Sol^{n}'s to Select Problems from Chapter 2 of Oneil's 2nd Revised Ed. of Elementary Differential Geometry (H22)

p. 50 §2.1

(1.) I assume the reader can calculate $V \cdot W$, $V \times W$ etc...

(2.) Prove that $d: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$d(P,Q) = \|Q-P\| = \sqrt{(Q-P)\cdot(Q-P)}$$

Has properties below:

(a.) $d(P,q) \geq 0$ and $d(P,q) = 0$ iff $P = q$

$d(P,q) = \sqrt{(Q-P)\cdot(Q-P)} \geq 0$ by def^{t} of $\sqrt{\ }$.

Suppose $P = q$ then $d(P,P) = \sqrt{(P-P)\cdot(P-P)} = 0$.

Suppose $d(P,Q) = 0$ then $\sqrt{(P-Q)\cdot(P-Q)} = 0$

$\Rightarrow (P-Q)\cdot(P-Q) = (P_1-Q_1)^2 + (P_2-Q_2)^2 + (P_3-Q_3)^2 = 0$

If $P_i \neq Q_i$ for some i then $\rightarrow\!\!\!\!\times$ the eq^{n} above

thus $P_i = Q_i \quad \forall i = 1,2,3 \Rightarrow P = Q$.

(b.) $d(P,Q) = d(Q,P)$

Note $(P-Q)\cdot(P-Q) = (-1)^2(P-Q)\cdot(P-Q)$
$= -1(P-Q)\cdot(-1)(P-Q)$
$= (Q-P)\cdot(Q-P)$ thus $d(P,Q)^2 = d(Q,P)^2$

And as $d(P,Q) \geq 0$ it follows $d(P,Q) = d(Q,P)$.

(c.) requires thought. See my Math 231 notes or ask Minh.

(3.) $e_1 = \frac{1}{\sqrt{6}}(1,2,1)$, $e_2 = \frac{1}{\sqrt{8}}(-2,0,2)$, $e_3 = \frac{1}{\sqrt{3}}(1,-1,1)$

$$e_1 \cdot e_1 = \frac{1}{6}(1+4+1) = 1 = e_2 \cdot e_2 = e_3 \cdot e_3$$

Thus $\|e_1\| = \|e_2\| = \|e_3\| = 1$. Moreover,

$$e_1 \cdot e_2 = \frac{1}{\sqrt{48}}(-2+0-2) = 0.$$
$$e_1 \cdot e_3 = \frac{1}{\sqrt{18}}(1-2+1) = 0.$$
$$e_2 \cdot e_3 = \frac{1}{\sqrt{24}}(-2+0+2) = 0.$$

Thus $e_i \cdot e_j = \delta_{ij}$ hence $\{e_1, e_2, e_3\}$ is a frame field.

$$\begin{aligned} V &= (6,1,-1) \\ &= (V\cdot e_1)e_1 + (V\cdot e_2)e_2 + (V\cdot e_3)e_3 \\ &= \frac{1}{\sqrt{6}}(6+2-1)e_1 + \frac{1}{\sqrt{8}}(-12+0-2)e_2 + \frac{1}{\sqrt{3}}(6-1-1)e_3 \\ &= \boxed{\frac{3}{\sqrt{6}}e_1 - \frac{14}{\sqrt{8}}e_2 + \frac{4}{\sqrt{3}}e_3} \end{aligned}$$

yes $\frac{3}{\sqrt{6}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}$ etc...

(4.) Let u, v, w be points in $\mathbb{R}^3$ (or vectors if you like)

(a.)
$$\begin{aligned} u\cdot(v\times w) &= \sum_i u_i (v\times w)_i \\ &= \sum_i u_i \sum_{jk} \epsilon_{jki} v_j w_k \\ &= \sum_{i,j,k} \epsilon_{jki} v_j w_k u_i \\ &= \det[v|w|u] \\ &= \det[u|v|w] = \det\left[\frac{\underline{u}}{\frac{v}{w}}\right]. \end{aligned}$$

def$^{\underline{n}}$ of determinant.

pg. 51 # 4 continued

(b.) $u \cdot (v \times w) \neq 0$ iff u, v, w are LI.

this follows from (a.) and the fact that $\{u, v, w\}$ is LI in $\mathbb{R}^3$ iff $\det\left[\frac{u}{\frac{v}{w}}\right] \neq 0$.

