Sol'ns to Select Problems from Elementary Differential Geometry
Revised 2nd Edition, Barret O'Neil

pg. 5

(1.) $f = x^2 y$ and $g = y\sin z$ functs on $\mathbb{R}^3$. Evaluate expressions,

(a.) $fg^2 = (x^2y)(y\sin z)^2 = \underline{x^2y^3\sin^2 z}$.

(b.) $\frac{\partial f}{\partial x}g + \frac{\partial g}{\partial y}f = (2xy)(y\sin z) + (\sin z)x^2y = \underline{(2xy^2 + x^2y)\sin z}$.

(c.) $\frac{\partial^2(fg)}{\partial y\,\partial z} = \frac{\partial}{\partial y}\frac{\partial}{\partial z}\left(x^2y^2\sin z\right) = \frac{\partial}{\partial y}\left[x^2y^2\cos z\right] = \underline{2x^2y\cos z}$.

(d.) $\frac{\partial}{\partial y}\left(\sin(f)\right) = \cos(f)\frac{\partial f}{\partial y} = \underline{x^2\cos(x^2y)}$.

(2.) $f = x^2y - y^2z$. Find values of f at given points,

(a.) $f(1,1,1) = 1^2 1 - 1^2 1 = \underline{0}$.

(b.) $f(+3,-1,\tfrac{1}{2}) = 9(-1) - (-1)^2(\tfrac{1}{2}) = -9 - \tfrac{1}{2} = \underline{-19/2}$.

(c.) $f(a,1,1-a) = a^2(1) - 1^2(1-a) = \underline{a^2 - 1 + a}$.

(d.) $f(t,t^2,t^3) = t^2t^2 - t^4t^3 = \underline{t^4 - t^7}$.

(3.) Find $\partial f/\partial x$ in terms of x, y, z

(a.) $\frac{\partial}{\partial x}\left[x\sin(xy) + y\cos(xz)\right] = \sin(xy) + x\cos(xy)\frac{\partial}{\partial x}[xy] + y(-\sin xz)\frac{\partial}{\partial x}(xz)$

$= \underline{\sin(xy) + xy\cos(xy) - yz\sin(xz)}$.

(b.) $f = \sin g$, $g = e^h$, $h = x^2 + y^2 + z^2$

$\frac{\partial f}{\partial x} = \cos(g)\frac{\partial g}{\partial x} = \cos(g)e^h\frac{\partial h}{\partial x} = (\cos g)(e^h)(2x)$

$= \underline{2x\cos\left(\exp(x^2+y^2+z^2)\right)e^{x^2+y^2+z^2}}$.

(4.) $f = h(g_1, g_2, g_3)$, $h : \mathbb{R}^3 \to \mathbb{R}$, $g_1, g_2, g_3 : \mathbb{R}^3 \to \mathbb{R}$.

(a.) $\frac{\partial f}{\partial x}$ given $f = h(x+y, y^2, x+z)$ where $h = x^2 - yz$

$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}\Big|_p \frac{\partial}{\partial x}(x+y) + \frac{\partial h}{\partial y}\Big|_p \frac{\partial}{\partial x}(y^2) + \frac{\partial h}{\partial z}\Big|_p \frac{\partial}{\partial x}(x+z)$ $\quad p = (\underbrace{x+y}_{P_1}, \underbrace{y^2}_{P_2}, \underbrace{x+z}_{P_3})$

chain-rule

$= 2P_1(1) + (-P_2)(1)$

$= \boxed{2(x+y) - y^2}$

pg. 6 (H2)

(4.) (b.) $f = h(\underbrace{e^3, e^{x+y}, e^x}_{p})$, $h = x^2 - yz$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial x}\Big|_p \frac{\partial}{\partial x}(e^3) + \frac{\partial h}{\partial y}\Big|_p \frac{\partial}{\partial x}(e^{x+y}) + \frac{\partial h}{\partial z}\Big|_p \frac{\partial}{\partial x}(e^x)$$

$$= 2x|_p\,(0) + -z|_p\, e^{x+y} - y|_p\, e^x$$

$$= -e^x e^{x+y} - e^{x+y} e^x$$

$$= \underline{-2e^{2x+y}}.$$

(c.) $h(x,-x,x) = f$ $P_1 = x$, $P_2 = -x$, $P_3 = x$

$$\frac{\partial f}{\partial x} = 2x|_p \frac{\partial}{\partial x}(x) + (-z)|_p \frac{\partial}{\partial x}(-x) + (-y)|_p \frac{\partial}{\partial x}(x)$$

$$= 2x + x + x$$

$$= \underline{4x}.$$

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pg. 11

(1.) $V_p = (-2,1,-1)_p$ and $W_p = (0,1,3)_p$

(a.) $3V_p - 2W_p = 3(-2,1,-1)_p - 2(0,1,3)_p$

$$= (-6,3,-3)_p + (0,-2,-6)_p$$

$$= (-6,1,-9)_p$$

$$= \underline{-6U_1 + U_2 - 9U_3}.$$ or at p.

(b.) no, I refuse.

(2.) Let $V = xU_1 + yU_2$ and $W = 2x^2U_2 - U_3$. Find $W - xV$

$$W - xV = (2x^2U_2 - U_3) - x(xU_1 + yU_2)$$

$$= \underline{-x^2U_1 + (2x^2 - xy)U_2 - U_3}.$$

$$(W - xV)_p = -U_1 + (2 - (-1)(0))U_2 - U_3 = \underline{-U_1 + 2U_2 - U_3}.$$

$$= \underline{(-1,2,-1)_p}.$$

$p = (-1,0,2)$

p. 11 (3.) In each case, express V in standard form $\sum_{i=1}^{3} v_i U_i$

(a.) $2z^2 U_1 = 7V + xy U_3$

$$\Rightarrow V = \frac{1}{7}\left(2z^2 U_1 - xy U_3\right) = v_1 U_1 + v_2 U_2 + v_3 U_3$$

$$\underline{v_1 = 2z^2/7 \,,\quad v_2 = 0 \,,\quad v_3 = \frac{-xy}{7}}.$$

(b.) $V(p) = (p_1, p_3 - p_1, 0)_p$

$$= p_1 (1, 0, 0)_p + (p_3 - p_1)(0, 1, 0)_p$$

$$= \left(x_1 U_1 + (x_3 - x_1) U_2\right)[p]$$

$x_1 = x$, $x_2 = y$, $x_3 = z$ notation

$$\Rightarrow \underline{v_1 = x_1 \,,\quad v_2 = x_3 - x_1 \,,\quad v_3 = 0}.$$

(c.) $V = 2(x U_1 + y U_2) - x(U_1 - y^2 U_3)$

$$= \underbrace{(2x - x)}_{v_1} U_1 + \underbrace{2y}_{v_2} U_2 + \underbrace{xy^2}_{v_3} U_3$$

