.) For R commutative, show $N(R)$ is an ideal of R,

(b.) For R commutative, show $N(R/N(R)) = 0$,

(c.) Is $N(R)$ an ideal if R is not commutative ?

Problem 28: Find all monic irreducible quadratics in $\mathbb{Z}_3[x]$

Problem 29: Factor $x^5 + x^4 + 1$ into irreducible factors over $\mathbb{Z}_2[x]$.

Problem 30: Dummit and Foote, §7.2#2

Problem 31: Dummit and Foote, §7.2#3

Problem 32: Dummit and Foote, §7.2#5

Problem 33: Dummit and Foote, §7.2#7

Problem 34: Dummit and Foote, §7.3#15

Problem 35: Dummit and Foote, §9.1#7