(c.) $u \cdot (v \times w) = u \cdot (-w \times v) = -u \cdot (w \times v)$.

Also, $\det[u/v/w] = -\det[v/u/w] = -v \cdot (u \times w)$

$= -\det[w/v/u] = -w \cdot (v \times u)$

These all follow from prop. of det. essentially. They're not so fun if we brute-force it.

(d) $u \cdot (v \times w) = (u \times v) \cdot w$

$\underset{\text{by (c.)}}{u \cdot (v \times w) = -w \cdot (v \times u)} = -w \cdot (-u \times v)$: prop. of $\times$

$= (u \times v) \cdot w$: prop. of dot-product.

(5) $v \times w \neq 0$ iff v, w are LI and show $\|v \times w\|$ is area of .

β $v \times w \neq 0$. If $v = kw$ then $v \times w = kw \times w = 0$ $\rightarrow\leftarrow$ thus $\{v, w\}$ is LI. Conversely, β $\{v, w\}$ is LI. Extend to basis for $\mathbb{R}^3$ by $u \in \operatorname{span}\{v, w\}^{\perp} = W$ thus $u \cdot v = 0$ and $u \cdot w = 0$. Consider,

$$v \times w = c_1 u + c_2 v + c_3 w$$

Clearly $c_2 = 0$ and $c_3 = 0$ as $v \cdot (v \times w) = 0$ & $w \cdot (v \times w) = 0$
$\Rightarrow c_2 + c_3 (w \cdot v) = 0$ and $c_3 + c_2 (w \cdot v) = 0$ $\Rightarrow c_2 = c_3 = 0$.
Note, $v \times w = c_1 u$ hence $v \times w \neq 0$ as $u = 0$ $\rightarrow\leftarrow$ $\{u, v, w\}$ a basis. //

(6.) Suppose e_1, e_2, e_3 is a frame. Show $e_1 \cdot (e_2 \times e_3) = \pm 1$.

$$\|e_2 \times e_3\| = \|e_2\| \|e_3\| \sin 90^\circ = 1 \quad \text{as } e_2 \cdot e_3 = 0 \text{ is given.}$$

However, $e_2 \times e_3 = c_1 e_1 + c_2 e_2 + c_3 e_3$ as $\{e_1, e_2, e_3\}$ is basis.

Note, $(e_2 \times e_3) \cdot e_2 = 0$ and $(e_2 \times e_3) \cdot e_3 = 0$

thus $c_2 = c_3 = 0$ hence $e_2 \times e_3 = c_1 e_1$ but

$$\|e_2 \times e_3\| = \|c_1 e_1\| = |c_1| \|e_1\| = |c_1| = 1 \quad \therefore \; c_1 = \pm 1$$

Therefore, $e_1 \cdot (e_2 \times e_3) = e_1 \cdot (\pm 1 e_1) = \pm e_1 \cdot e_1 = \pm 1.$ //

Recall (4) $e_1 \cdot (e_2 \times e_3) = \det \underbrace{\begin{bmatrix} e_1 \\ \hline e_2 \\ \hline e_3 \end{bmatrix}}_{\text{attitude matrix for frame, O'neil tells us it's orthogonal.}} = \pm 1.$

Comment: BETTER WAY:

$$\underbrace{R^T R = I}_{\text{defines orthogonal matrix}} \Rightarrow \det(R^T)\det(R) = \det(I)$$

$$\Rightarrow \det(R)\det(R) = 1$$

$$\Rightarrow \underline{\det(R) = \pm 1} \quad (\text{as } \det(R) \in \mathbb{R}).$$

(7.) If u is unit vector then $\text{Proj}_u(v) = (v \cdot u)u = \|v\| \cos\theta \, u$

Show $v = v_1 + v_2$ where $v_1 \cdot v_2 = 0$ and $v_1 = \text{Proj}_u(v)$.