(d.) At each point p, $V(p)$ is vector from $p = (p_1, p_2, p_3)$ to the point $(1 + p_1, p_2 p_3, p_2)$

$$V(p) = (1 + p_1, p_2 p_3, p_2) - (p_1, p_2, p_3), \quad \text{at } p$$

$$= (1, p_2 p_3 - p_2, p_2 - p_3)_p$$

$$= 1\, U_1(p) + (p_2 p_3 - p_2) U_2(p) + (p_2 - p_3) U_3(p)$$

$$\Rightarrow V = \underbrace{1}_{v_1} \cdot U_1 + \underbrace{(yz - y)}_{v_2} U_2 + \underbrace{(y - z)}_{v_3} U_3$$

(e.) at each pt. p, $V(p)$ is vector from p to origin

$$V(p) = -p \quad \text{at } p.$$

$$= -p_1 U_1 - p_2 U_2 - p_3 U_3$$

$$= (-x U_1 - y U_2 - z U_3)(p)$$

$$\Rightarrow \underline{v_1 = -x \,,\quad v_2 = -y \,,\quad v_3 = -z}.$$

(4.) If $V = y^2 U_1 - x^2 U_3$ and $W = x^2 U_1 - z U_2$ find f, g such that $fV + gW \in \text{span}\{U_2, U_3\}$

$$(fy^2 U_1 - fx^2 U_3) + (gx^2 U_1 - gz U_2) = X$$

$$(\underbrace{fy^2 + gx^2}) U_1 - gz U_2 - fx^2 U_3 = X$$

need this to vanish if we want $X \in \text{span}\{U_2, U_3\}$.

Many possible choices, $\boxed{f = x^2,\ g = -y^2}$ is mine

(5.) $V_1 = U_1 - xU_3$, $V_2 = U_2$, $V_3 = xU_1 + U_3$

(a.) Show $\{V_1(p), V_2(p), V_3(p)\}$ is LI at each $p \in \mathbb{R}^3$.

$$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$$

$$\Rightarrow c_1(U_1 - xU_3) + c_2 U_2 + c_3(xU_1 + U_3) = 0$$

$$\Rightarrow (c_1 + c_3 x) U_1 + c_2 U_2 + (c_3 - xc_1) U_3 = 0$$

But, $\{U_1, U_2, U_3\}$ are LI at each p hence,

$$\underbrace{c_1 + c_3 x = 0, \quad c_2 = 0 \quad \text{and} \quad c_3 - xc_1 = 0}$$

$$\Rightarrow c_1 = 0 \text{ and } c_3 = 0 \qquad \text{hence } c_1 = c_2 = c_3 = 0$$

for each $p \in \mathbb{R}^3$ so LI follows at each $p \in \mathbb{R}^3$.

(6.) Express $xU_1 + yU_2 + zU_3$ as linear comb. of V_1, V_2, V_3

I'd advocate an easier notation here,

$$V_1 = (1, 0, -x) = \begin{bmatrix} 1 \\ 0 \\ -x \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}$$

We wish to solve,

$$a\begin{bmatrix} 1 \\ 0 \\ -x \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{array}{l} \longrightarrow a + cx = x \\ \longrightarrow b = y \\ \longrightarrow -ax + c = z \end{array}$$

observe $\left.\begin{array}{l} ax + cx^2 = x^2 \\ -ax + c = z \end{array}\right|$ $\quad c(x^2+1) = x^2 + z \Rightarrow c = \dfrac{x^2+z}{x^2+1}$

$$a = x - cx = x - \left(\frac{x^2+z}{x^2+1}\right)x$$

(S6) (continued)

pg. 11

$$x\,U_1 + y\,U_2 + z\,U_3 =$$

$$= x\left(1 - \frac{x^2+z}{x^2+1}\right)V_1 + y\,V_2 + \left(\frac{x^2+z}{x^2+1}\right)V_3$$

$$= x\left(\frac{x^2+1-x^2-z}{x^2+1}\right)V_1 + y\,V_2 + \left(\frac{x^2+z}{x^2+1}\right)V_3$$

$$= \underline{\frac{x(1-z)}{x^2+1}\,V_1 + y\,V_2 + \frac{x^2+z}{x^2+1}\,V_3}\,.$$

Where $V_1 = U_1 - xU_3$, $V_2 = U_2$, $V_3 = xU_1 + U_3$. Check it

$$U_1 : \left(\frac{x(1-z)}{x^2+1} + \left(\frac{x^2+z}{x^2+1}\right)(+x)\right) = \frac{x - xz + x^3 + zx}{x^2+1} = \frac{x(x^2+1)}{x^2+1} = x\,.$$

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pg. 15

(1.) V_p be defined with $V = (2,-1,3)$, $p = (2,0,-1)$

use def 2 to calculate $V_p[f]$ for

(a.) $f = y^2 z$ (Def^n/ $V_p[f] = \frac{d}{dt}\left[f(p+tv)\right]\Big|_{t=0}$.

$$p + tv = (2,0,-1) + t\,(2,-1,3) = (2+2t,\,-t,\,-1+3t)$$

$$f(p+tv) = (-t)^2(-1+3t) = 3t^3 - t^2$$

$$\frac{d}{dt}\left[f(p+tv)\right] = \frac{d}{dt}\left(3t^3 - t^2\right) = 9t^2 - 2t$$

$$\frac{d}{dt}\left[f(p+tv)\right]\Big|_{t=0} = \underline{0}.$$

(b.) $f = x^7 \Rightarrow f(p+tv) = (2+2t)^7$

$$\frac{d}{dt}\left[f(p+tv)\right]\Big|_{t=0} = \left(2\cdot 7(2+2t)^6\right)\Big|_{t=0} = 7(2)^7 = \underline{896}.$$

(c.) $f = e^x \cos y \Rightarrow f(p+tv) = e^{2+2t}\cos(-t)$

$$\frac{d}{dt}\left[f(p+tv)\right] = 2e^{2+2t}\cos t - e^{2+2t}\sin t$$

$\Rightarrow \boxed{V_p[f] = 2e^2}$ evaluate at $t=0$

#6

(2.) Again calculate $V_p[f]$ but, now with Lemma 3.2 which says $V_p = (V_1, V_2, V_3)_p$ gives

$$V_p[f] = V_1 \frac{\partial f}{\partial x} + V_2 \frac{\partial f}{\partial y} + V_3 \frac{\partial f}{\partial z}. \quad \left(V_1 = 2,\ V_2 = -1,\ V_3 = 3 \text{ given here}\right)$$

