

Show **your** work carefully. 3pts per problem. Be sure to write both my problem number as well as the number from Dummit and Foote on your solution. Thanks!

Problem 71: Dummit and Foote, §10.1#5 (submodule proof)

Problem 72: Dummit and Foote, §10.1#8 (torsion)

Problem 73: Dummit and Foote, §10.1#9 (annihilator)

Problem 74: Dummit and Foote, §10.1#22 (on the natural characterization of R-algebras)

Problem 75: Dummit and Foote, §10.2#1 (who says I never give gifts ?)

Problem 76: Dummit and Foote, §10.2#9 (fuzzy comment: commutative ring and dual space to R-module)

Problem 77: Dummit and Foote, §10.2#10 (fuzzy comment: R-module regular representation)

Problem 78: Let A_1, A_2 be R-modules and let B_1 be a submodule of A_1 whereas B_2 is a submodule of A_2 . Prove that

$$(A_1 \times A_2)/(B_1 \times B_2) \cong (A_1/B_1) \times (A_2/B_2)$$

Problem 79: Dummit and Foote, §10.3#4 (torsion module)

Problem 80: Dummit and Foote, §10.3#13 (isomorphism, looks like dual space)

Problem 81: Dummit and Foote, §10.3#15 (central idempotent)

Problem 82: Dummit and Foote, §10.4#3 (fun with $\otimes \mathbb{R}$ and \mathbb{C})

Problem 83: Dummit and Foote, §10.4#6 (fun with \otimes , field of fractions)

Problem 84: Dummit and Foote, §10.4#11 (fun with \otimes , simple tensor)

Problem 85: Dummit and Foote, §10.4#25 (fun with \otimes , isomorphism of algebras)

Problem 86: Let \mathcal{B} denote the set of bilinear mappings on the vector space V over characteristic zero field F with basis $\beta = \{e_1, \dots, e_n\}$. Let β^* be the dual basis $\{e^1, \dots, e^n\}$ defined by $e^i(e_j) = \delta_{ij}$. For any $\alpha, \beta \in V^*$ we define

$$(\alpha \otimes \beta)(x, y) = \alpha(x)\beta(y)$$

for all $x, y \in V$. Furthermore, $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$

(a.) If $\mathcal{A} = \{b \in \mathcal{B} \mid b(x, y) = -b(y, x) \text{ for all } x, y \in V\}$ then find the basis and calculate the dimension of \mathcal{A}

(b.) Show \mathcal{A} is isomorphic to $\Lambda^2(V)$ as defined in Dummit and Foote §11.5

Problem 87: Dummit and Foote, §12.1#2 (meaning of rank in module over integral domain)

Problem 88: Dummit and Foote, §12.1#3 (rank interacting with \oplus)

Problem 89: Dummit and Foote, §12.1#4 (rank interacting with quotient of modules)

Problem 90: Dummit and Foote, §12.1#6 (on constructing an example of rank 1 module which is not a free R -module)

Remark: exercises §12.1#15–20 develop the Smith Normal form calculation for the relation matrix which defines the module structure over R . We will apply the results to some examples in class as well as corresponding homework. This justifies the row and column operations I shared (and will soon share for context outside $R = F[x]$, as in we look at $R = \mathbb{Z}$ examples etc...)

Problem 91: Suppose A is a matrix for which $(x^2 - 2)^2(x - 3)^3$ is the characteristic polynomial.

- (a.) Find the possible invariant factors for the $\mathbb{Q}[x]$ -module of A and for each case also list the elementary divisors in $\mathbb{Q}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A
- (b.) Find the possible invariant factors for the $\mathbb{R}[x]$ -module of A and for each case also list the elementary divisors in $\mathbb{R}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A

Problem 92: Suppose A is a matrix for which $(x^3 - 2)^2(x - 3)^2$ is the characteristic polynomial.

- (a.) Find the possible invariant factors for the $\mathbb{Q}[x]$ -module of A and for each case also list the elementary divisors in $\mathbb{Q}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A
- (b.) Find the possible invariant factors for the $\mathbb{R}[x]$ -module of A and for each case also list the elementary divisors in $\mathbb{R}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A
- (c.) Find the possible invariant factors for the $\mathbb{C}[x]$ -module of A and for each case also list the elementary divisors in $\mathbb{C}[x]$ and give the rational canonical form of A and where possible give the Jordan form of A

Problem 93: Suppose $G = \{(a, b, c) \in \mathbb{Z}^3 \mid 2a + 4b + 4c = 0, 6b + 6c = 0, 8c = 0\}$. Find the decomposition of G as a \mathbb{Z} -module in invariant factor form as well as elementary divisor form.