Simple, let $v_2 = v - v_1 = v - (v \cdot u)u = \underbrace{\text{Orth}_u(v)}_{\text{my usual notation for this.}}$

$$v_1 \cdot v_2 = (v \cdot u)u \cdot [v - (v \cdot u)u]$$

$$= (v \cdot u)(u \cdot v) - (v \cdot u)^2 u \cdot u$$

$$= 0.$$

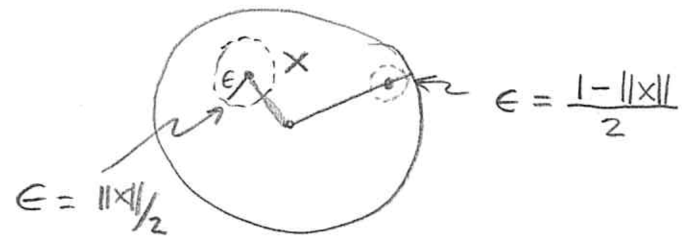
(H26)

(8) Volume of $\|$-piped is $\pm\, U \cdot (V \times W)$. See 231 notes it's in there.

(9.) Prove the nbhds below are open.

(a.) $U = \{ P \mid \|P\| < 1 \}$

Let $X \in U$ and consider, $\|X\| < 1$



Let $\epsilon = \min\{ \|X\|/2, \frac{1-\|X\|}{2} \}$. If $y \in B_\epsilon(0)$ then $\|y\| < \epsilon$ and there are two cases to consider.

(1.) $\epsilon = \|X\|/2$. Note $\|y\| < \epsilon = \frac{\|X\|}{2} < \frac{1}{2}$
thus $\|y\| < 1/2 \Rightarrow y \in U$.

(2.) $\epsilon = \frac{1 - \|X\|}{2}$. Note $\|y\| < \epsilon = \frac{1 - \|X\|}{2} < \frac{1}{2}$
and again we obtain $\|y\| < 1/2 \Rightarrow y \in U$.

Thus each point in U is an interior point as it possesses an ϵ-ball nbhd contained in U. It follows that U is open.

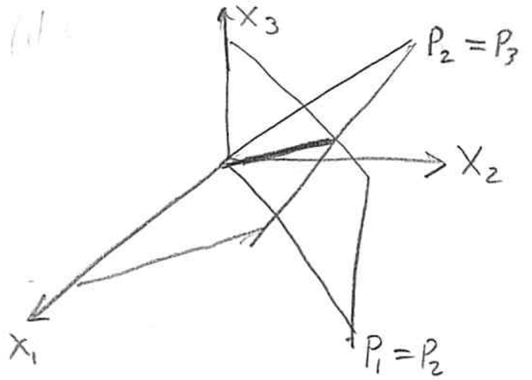
(b.) I leave to reader. The argument is likely easier than (a.).

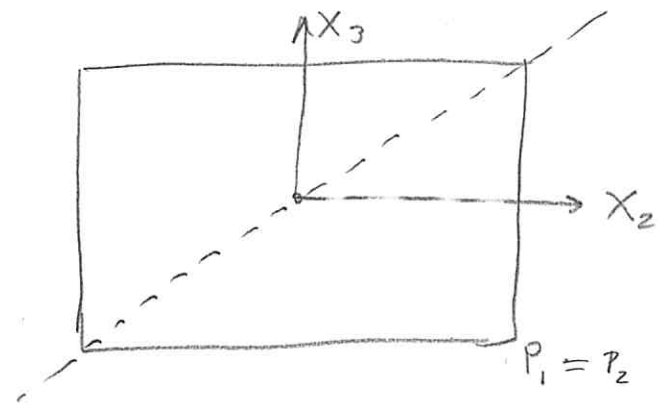
(H27)

(10) (a.) $P_1^2 + P_2^2 + P_3^2 = 1$ is unit-sphere it is <u>not</u> open.

(b.) $P_3 \neq 0$ is union of $\underbrace{(\mathbb{R}^2 \times (-\infty, 0))}_{\text{open}} \cup \underbrace{(\mathbb{R}^2 \times (0, \infty))}_{\text{open}}$ hence open.