$P = (2, 0, -1)$

(a.) $f = y^2 z$

$$V_p[f] = \left(2\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + 3\frac{\partial}{\partial z}\right)(y^2 z)\Big|_p = \left(-2yz + 3y^2\right)\Big|_p$$
$$= -2(0)(-1) + 3(0)$$
$$= \boxed{0}$$

(b.) $f = x^7$

$$V_p[f] = \left(2\partial_x - \partial_y + 3\partial_z\right)(x^7)\Big|_p = 14x^6\Big|_p = 14\cdot 2^6$$
$$= 7\cdot 2^7$$
$$= \underline{896}.$$

(c.) $f = e^x \cos(y)$

$$V_p[f] = \left(2\partial_x - \partial_y + 3\partial_z\right)(e^x \cos y)\Big|_p$$
$$= \left(2e^x \cos y + e^x \sin y\right)\Big|_{p=(2,0,-1)}$$
$$= 2e^2 \cos(0) + e^2 \sin(0)$$
$$= \boxed{2e^2.}$$

(3.) <u>Let $V = y^2 U_1 - x U_3$ and let $f = xy$, $g = z^3$. Calculate:</u>

(a.) $V[f] = (y^2 U_1 - x U_3)[xy] = y^2 \underbrace{U_1[xy]}_{y} - x \underbrace{U_3[xy]}_{0} = \underline{y^3}.$

(b.) $V[g] = (y^2 U_1 - x U_3)[z^3] = \underline{-3xz^2}.$

(f.) $V[V[f]] = V[y^3] = 0$ (as V has no U_2 term.)

4) on pg. 15) Prove $V = \sum_{i=1}^{3} V[x_i]U_i$ where x_1, x_2, x_3 are natural coord. fncts. (H7)

Let $V = \sum_i v_i U_i$ and calculate $V[x_j]$

$$V[x_j] = \left(\sum_i v_i U_i\right)[x_j] = \sum_i v_i U_i[x_j] = \sum_i v_i \frac{\partial x_j}{\partial x_i} = v_j$$

Thus $v_j = V[x_j]$ and it follows $\underline{V = \sum_{i=1}^{3} V[x_i]U_i}$.

5.) Suppose $V[f] = W[f]$ $\forall f \in C^\infty(\mathbb{R}^3)$, let $f = x_j$

then $V[x_j] = W[x_j] \Rightarrow$ $\underbrace{V = \sum_i V[x_i]U_i = \sum_i W[x_i]U_i}_{\text{by Problem 4.}} = W$ //

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Pg. 22 §1.4)

(1.) Find velocity vector of curve (and for $t = 0, \pi/2, \pi$)

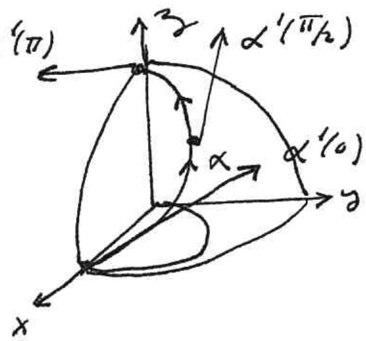
$$\alpha(t) = (1+\cos t, \sin t, 2\sin \tfrac{t}{2})$$

$$\alpha'(t) = (-\sin t, \cos t, \cos(t/2))_{\alpha(t)}$$

$$\alpha'(0) = (0, 1, 1)_{(2,0,0)}$$

$$\alpha'(\pi/2) = (-1, 0, 2/\sqrt{2})_{(1,1,2/\sqrt{2})}$$

$$\alpha'(\pi) = (0, -1, 0)_{(0,0,2)}$$



(2.) $\underline{\alpha(0) = (1,0,5) \text{ and } \alpha'(t) = (t^2, t, e^t) \text{ find } \alpha.}$

integrate: $\alpha(t) = (\frac{1}{3}t^3 + c_1, \frac{1}{2}t^2 + c_2, e^t + c_3)$

$\alpha(0) = (c_1, c_2, 1 + c_3) = (1, 0, 5) \rightarrow c_1 = 1, \; c_2 = 0, \; c_3 = 4$

$$\therefore \underline{\alpha(t) = (1 + t^3/3, t^2/2, 4 + e^t)}.$$

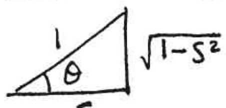
p.22 §1.4

(H8)

(3.) Let $h(s) = \cos^{-1}(s)$ for $0 < s < 1$ find $\beta = \alpha(h)$ where

$\alpha(t) = (1+\cos t,\ \sin t,\ 2\sin(t/2))$

$$\beta(s) = \alpha(\cos^{-1}(s)) = \left(1+\cos(\cos^{-1}(s)),\ \sin(\cos^{-1}(s)),\ 2\sin\left(\tfrac{1}{2}\cos^{-1}(s)\right)\right)$$
$$= \boxed{\left(1+s,\ \sqrt{1-s^2},\ \sqrt{2}\sqrt{1-s}\right)}$$



$\cos\theta = s$

$\theta = \cos^{-1}(s)$

$\sin\theta = \sqrt{1-s^2}$

$$\sin(\theta/2) = \sqrt{\sin^2(\theta/2)} = \sqrt{\tfrac{1}{2}(1-\cos\theta)} = \sqrt{\tfrac{1-s}{2}}$$

(4.) $\alpha(t) = (e^t,\ e^{-t},\ \sqrt{2}\,t)$ reparametrize via $h(s) = \ln(s)$ for $s>0$
Verify Lemma 4.5 which says $\beta'(s) = \frac{dh}{ds}\alpha'(h(s))$

$$\beta = \alpha(h) \Rightarrow \beta(s) = (e^{\ln s},\ e^{-\ln(s)},\ \sqrt{2}\ln(s)) = (s,\ 1/s,\ \sqrt{2}\ln(s)).$$

$$\alpha'(t) = (e^t,\ -e^{-t},\ \sqrt{2})$$
$$\alpha'(h(s)) = (s,\ 1/s,\ \sqrt{2})$$
$$\frac{dh}{ds}\alpha'(h(s)) = \frac{1}{s}(s,\ 1/s,\ \sqrt{2}) = (1,\ 1/s^2,\ \sqrt{2}/s) \neq \beta'(s)$$

(5.) Find eq^{n} of line through $(1,-3,-1)$ and $(6,2,1)$
does this line meet the line connecting pts. $(-1,1,0)$ and $(-5,-1,-1)$.