Problem 94: Dummit and Foote, §12.3#17 (every square matrix similar to its transpose)

Problem 95: Dummit and Foote, §12.3#22 (diagonalizability of matrix over \mathbb{C})

Problem 96: Let V be a vector space of dimension n over the field F . Suppose E is an extension field of F for which $\delta = \{w_1, \dots, w_m\}$ serves as an F -basis of E . Let $\beta = \{v_1, \dots, v_n\}$ form an F -basis for V . We wish to study the E -vector space $E \otimes_F V$ in this problem. Notice,

$$c(a \otimes v) = (ca) \otimes v$$

defines the scalar multiplication of E on $E \otimes_F V$. Prove the following:

- (a.) If $\gamma = 1 \otimes \beta = \{1 \otimes v_1, \dots, 1 \otimes v_n\}$ then γ is an E -basis for $E \otimes V$. Notice by the theory developed in Dummit and Foote we know $\{w_i \otimes v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ serves as an F -basis for $E \otimes_F V$.
- (b.) Suppose $T : V \rightarrow V$ is a linear transformation. Let $T_E : E \otimes_F V \rightarrow E \otimes_F V$ be defined by $T(c \otimes x) = c \otimes T(x)$ for all $c \otimes x \in E \otimes_F V$. Show T_E is a linear transformation on $E \otimes_F V$ and show that $[T]_{\beta, \beta}$ and $[T_E]_{\gamma, \gamma}$ are related.

Problem 97: The complexification of a $2k$ -dimensional vector space V over \mathbb{R} is given by $V_{\mathbb{C}} = \mathbb{C} \otimes V$. We suppress explicit \otimes notation via the following conventions: for $x \in V$ and $a + ib \in \mathbb{C}$,

$$1 \otimes x = x \quad \& \quad i \otimes x = ix$$

hence $(a + ib) \otimes x = (a + ib)x = ax + ibx$ and $T_{\mathbb{C}}(x + iy) = T(x) + iT(y)$ for all $x, y \in V$ defines $T_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$.

(a.) Suppose $\Gamma = \{v_1, \dots, v_k\}$ is a k -chain with eigenvalue $\lambda = \alpha + i\beta$ for $T_{\mathbb{C}}$. Let $v_j = a_j + ib_j$ for $j = 1, \dots, k$ where $a_j, b_j \in V$. Prove $\Upsilon = \{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ is \mathbb{R} -linearly independent given that $\beta \neq 0$.

(b.) Extend Γ to a basis δ for $V_{\mathbb{C}}$ such that $[T_{\mathbb{C}}]_{\delta, \delta}$ is a Jordan form

(c.) Show $[T]_{\Upsilon, \Upsilon} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \otimes I_k + I_2 \otimes N_k$

Problem 98: Consider a real associative unital algebra \mathcal{A} of finite dimension with multiplication \star . If $\mathcal{R}_{\mathcal{A}}$ is the set of linear transformations on \mathcal{A} for which $T(x \star y) = T(x) \star y$ for all $x, y \in \mathcal{A}$ then prove $\mathcal{R}_{\mathcal{A}}$ is real algebra which is isomorphic to \mathcal{A} .

Problem 99: Consider a real associative unital algebra \mathcal{A} of finite dimension n . Prove there exists a matrix subalgebra of $\mathbb{R}^{n \times n}$ which is isomorphic to \mathcal{A} .

Problem 100: Consider a real associative unital algebra \mathcal{A} of finite dimension n . Prove every element of \mathcal{A} is either a zero, a unit, or a zero-divisor.

Problem 101: Find the matrix regular representation for each real associative algebra given below:

- (a.) $\mathbb{C} \times \mathbb{C}$
- (b.) $\mathcal{H} \times \mathbb{C}$
- (c.) $\mathbb{R}^{2 \times 2}$

Find the group of units for each algebra above.

Problem 102: Let \mathcal{A} and \mathcal{B} be real associative algebras of finite dimension and suppose $M_{\beta_1}(\mathcal{A})$ and $M_{\beta_2}(\mathcal{B})$ are the representations of \mathcal{A} and \mathcal{B} with respect to bases β_1, β_2 . Show that $\beta = (\beta_1 \times 0) \cup (0 \times \beta_2)$ basis for $\mathcal{A} \times \mathcal{B}$ has

$$M_{\beta}(\mathcal{A} \times \mathcal{B}) = M_{\beta_1}(\mathcal{A}) \oplus M_{\beta_2}(\mathcal{B}).$$

Problem 103: The algebras below are all 9-dimensional over \mathbb{R} . Determine which algebras are isomorphic and either provide the isomorphism or give evidence that no isomorphism can be found.

$$\mathbb{R}^{3 \times 3} \quad \mathcal{H}_9 = \mathbb{R} \oplus j\mathbb{R} \oplus \dots \oplus j^8\mathbb{R} \quad \mathcal{C}_9 = \mathbb{R} \oplus k\mathbb{R} \oplus \dots \oplus k^8\mathbb{R} \quad \Gamma_9 = \mathbb{R} \oplus \varepsilon\mathbb{R} \oplus \dots \oplus \varepsilon^8\mathbb{R}$$

where $j^9 = 1$ and $k^9 = -1$ and $\varepsilon^9 = 0$.

Problem 104: The algebras below are all 4-dimensional over \mathbb{R} . Determine which algebras are isomorphic and either provide the isomorphism or give evidence that no isomorphism can be found.

- (i.) the matrix algebra $\mathbb{R}^{2 \times 2}$

- (ii.) $\mathcal{H} \times \mathcal{H}$ where \mathcal{H} is the hyperbolic numbers
- (iii.) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^4$, the direct product algebra
- (iv.) $\mathbb{R} \times \mathcal{H}_3$ the direct product of \mathbb{R} with the 3-hyperbolic numbers
- (v.) $\mathbb{C} \times \mathcal{H}$