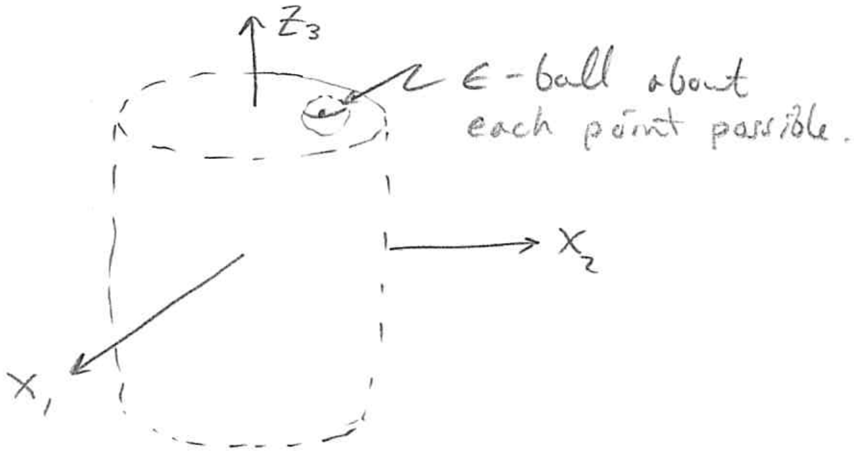
(c.) $P_1 = P_2 \neq P_3$





this is a plane with a line deleted.
it is not open in $\mathbb{R}^3$. Every ϵ-ball contains points outside set. No interior points.

(d.) $\underbrace{P_1^2 + P_2^2 < 9}$
fuzzy cylinder is open



(11) Show $\nabla f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} U_i$, Umm... isn't this the def$^{\underline{n}}$?

Maybe he wants us to say,

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x} \underbrace{\langle 1,0,0 \rangle}_{U_1} + \frac{\partial f}{\partial y} \underbrace{\langle 0,1,0 \rangle}_{U_2} + \frac{\partial f}{\partial z} \underbrace{\langle 0,0,1 \rangle}_{U_3}.$$

Oh, nevermind, turn page to 52

p. 52 § 21

#11 continued

(a) $V[f] = df(V) = V \cdot (\nabla f)(P)$ for any $V \in T_p\mathbb{R}^3$

$$V[f] = V_1 \frac{\partial f}{\partial x} + V_2 \frac{\partial f}{\partial y} + V_3 \frac{\partial f}{\partial z}$$

previous Chpt. 1 work. See Ex. 5.3 pg. 24 if you don't believe me.

$$= (V_1, V_2, V_3)_p \cdot (\nabla f)_p$$

$$= V \cdot (\nabla f)(P).$$

(b.) $\|\nabla f(P)\| = \sqrt{(\nabla f)\cdot(\nabla f)} = \sqrt{\sum_i \left(\frac{\partial f}{\partial x_i}(P)\right)^2}$

Clearly $u = \widehat{\nabla f(P)} = \frac{\nabla f(P)}{\|\nabla f(P)\|}$ gives $\|\nabla f(P)\|$ max. value.

$$u[f] = \frac{\nabla f(P)}{\|\nabla f(P)\|} \cdot (\nabla f)(P) = \frac{\nabla f(P) \cdot \nabla f(P)}{\|\nabla f(P)\|} = \frac{\|\nabla f(P)\|^2}{\|\nabla f(P)\|} = \|\nabla f(P)\|.$$

(12) Angle Functions: Let $f, g \in C^\infty(I)$, $I \subseteq \mathbb{R}$, also $\&$ $f^2 + g^2 = 1$ and $\vartheta_0 \in \mathbb{R}$ such that

$$f(0) = \cos\vartheta_0 \quad \& \quad g(0) = \sin\vartheta_0$$

Define $\vartheta = \vartheta_0 + \int_0^t (fg' - gf')\,du$. Prove $f = \cos\vartheta$, $g = \sin\vartheta$.

Consider $(f - \cos\vartheta)^2 + (g - \sin\vartheta)^2 = h$

$$h' = 2(f - \cos\vartheta)(f' + \sin\vartheta\,\vartheta') + 2(g - \sin\vartheta)(g' - \cos\vartheta\,\vartheta')$$

But $\vartheta' = \frac{d}{dt}\int_0^t (fg' - gf')\,du = fg' - gf'$ at t.