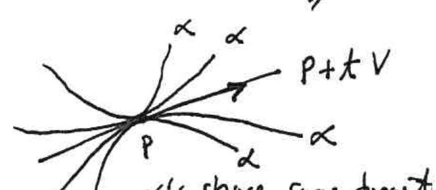
$$\alpha(t) = (1,-3,-1) + t(5,5,2) \quad 0 \le t \le 1$$
$$\beta(t) = (-1,-1,0) + t(-4,-2,-1) \quad 0 \le t \le 1$$

Does $\alpha(t) \stackrel{?}{=} \beta(\lambda)$ for some $t, \lambda \in [0,1]$?

$$\left.\begin{array}{l} 1+5t = -1-4\lambda \\ -3+5t = -1-2\lambda \end{array}\right\} \longrightarrow 4 = -2\lambda \therefore \lambda = -2 \notin [0,1]$$
$$-1+2t = -\lambda$$

thus no intersection.

(6.) Lemma 4.6 says $\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$.



α's share same tangent

$V_p[f] = \frac{d}{dt}\left(f(P+tv)\right)\Big|_{t=0}$ was the def^{n} given.

However, $\frac{d}{dt}\left(f(\alpha(t))\right) = \alpha'(t)[f]$ hence $\frac{d}{dt}\left(f(\alpha(t))\right)\Big|_{t=0} = \alpha'(0)[f]$

thus any α such that $\alpha'(0) = V_p$ will produce same result for $V_p[f]$.

p. 23 §1.4

(7.)(a) $\alpha(t) = (t, 1+t^2, t) \longrightarrow \alpha'(0) = (1,0,1)_p = V_p$

$\beta(t) = (\sin t, \cos t, t) \longrightarrow \beta'(0) = (1,0,1)_p = V_p$

$\gamma(t) = (\sinh t, \cosh t, t) \longrightarrow \gamma'(0) = (1,0,1)_p = V_p$

where $p = (0,1,0)$, $v = (1,0,1)$.

(b.) Let $f = x^2 - y^2 + z^2$ calculate $V_p[f]$ by calculating $\frac{d(f(\alpha))}{dt}$ at $t=0$ and the same for β, γ,

$$f(\alpha(t)) = f(\underbrace{t}_{x}, \underbrace{1+t^2}_{y}, \underbrace{t}_{z}) = t^2 - (1+t^2)^2 + t^2 = 2t^2 - (1+t^2)^2$$

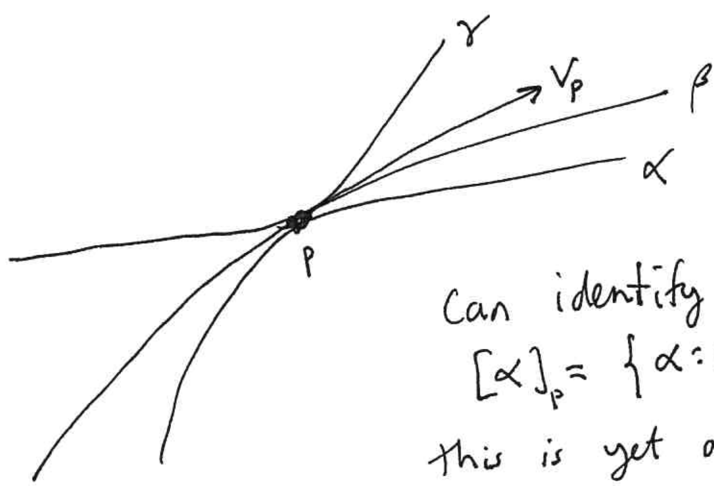
$$\frac{d}{dt}\left[f(\alpha(t))\right] = 4t - 2(1+t^2)(2t) \Rightarrow (f\circ\alpha)'(0) = 0,$$

Or, in view of lemma 4.6, $\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$ for $t=0$,

$$\begin{aligned}\frac{d(f(\alpha))}{dt}(0) &= \alpha'(0)[f] \\ &= \Big((U_1 + U_3)[x^2 - y^2 + z^2]\Big)(\alpha(0)) \\ &= \big(U_1(0,1,0) + U_3(1,0,1)\big)[x^2 - y^2 + z^2] \\ &= \left[\frac{\partial}{\partial x}(x^2 - y^2 + z^2) + \frac{\partial}{\partial z}(x^2 - y^2 + z^2)\right](0,1,0) \\ &= (2x + 2z)\Big|_{(0,1,0)} \\ &= \boxed{0.}\end{aligned}$$

or could write

Similar calculations hold for β & γ pictorially,



Can identify V_p with

$[\alpha]_p = \{\alpha : \mathbb{R}^3 \to \mathbb{R}^3 \mid \alpha(0) = p, \alpha'(0) = v\}$

this is yet another way of viewing tangent vectors.

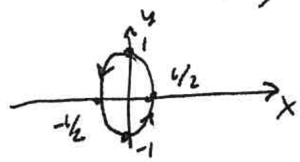
p. 23 §1.4

(H10)

#8 Sketch and parametrize the following curves in $\mathbb{R}^3$,

(a.) $C: 4x^2+y^2=1$

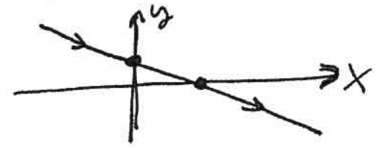
Let $\alpha(t) = \left(\frac{1}{2}\cos t,\ \sin t\right)$



ellipse.

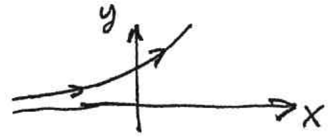
(b.) $C: 3x+4y=1$

Let $x=t$ then $4y=1-3t$ $\therefore$ $y=\frac{1}{4}(1-3t)$ thus $\alpha(t)=\left(t, \frac{1}{4}(1-3t)\right)$



(c.) $C: y=e^x$

$\alpha(t) = (t, e^t)$

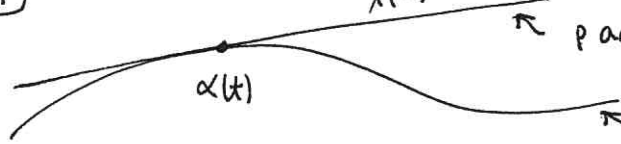


Remark: C has no explicit orientation as given. However, once I write the formula for α the resulting curve has the indicated direction. Sometimes one is given $C: f(x,y)=k$ paired with a sense of direction, in such a case we must choose a $\alpha(t)$ which shares the given orientation.