(JUMP!) It follows that $h' = 0$ but $h(0) = 0$ thus $h(t) = 0$ $\forall t \in I$ and it follows that $f = \cos\vartheta$ and $g = \sin\vartheta$

§2.2 pg. 57 H29

(1) Let $\alpha(t) = (2t, t^2, t^3/3)$

(a.) $\alpha'(t) = (2, 2t, t^2)_{\alpha(t)}$ $\quad\quad$ $\alpha'(1) = (2,2,1)_{(2,1,1/3)}$

velocity

$\|\alpha'(t)\| = \sqrt{4+4t^2+t^4}$ $\quad\quad$ $\|\alpha'(1)\| = \sqrt{9} = 3.$

speed

$\alpha''(t) = (0, 2, 2t)_{\alpha(t)}$ $\quad\quad$ $\alpha''(1) = (0,2,2)_{(2,1,1/3)}$

acceleration.

(b.) $$s(t) = \int_0^t \sqrt{u^4+4u^2+4}\,du$$
$$= \int_0^t \sqrt{(u^2+2)^2}\,du$$
$$= \int_0^t (u^2+2)\,du$$
$$= \boxed{\tfrac{1}{3}t^3 + 2t = s(t)}$$

$$\Delta s = s(1) - s(-1) = \tfrac{1}{3} + 2 - (-\tfrac{1}{3} - 2) = \tfrac{2}{3} + 4 = \boxed{\tfrac{14}{3}}$$

(2.) Show constant speed $\Leftrightarrow \alpha''(t)\cdot\alpha'(t) = 0 \quad \forall t.$

Observe, $\|\alpha'(t)\|^2 = \alpha'(t)\cdot\alpha'(t).$

If $\|\alpha'(t)\| = c \quad \forall t \in I \Rightarrow \alpha''(t)\cdot\alpha'(t) + \alpha'(t)\cdot\alpha''(t) = 0 \quad \forall t \in I$
$\Rightarrow \alpha''(t)\cdot\alpha'(t) = 0 \quad \forall t \in I.$

If $\alpha''(t)\cdot\alpha'(t) = 0 \quad \forall t \Rightarrow \alpha''(t)\cdot\alpha'(t) + \alpha'(t)\cdot\alpha''(t) = 0 \quad \forall t \in I$
$\Rightarrow \frac{d}{dt}[\alpha'(t)\cdot\alpha'(t)] = 0 \quad \forall t \in I$
$\Rightarrow \|\alpha'(t)\|^2 = c \quad \forall t \in I$
$\Rightarrow \|\alpha'(t)\| = \sqrt{c} \quad \forall t \in I.$

(3.) $\alpha(t) = (\cosh t, \sinh t, t)$ find $s(t)$. $\quad\quad$ $\cosh^2 t - \sinh^2 t = 1$

$\alpha'(t) = (\sinh t, \cosh t, 1) \Rightarrow \|\alpha'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t}$

Thus $s(t) = \int_0^t \sqrt{2}\cosh(u)\,du = \boxed{\sqrt{2}\sinh(t) = s(t)}$

(3 continued) $s = \sqrt{2}\sinh t \Rightarrow t = \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)$

$$\underbrace{\bar{\alpha}(s) = \left(\cosh\left(\sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right), \frac{s}{\sqrt{2}}, \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right)}_{\text{unit-speed reparametrization of } \alpha.}$$

(4) I skip.

(5.) Suppose β_1, β_2 are unit-speed reparametrizations of the same curve α. Show $\exists\, s_o$ such that $\beta_2(s) = \beta_1(s+s_o)$ $\forall s$. What is the geometric significance of s_o?

Let $I = \text{dom}\,\alpha$. By assumption $\exists h_1 : I \to J_1$, $h_2 : I \to J_2$ for which $\beta_1(h_1(t)) = \alpha(t)$ and $\beta_2(h_2(t)) = \alpha(t)$

By chain-rule, $\alpha'(t) = \beta_1'(h_1(t))\frac{dh_1}{dt} = \beta_2'(h_2(t))\frac{dh_2}{dt}$

$$\Rightarrow \left\|\beta_1'(h_1(t))\frac{dh_1}{dt}\right\| = \left\|\beta_2'(h_2(t))\frac{dh_2}{dt}\right\|$$

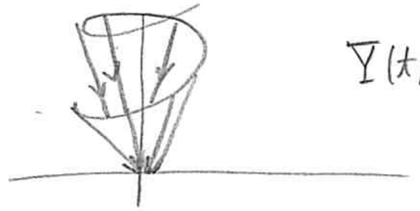
$$\Rightarrow \left|\frac{dh_1}{dt}\right| \overbrace{\cancel{\|\beta_1'(h_1(t))\|}}^{1} = \left|\frac{dh_1}{dt}\right| \overbrace{\cancel{\|\beta_2'(h_2(t))\|}}^{1} \quad \beta_1, \beta_2 \text{ unit-speed,}$$

Therefore, $\left|\frac{dh_1}{dt}\right| = \left|\frac{dh_2}{dt}\right|$. But, we assume h_1, h_2 increasing fncts thus $\frac{dh_1}{dt}, \frac{dh_2}{dt} > 0$ and it follows $\frac{dh_1}{dt} = \frac{dh_2}{dt}$ $\forall t \in I$. If I connected then we find $h_2 = h_1 + s_o$ for some $s_o \in \mathbb{R}$. Thus $\beta_1(h_1(t)) = \beta_2(h_1(t) + s_o)$ hence $\beta_1(s) = \beta_2(s+s_o)$ $\forall s \in J_1$.

Remark: If $\frac{dh_1}{dt}, \frac{dh_2}{dt}$ could be negative then $\exists$ a few more choices to relate β_1, β_2 ...

(6.) Let Y be a vector field on the helix $\alpha(t) = (\cos t, \sin t, t)$. Express Y in terms of the natural cartesian frame.

(a.) $Y(t)$ is vector from $\alpha(t)$ to the origin.



$$Y(t) = -\alpha(t)_{\alpha(t)}$$
$$= -\alpha_1 U_1(\alpha(t)) - \alpha_2 U_2(\alpha(t)) - \alpha_3 U_3(\alpha(t))$$
$$= -\alpha_1 U_1 - \alpha_2 U_2 - \alpha_3 U_3.$$

omit whatever you like here... notation so annoying...

$$= \boxed{-\cos t\, U_1 + \sin t\, U_2 + t\, U_3}$$

(b.) $Y(t) = \alpha'(t) - \alpha''(t)$

$$= (-\sin t, \cos t, 1)_p - (-\cos t, -\sin t, 0)_p \leftarrow p = (\cos t, \sin t, t)$$
$$= \boxed{(\cos t - \sin t)U_1 + (\cos t + \sin t)U_2 + U_3} \quad (\text{at } p)$$

(c.) $Y(t)$ with unit-length $\perp$ to $\alpha'(t)$ & $\alpha''(t)$. Natural approach, $Y(t)$ is normalized $\alpha' \times \alpha''$.

(d.) $Y(t)$ is vector from $\alpha(t)$ to $\alpha(t+\pi)$

$$Y(t) = \alpha(t+\pi) - \alpha(t) \quad \text{at } P = \alpha(t).$$
$$= (\cos(t+\pi), \sin(t+\pi), t+\pi) - (\cos t, \sin t, t) \text{ at } P,$$
$$= (\cos t \cos\pi, \sin t \cos\pi, t+\pi) - (\cos t, \sin t, t) \text{ at } P,$$
$$= (-2\cos t, -2\sin t, \pi)_P$$
$$= -2\cos t\, U_1 - 2\sin t\, U_2 + \pi U_3 \quad \text{at } P = (\cos t, \sin t, t).$$

(7.) This problem is simply u-subst. Th$^{\underline{m}}$ applied to arclength integral.

p. 58 §2.2 (H32)

(8) Let Y be vector field on α.

Let $\alpha(h)$ be a reparametrization of α.

Show $Y(h)$ is a vector field on $\alpha(h)$ and prove $Y(h)' = h' Y'(h)$

Note, $Y(h)(t) = Y(h(t)) \in T_{\alpha(h(t))}\mathbb{R}^3$ and $\alpha(h(t))$ is on $\alpha(h)$ so it gives us a vector at each $\alpha(h(t))$ thus $Y(h)$ is a vector field along the curve $\alpha(h)$.

$$Y(h(t)) = \sum_{i=1}^{3} y_i(h(t))\, U_i(\alpha(h(t)))$$

$$\frac{d}{dt}\big(Y(h(t))\big) = \sum_{i=1}^{3} \frac{dy_i}{dt}\Big|_{h(t)} \frac{dh}{dt} U_i(\alpha(h(t))) = \frac{dh}{dt} Y'(h(t))$$

(10.) Let $\alpha, \beta : I \to \mathbb{R}^3$ be curves s.t $\alpha'(t) \parallel \beta'(t)$ $\forall t \in I$.