#9 $l(u) = \alpha(t) + u\alpha'(t)$ technically $\alpha'(t) = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt}\right)_{\alpha(t)}$

but here he says delete

parameter of line is "u"



parameter of α is t

$$\alpha(t) = (2\cos t,\ 2\sin t,\ t)$$
$$\alpha'(t) = (-2\sin t,\ 2\cos t,\ 1)$$
$$\alpha'(0) = (0,\ 2,\ 1) \longrightarrow l(u) = (2,0,0) + u(0,2,1)$$
$$\alpha'(\pi/4) = (-2/\sqrt{2},\ 2/\sqrt{2},\ 1) = (-\sqrt{2},\ \sqrt{2},\ 1) \longrightarrow l(u) = (\sqrt{2}, \sqrt{2}, \tfrac{\pi}{4}) + u(-\sqrt{2}, \sqrt{2}, 1)$$

- oh, this is Math 231, let's get on with it already...

(1.) Let $V = (1, 2, -3)$ and $p = (0, -2, 1)$. Evaluate the following 1-forms on the tangent vector V_p.

(a.) $(y^2 dx)(V_p) = \underbrace{y^2}_{\text{at } p} V_p[x] = 4\underbrace{V_p[x]} = 4(1) = \boxed{4.}$

x-component of V_p by #4 of §1.3 pg. 15, see H7

(b.) $(z\,dy - y\,dz)(V_p) =$

$= 1 \cdot V_p[y] + 2\, V_p[z]$

$= 1 \cdot V_2 + 2 \cdot V_3$

$= 2 + 2(-3)$

$= \boxed{-4}$

error, why? see I confused p and V. Watch for this!

had error

(c.) $((z^2-1)dx - dy + x^2 dz)(V_p) = 8\,V_p[x] - V_p[y] + 0\,V_p[z]$

$= 8(1) - (2)$

$= \cancel{\boxed{6}}$

corrected (c.) $((z^2-1)dx - dy + x^2 dz)(V_p) = (1-1)V_p[x] - V_p[y] + 0\,V_p[z]$

$= 0 - 2 + 0$

$= \boxed{-2.}$

(2.) If $\phi = \sum_{i=1}^{3} f_i dx_i$ and $V = \sum_{j=1}^{3} V_j U_j$ show $\phi(V) = \sum_{i=1}^{3} f_i V_i$

Proof: $\phi(V) = \left(\sum_{i=1}^{3} f_i\, dx_i\right)(V)$

$= \sum_{i=1}^{3} f_i\, dx_i(V)$

$= \sum_{i=1}^{3} f_i\, V[x_i]$

$= \sum_{i=1}^{3} f_i V_i$

by #4 of §1.3 see H7.

(3.) $\phi = x^2 dx - y^2 dz$ and $V = (x, y, z)_{(x,y,z)} = x U_1 + y U_2 + z U_3$

$\phi(V) = x^2(x) + 0(y) - y^2(z) = \underline{x^3 - zy^2}.$

pg. 27 §1-5

#3 continued, $\phi(W) = (x^2dx - y^2dz)\Big[\underbrace{xy(U_1 - U_3) + yz(U_1 - U_2)}_{W}\Big]$ H12

$$= x^2 W[x] - y^2 W[z]$$
$$= x^2(xy + yz) - y^2(-xy)$$
$$= \underline{x^3y + x^2yz + xy^3}.$$

using that $W[x]$ selects coeff. of U_1 and $W[z]$ selects coeff. of U_3 in W.

$$\phi\Big(\underbrace{\tfrac{1}{x}V + \tfrac{1}{y}W}_{\Xi}\Big) = (x^2dx - y^2dz)(\Xi)$$
$$= x^2\,\Xi(x) - y^2\,\Xi(z)$$
$$= x^2\left(\tfrac{1}{x}V_1 + \tfrac{1}{y}W_1\right) - y^2\left(\tfrac{1}{x}V_3 + \tfrac{1}{y}W_3\right)$$
$$= x^2\left(\tfrac{1}{x}(x) + \tfrac{1}{y}(xy+yz)\right) - y^2\left[\tfrac{1}{x}(z) + \tfrac{1}{y}(-xy)\right]$$
$$= \underline{x^2(1 + x + z) - y^2(z/x - x)}.$$
$$= \underbrace{(x^2y + x^2yz + xy^3)}_{\phi(W)}/y + \underbrace{(x^3 - zy^2)}_{\phi(V)}/x$$

Same as answer suggested by hint.

(4.) (a.) $df^5 = 5f^4df$

(b.) $d(\sqrt{f}) = \dfrac{1}{2\sqrt{f}}df$

(c.) $d(\log(1+f^2)) = \left(\dfrac{1}{1+f^2}\right)2f\,df.$

(5.) Express df for functions given below as $\sum f_i\,dx_i$

(a.) $f = (x^2+y^2+z^2)^{1/2}$ note $\dfrac{\partial f}{\partial x_i} = \dfrac{x_i}{f}$ for $x_i = x, y, z$.

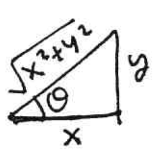
thus $\boxed{df = \frac{1}{f}(x\,dx + y\,dy + z\,dz)}$.

Alternatively, $f^2 = x^2+y^2+z^2 \Rightarrow 2f\,df = 2x\,dx + 2y\,dy + 2z\,dz$

(6.) $\theta = \tan^{-1}(y/x) \Longrightarrow \tan\theta = y/x$

$$\sec^2\theta\,d\theta = \frac{1}{x}dy - \frac{y}{x^2}dx = \frac{x\,dy - y\,dx}{x^2}$$

Hence, $d\theta = \dfrac{x\,dy - y\,dx}{\sec^2\theta\,x^2} = \dfrac{x\,dy - y\,dx}{\left(\frac{x^2+y^2}{x^2}\right)x^2} = \boxed{\dfrac{x\,dy - y\,dx}{x^2+y^2}}$



$\sec\theta = \dfrac{\sqrt{x^2+y^2}}{x}$

(you could just diff. $\tan^{-1}(y/x)$ directly, I'm playing around here.)

p-27 §1.5 / Given $V_p = (1,2,-3)_{(0,-2,1)}$ find df for f (H13)

6.) ~~(ea)~~ given below and calculate $V_p[f]$.

(a.) $f = xy^2 - yz^2$

$$df = y^2dx + 2xydy - z^2dy - 2yzdz = (y^2)dx + (2xy - z^2)dy - 2yzdz$$

$$V_p[f] = V_1f_1 + V_2f_2 + V_3f_3 \quad , \quad \underbrace{df|_p}_{\uparrow\,(0,-2,1)} = \underbrace{4dx - dy + 4dz}_{f_1 = 4,\; f_2 = -1,\; f_3 = 4}$$

$$= 1(4) + 2(-1) + (-3)(4)$$
$$= 4 - 2 - 12$$
$$= \boxed{-10}$$

(b.) $f = xe^{yz}$

$$df = e^{yz}dx + yxe^{yz}dz + zxe^{yz}dy$$

$$df|_{(0,-2,1)} = e^{-2}dx$$ as $\underline{x = 0}$ kills the dz & dy terms

Hence, $V_p[f] = V_1f_1 = 1 \cdot e^{-2} = \boxed{e^{-2}}$

(c.) $f = \cos(xy)\sin(xy)$

$$df = \left[-\sin^2(xy)\frac{\partial}{\partial x}(xy) + \cos^2(xy)\frac{\partial}{\partial x}(xy)\right]dx + \left[-\sin^2 xy\frac{\partial}{\partial y}(xy) + \cos^2(xy)\frac{\partial}{\partial y}(xy)\right]dy$$

$$= \left[\cos^2(xy) - \sin^2(xy)\right]y\,dx + \left[\cos^2(xy) - \sin^2(xy)\right]x\,dy$$

$$= \underline{\left[\cos^2(xy) - \sin^2(xy)\right](y\,dx + x\,dy)}.$$

I'm moving on, I leave $V_p[f]$ to reader.

7.) For which of the expressions below is it reasonable to suppose ϕ is one form and $\phi(V_p)$ is given by (for $V_p = (V_1, V_2, V_3)_p$)

(a.) $V_1 - V_3$ sure, $\phi = dx - dz$

(b.) $P_1 - P_3$ no, notice $V_p = 0$ does not give $\phi(0) = 0$, this formula is not linear on $T_p\mathbb{R}^3$.

(c.) $V_1P_3 + V_2P_1$ sure $\phi = z\,dx + x\,dy$

(d.) $V_p[x^2+y^2] = (d(x^2+y^2))[V_p]$ so yes, $\phi = 2x\,dx + 2y\,dy$ will do.

7 continued

(e.) 0 sure, let $\phi = 0$.

(f.) $(P_1)^2$, no, again this makes ϕ proposed nonlinear.

8.) Prove Lemma 5.6 directly from def$^{\underline{n}}$ of d

Lemma 5.6: $d(fg) = g\,df + f\,dg$

Proof:
$$\begin{aligned} d(fg)(V_p) &= V_p[fg] \\ &= V_p[f]\,g(p) + f(p)\,V_p[g] \\ &= g(p)\big(df(V_p)\big) + f(p)\big(dg(V_p)\big) \\ &= (g\,df + f\,dg)(V_p) \end{aligned}$$

Leibniz prop of V_p acting on fg.

addition of one forms & scalar mult. by facts defined pointwise principle.

But, p and V_p arbitrary hence Lemma 5.6 follows. //

9.) $\phi = 0$ iff $\phi(V_p) = 0 \ \forall V_p \in T_p\mathbb{R}^3$ (here $\phi \in (T_p\mathbb{R}^3)^*$ so, as he says ϕ is "at p")

A point p for which $df = 0$ for a function $f : \mathbb{R}^3 \to \mathbb{R}$ is called a critical point. Show p critical iff $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$ at p.

Proof: Assume $df = 0$ at p. Then
$df(U_i) = 0$ at $p \Rightarrow U_i[f] = 0$ at $p \Rightarrow \underbrace{\frac{\partial f}{\partial x_i}(p) = 0}_{i=1,2,3.}$. //

Let $f = (1-x^2)y + (1-y^2)z$
$$df = -2xy\,dx + (1-x^2-2yz)\,dy + (1-y^2)\,dz$$

Need simultaneously
$2xy = 0 \Rightarrow \underline{x = 0 \text{ or } y = 0}$.
$1 - x^2 - 2yz = 0$
$1 - y^2 = 0 \Rightarrow \underline{y = \pm 1}$.

not allowed.

Thus $y = \pm 1 \Rightarrow x = 0$ which leaves $1 - 2yz = 0$
So $2yz = 1 \Rightarrow z = \frac{1}{2y} = \frac{\pm 1}{2}$ hence $(0, \pm 1, \pm 1/2)$
or $(0, 1, 1/2)$ and $(0, -1, -1/2)$ these are critical for f.

(H15)

10.) Prove local max/min of f are critical points of f.

We say $f(p)$ is local max iff $\forall q \in Nbhd(p)$, $f(q) \leq f(p)$.

Let α be curve with $\alpha(0) = p$ and consider t such that $\alpha(t) \in Nbhd(p)$. We have, for all $t \in [-\delta, \delta]$ (for some $\delta > 0$.)

$$f(\alpha(t)) \leq f(\alpha(0))$$

Let $g = f \circ \alpha$ and note $g(0)$ is local max in sense of single-variable calculus. Thus, by Fermat's Thm $g'(0) = 0$. But, by chain rule,

$$\frac{d}{dt}\left[f(\alpha(t))\right] = \frac{\partial f}{\partial x}(\alpha(t))\frac{d\alpha_1}{dt} + \frac{\partial f}{\partial y}(\alpha(t))\frac{d\alpha_2}{dt} + \frac{\partial f}{\partial z}(\alpha(t))\frac{d\alpha_3}{dt}$$

For $t=0$, we obtain $\alpha(0) = p$ by construction hence,

$$\frac{\partial f}{\partial x}(p)V_1 + \frac{\partial f}{\partial y}(p)V_2 + \frac{\partial f}{\partial z}(p)V_3 = 0$$

where I denoted $\frac{d\alpha_i}{dt}(0) = V_i$. Notice, this holds $\forall \alpha$ in this nbhd of p so it follows $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0$.

Use $\alpha(t) = p + tV_i$ if you prefer.

11.) Explain how $(df)(V_p)$ gives linear approx. to $\Delta f = f(p+v) - f(p)$

1.) $df \in (T_p\mathbb{R}^3)^*$ hence by construction is linear.

2.) $df(V_p) = V_p[f] = \frac{d}{dt}\left(f(p+tV)\right)\Big|_{t=0}$

$$= \lim_{h\to 0}\left[\frac{f(p+hV) - f(p)}{h}\right]$$

p. 32 § 1.6

(H16)

1.) $\phi = yz\,dx + dz \quad , \psi = \sin z\, dx \, , \qquad \xi = dy + z\,dz$

(a.) $\phi \wedge \psi = (yz\,dx + dz) \wedge (\sin z\, dx)$

$= yz \sin z \underbrace{dx \wedge dx}_{0} + \sin z\, dz \wedge dx$

$= \underline{\sin z\, dz \wedge dx} . = \Phi_{\langle 0, \sin z, 0 \rangle}$

our notation from Math 332

$\Phi_{\langle a,b,c \rangle} = a\,dy\wedge dz + b\,dz \wedge dx + c\,dx\wedge dy$

the so-called flux-form mapping.