Prove α, β are "parallel" in the sense $\beta(t) = \alpha(t) + p$ $\forall t$.

Remark: I would say α & β are related by a translation See Chapter 3.

If $\alpha'(t) \parallel \beta'(t)$ then $\exists\, k \in \mathbb{R}$ such that

$\beta'(t) = k\alpha'(t) \Rightarrow \frac{d\beta_i}{dt} = k\frac{d\alpha_i}{dt}$ well, this is not enough, I believe he indicates $k=1$ then the result is almost immediate.

$$\frac{d\beta_i}{dt} = \frac{d\alpha_i}{dt} \quad \forall t \in I \Rightarrow \beta_i = \alpha_i + p_i \text{ for } i = 1,2,3$$

$$\Rightarrow \underline{\beta = \alpha + p}.$$

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(11) skip, but, interesting.

(2.) Consider $\beta(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right)$ for $s \in (-1,1)$.
Show β is unit-speed and calculate κ, τ, T, N, B for β.

$$\beta'(s) = \left(\tfrac{1}{2}(1+s)^{1/2}, -\tfrac{1}{2}(1-s)^{1/2}, \tfrac{1}{\sqrt{2}}\right) = T(s).$$

$$\|\beta'(s)\|^2 = \tfrac{1}{4}(1+s) + \tfrac{1}{4}(1-s) + \tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{2} = 1. \quad \text{(unit-speed ✓)}$$

$$\beta''(s) = \left(\tfrac{1}{4}(1+s)^{-1/2}, \tfrac{1}{4}(1-s)^{-1/2}, 0\right)$$

$$\|\beta''(s)\| = \sqrt{\frac{1}{16}\left(\frac{1}{1+s} + \frac{1}{1-s}\right)} = \frac{1}{4}\sqrt{\frac{1-s+1+s}{1-s^2}} = \frac{1}{\sqrt{8(1-s^2)}}$$

Thus $\boxed{\kappa(s) = \dfrac{1}{\sqrt{8(1-s^2)}}}$ and $N(s) = \sqrt{8(1-s^2)}\;\beta''(s)$.

oh, to be explicit,

$$N(s) = \left(\frac{1}{4}\sqrt{\frac{8(1-s^2)}{1+s}}, \frac{1}{4}\sqrt{\frac{8(1-s^2)}{1-s}}, 0\right)$$

$$\boxed{N(s) = \left(\frac{1}{\sqrt{2}}\sqrt{1-s}, \frac{1}{\sqrt{2}}\sqrt{1+s}, 0\right)}$$ better f-la.

Next, the dreaded binormal,

$$B = T \times N = \left(\tfrac{1}{2}\sqrt{1+s}, -\tfrac{1}{2}\sqrt{1-s}, \tfrac{1}{\sqrt{2}}\right) \times \left(\sqrt{\tfrac{1-s}{2}}, \sqrt{\tfrac{1+s}{2}}, 0\right)$$

$$= \left(-\tfrac{1}{2}\sqrt{1+s}, \tfrac{1}{2}\sqrt{1-s}, \tfrac{1}{2\sqrt{2}}(1+s) + \tfrac{1}{2\sqrt{2}}(1-s)\right)$$

$$= \boxed{\left(-\tfrac{1}{2}\sqrt{1+s}, \tfrac{1}{2}\sqrt{1-s}, \tfrac{1}{\sqrt{2}}\right) = B(s)}$$

As a check on our work, $\|B\| = \frac{1}{4}(1+s+1-s) + \frac{1}{2} = 1.$
Finally the torsion, $\tau = -N \cdot \frac{dB}{ds}$,

$$\frac{dB}{ds} = \left(-\tfrac{1}{4}(1+s)^{-1/2}, -\tfrac{1}{4}(1-s)^{-1/2}, 0\right)$$

$$-N \cdot \frac{dB}{ds} = \frac{1}{4\sqrt{2}}\sqrt{\frac{1-s}{1+s}} + \frac{1}{4\sqrt{2}}\sqrt{\frac{1+s}{1-s}} = \frac{1}{4\sqrt{2}}\left(\frac{1-s+1+s}{\sqrt{1-s^2}}\right) = \boxed{\frac{1}{\sqrt{8(1-s^2)}} = \tau}$$

Remark: the previous problem has $\tau/\kappa = 1 \neq 0$ $\therefore$ β of (2.) (in §2.3) is a cylindrical helix (see pg. 77 or Th$^{\underline{m}}$ 4.6 p. 76)

(3.) I leave to others.