$\psi \wedge \xi = (\sin z\, dx) \wedge (dy + z\,dz)$

$= \underline{\sin z\, dx\wedge dy + z \sin z\, dx \wedge dz} . = \Phi_{\langle 0, -z\sin z, \sin z \rangle}$

$\xi \wedge \phi = (dy + z\,dz) \wedge (yz\,dx + dz)$

$= \underline{yz\, dy\wedge dx + dy \wedge dz + yz^2\, dz \wedge dx} . = \Phi_{\langle 1, yz^2, -yz \rangle}$

(b.) $d\phi = d(yz) \wedge dx + d(dz)$

$= (z\,dy + y\,dz) \wedge dx$

$= \underline{z\,dy\wedge dx + y\,dz\wedge dx} .$

$= -z\,dx\wedge dy + y\,dz \wedge dx = \Phi_{\langle 0, y, -z \rangle}$

$d\psi = d(\sin z) \wedge dx$

$= \underline{\cos z\, dz \wedge dx} .$

$= \Phi_{\langle 0, \cos z, 0 \rangle}$

$d\xi = d(dy) + dz \wedge dz = \underline{0} .$

Proved next page (H17)

2.) left to reader.

3.) $d(df) = d\left(\sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i \right) = \sum_{i=1}^{3} d(\partial_i f) \wedge dx_i$

By fundamental Lemma of tensor arithmetic

$= \sum_{i=1}^{3} \sum_{j=1}^{3} \underbrace{\partial_j (\partial_i f)}_{\text{symmetric in } i,j} \underbrace{dx_j \wedge dx_i}_{\text{antisym. in } i,j} = 0$

Lemma:

Suppose $S_{ij} = S_{ji}$ and $A_{ij} = -A_{ji}$ $\forall\, i,j \in \mathbb{N}_n$

then $\sum_i \sum_j S_{ij} A_{ij} = 0$.

Proof: $I = \sum_i \sum_j S_{ij} A_{ij} = \sum_i \sum_j - S_{ji} A_{ji} \underset{\substack{\uparrow \\ \text{property} \\ \text{of finite sums.}}}{=} -\sum_j \sum_i S_{ji} A_{ji} = -I$

thus $I = -I$ and so $I = 0$.

3 continued

$$d(f\,dg) = \underline{df \wedge dg} + f\, \cancel{d(dg)}^{\,0} \qquad (\text{by Th}^{\underline{m}}\ 6.4.1)$$

(5.) $\phi_1 \wedge \phi_2 \wedge \phi_3 = (f_{11}dx_1 + f_{12}dx_2 + f_{13}dx_3) \wedge (f_{21}dx_1 + f_{22}dx_2 + f_{23}dx_3) \wedge \phi_3$

$$= \big(f_{11}f_{22}\,dx_1 \wedge dx_2 + f_{11}f_{23}\,dx_1 \wedge dx_3 + f_{12}f_{21}\,dx_2 \wedge dx_1$$
$$+ f_{12}f_{23}\,dx_2 \wedge dx_3 + f_{13}f_{21}\,dx_3 \wedge dx_1 + f_{13}f_{22}\,dx_3 \wedge dx_2\big) \wedge \phi_3$$

$$= \Big[(f_{11}f_{22} - f_{12}f_{21})\,dx_1 \wedge dx_2 + (f_{11}f_{23} - f_{13}f_{21})\,dx_1 \wedge dx_3$$
$$+ (f_{12}f_{23} - f_{13}f_{22})\,dx_2 \wedge dx_3\Big] \wedge \big[f_{31}dx_1 + f_{32}dx_2 + f_{33}dx_3\big]$$

$$= f_{33} \det\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} dx_1 \wedge dx_2 \wedge dx_3$$

$$+ f_{32} \det\begin{bmatrix} f_{11} & f_{13} \\ f_{21} & f_{23} \end{bmatrix} dx_1 \wedge dx_3 \wedge dx_2$$

$$+ f_{31} \det\begin{bmatrix} f_{12} & f_{13} \\ f_{22} & f_{23} \end{bmatrix} dx_2 \wedge dx_3 \wedge dx_1$$

$$= \underbrace{\left(f_{31}\begin{vmatrix} f_{12} & f_{13} \\ f_{22} & f_{23} \end{vmatrix} - f_{32}\begin{vmatrix} f_{11} & f_{13} \\ f_{21} & f_{23} \end{vmatrix} + f_{33}\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}\right)}_{\text{Laplace Expansion by minors on } 3^{\underline{rd}} \text{ column.}} dx_1 \wedge dx_2 \wedge dx_3$$

$$= \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} dx_1 \wedge dx_2 \wedge dx_3 .$$

H18

6.) $$dx \wedge dy \wedge dz = d(r\cos\theta) \wedge d(r\sin\theta) \wedge dz$$
$$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \wedge dz$$
$$= \underline{r\, dr \wedge d\theta \wedge dz}.$$

7.) $$\eta = f\, dx \wedge dy + g\, dx \wedge dz + h\, dy \wedge dz$$
$$d\eta = df \wedge dx \wedge dy + dg \wedge dx \wedge dz + dh \wedge dy \wedge dz$$
definition of exterior derivative.

Let $\phi = a\,dx + b\,dy + c\,dz$

then $d\phi = da \wedge dx + db \wedge dy + dc \wedge dz$

thus $$d(d\phi) = d(da) \wedge dx + d(db) \wedge dy + d(dc) \wedge dz - da \wedge d(dx) - db \wedge d(dy) - dc \wedge d(dz)$$

Here I used $d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$ where α, β are one-forms, this is Th^{m} 6.4.3. Then we find $d(d\phi) = 0$ as $d(df) = 0$ from problem (3) of this section see (H16)

8.) Let $$\omega_{\vec{F}} = F_1\, dx + F_2\, dy + F_3\, dz$$
$$\Phi_{\vec{F}} = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy.$$

I encourage the reader to verify,

$$df = \omega_{\nabla f}$$
$$d\omega_{\vec{F}} = \Phi_{\nabla \times \vec{F}}$$
$$d\Phi_{\vec{G}} = (\nabla \cdot \vec{G})\, dx \wedge dy \wedge dz$$

and $\omega_{\vec{A}} \wedge \omega_{\vec{B}} = \Phi_{\vec{A} \times \vec{B}}$. I think this notation improves what is said in (8.). Much of this is explicitly calculated in my Math 332 notes... ask if need reference...

9.) $$df \wedge dg = \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \wedge \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy\right) = \left[f_x g_y - f_y g_x\right] dx \wedge dy.$$

p. 40 § 1.7)

(H19)

$$F(u,v) = (u^2-v^2, 2uv)$$

(1) Find $p \in \mathbb{R}^2$ s.t.

(a.) $F(p) = (0,0) \Rightarrow \underbrace{u^2-v^2=0}_{u=\pm v}, \ 2uv=0$

$u = \pm v \longrightarrow \pm 2u^2 = 0 \longrightarrow u=v=0$

$\therefore (0,0)$

(b.) $F(P) = (8,6) = (u^2-v^2, 2uv)$

$u^2-v^2 = 8$

$2uv = 6 \quad \therefore \ v = \frac{3}{u} \quad \Rightarrow \quad u^2 - \frac{9}{u^2} = 8$

$\Rightarrow \quad u^4 - 9 = 8u^2$

$\Rightarrow \quad u^4 - 8u^2 - 9 = 0$

$\Rightarrow \underbrace{(u^2-9)}_{u=\pm 3}\underbrace{(u^2+1)}_{\text{no sol}^{\text{ns}}} = 0$

$\hookrightarrow v = \frac{3}{\pm 3} = \pm 1$

$\therefore \boxed{(3,1), (-3,-1)}$

(c.) $F(P) = P$

$u^2-v^2 = u$ & $\underbrace{2uv = v}_{2u=1 \text{ or } v=0}$

if $v=0$ then $u^2=u \Rightarrow u^2-u = u(u-1) = 0 \Rightarrow u=0,1$

thus find $\boxed{(0,0), (1,0)}$ from $v=0$ case. If $v \neq 0$

then $u = 1/2$ thus $\frac{1}{4} - v^2 = \frac{1}{2} \Rightarrow v^2 = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4}$

thus no sol$^{\text{ns}}$ are found in $u = 1/2$ case.

(2.) actually, a lot of work.

(3.) Let $V = (V_1, V_2)$ be tangent to $\mathbb{R}^2$ at $P = (P_1, P_2)$.
Apply Def$^{\underline{n}}$ 7.4 directly to express $F_*(V)$ in terms of V and P's coordinates.

$F_*(v)$ is initial velocity of curve $t \mapsto F(P+tV)$

$F(u,v) = (u^2-v^2, 2uv)$ once again (given)

$$F(P+tV) = F(P_1+tV_1, P_2+tV_2)$$
$$= \left((P_1+tV_1)^2 - (P_2+tV_2)^2, 2(P_1+tV_1)(P_2+tV_2)\right)$$

$$\frac{d}{dt}\left[F(P+tV)\right]\Big|_{t=0} = \left(2(P_1+tV_1)V_1 - 2(P_2+tV_2)V_2, 2V_1(P_2+tV_2)+2V_2(P_1+tV_1)\right)\Big|_{t=0}$$
$$= \left(2P_1V_1 - 2P_2V_2, 2V_1P_2 + 2V_2P_1\right)_{F(P)}$$

(4.) $J_F = \left[\frac{\partial F}{\partial u} \Big| \frac{\partial F}{\partial v}\right] = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$

$J_F(P) = \begin{bmatrix} 2P_1 & -2P_2 \\ 2P_2 & 2P_1 \end{bmatrix}$

shows $(F_*)_p$ linear.

$$(F_*)_p(V_p) = \begin{bmatrix} 2P_1V_1 - 2P_2V_2 \\ 2P_2V_1 + 2V_2P_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2P_1 & -2P_2 \\ 2P_2 & 2P_1 \end{bmatrix}}_{J_F(P)} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$\det(J_F(P)) = 4(P_1^2 + P_2^2) \neq 0$ except for $P = 0$ where $P_1 = P_2 = 0$.

(5.) $F: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation, show $F_*(V_p) = F(V)_{F(P)}$
there are many ways to show this. I'll use Prop. 7.5,

$F_*(v) = (v[f_1], \dots, v[f_m])$ at $F(P)$ where $F = (f_1, \dots, f_m)$

$f_1(x) = A_{11}X_1 + \dots + A_{1n}X_n$

$f_j(x) = A_{j1}X_1 + \dots + A_{jn}X_n$

Observe $df_j = A_{j1}dx_1 + \dots + A_{jn}dX_n$
hence $v[f_j] = df_j[v] = A_{j1}dx_1[v] + \dots + A_{jn}dx_n[v] = A_{j1}V_1 + \dots + A_{jn}V_n$
It follows that

$$F_*(V) = (\text{row}_1(A)V, \dots, \text{row}_m(A)V) = \begin{bmatrix} \text{row}_1(A) \\ \vdots \\ \text{row}_m(A) \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = AV$$

thus F_* is linear with matrix $[f_1|f_2|\dots|f_n] \Rightarrow F_* = F$.

(6.) consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^3$
this is 1-1 and onto and f is smooth. However,
$f^{-1}(x) = \sqrt[3]{x}$ is not smooth at $x = 0$.

(7.) Show $F_*(V_p)[g] = V_p[g(F)]$

$$V_p[g(F)] = d(g \circ F)[V_p]$$

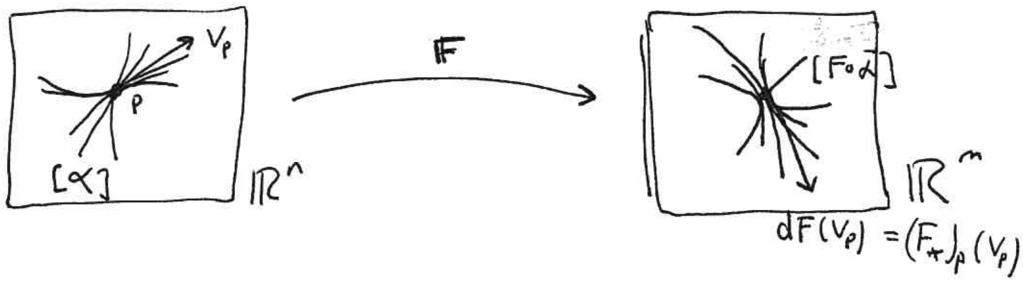
Chain-rule for differentials

$$= dg(dF[V_p])$$

our notation in Math 332 is $dF_p : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^m$
so $(F_*)_p = dF_p$.

$$= F_*(V_p)[g]$$

Some texts say $d F \underbrace{[\alpha]}_{\text{equivalence class of curves}} = \underbrace{[F \circ \alpha]}_{\text{new equivalence class of curves pushed forward by } dF}$



I leave 8, 9, 10 to reader,
I prefer to consider such question in 332 where
I've more background and a clear def$^{\underline{n}}$ of dF
in terms of Frechet quotient. Here we simplify
the basis for the theory by making directional
derivatives primary, it works for the purpose of
this text, but, I prefer to use Frechet as basic.