(4.) Prove that

$$T = N \times B = -B \times N \quad \text{(I)}$$
$$N = B \times T = -T \times B \quad \text{(II)}$$
$$B = T \times N = -N \times T \quad \text{(III)}$$

First the RHS follows from LHS of (I), (II) and (III) by $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. Next, (III) is the def$^{\underline{n}}$ of $B = T \times N$ so we're done with that. Consider $N \times B = c_1 T + c_2 N + c_3 B$ as $\{T, N, B\}$ forms basis. Notice $c_2 = 0$ and $c_3 = 0$ as $(N \times B) \cdot B = (N \times B) \cdot N = 0$ but, $(N \times B) \cdot T = c_1 T \cdot T = c_1$ and as $N \cdot B = 0$ it follows that $\|N \times B\| = 1$ $\therefore$ $c_1 = 1$ and so $\underline{T = N \times B}$. Similar argument shows (II) to be true.

(5.) If $A = \tau T + \kappa B$ for unit-speed β show that Frenet's f-la's yield $T' = A \times T$, $N' = A \times N$, $B' = A \times B$

$$A \times T = (\tau T + \kappa B) \times T = \kappa B \times T = \kappa N = T',$$
$$A \times N = (\tau T + \kappa B) \times N = \tau (T \times N) + \kappa (B \times N) = \tau B - \kappa T = N',$$
$$A \times B = (\tau T + \kappa B) \times B = \tau (T \times B) = -\tau N = B'.$$

Seems like A ought to have a name.

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(6.) $\gamma(s) = c + r\cos\left(\frac{s}{r}\right)e_1 + r\sin\left(\frac{s}{r}\right)e_2$ where $e_i \cdot e_j = \delta_{ij}$

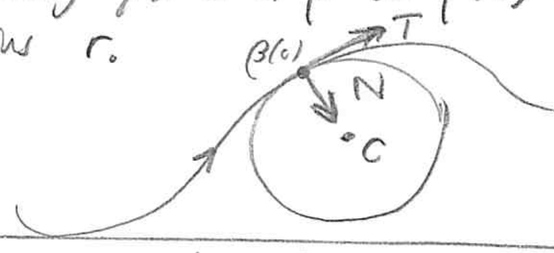
$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$

unit-speed parametrization of circle.

If β is unit-speed with $\kappa(0) > 0$, prove $\exists!$ circle γ which approximates β near $\beta(0)$ in the sense

$$\gamma(0) = \beta(0), \quad \gamma'(0) = \beta'(0), \quad \gamma''(0) = \beta''(0).$$

Show also γ lies in osculating plane of β at $\beta(0)$ and find its center c and radius r.



We can construct $\gamma(s)$ using the Frenet-frame at $\beta(0)$. Notice $N(0)$ points to c. Also, we saw $\kappa = 1/R$ for circle so use $r = 1/\kappa(0)$ for radius of circle. Thus, $c = \beta(0) + \frac{1}{\kappa(0)}N(0)$. Put it together,

$$\boxed{\begin{gathered}\gamma(s) = c - r\cos\left(\tfrac{s}{r}\right)N(0) + r\sin\left(\tfrac{s}{r}\right)T(0) \\ \text{with } c = \beta(0) + rN(0) \text{ and } r = 1/\kappa(0)\end{gathered}}$$

Let's check our construction,

$$\gamma(0) = c - rN(0) = \beta(0) + rN(0) - rN(0) \overset{\checkmark}{=} \beta(0).$$

$$\gamma'(0) = r\tfrac{1}{r}\cos\left(\tfrac{0}{r}\right)T(0) = T(0) \overset{\checkmark}{=} \beta'(0) \quad \left(\begin{gathered}\text{as } \beta \text{ is} \\ \text{unit-speed}\end{gathered}\right)$$

$$\gamma''(0) = \tfrac{1}{r}\cos(0)N(0) = \kappa(0)N(0) \overset{\checkmark}{=} \beta''(0).$$

To see γ is in plane with normal $B(0)$ observe,

$$(\gamma(s) - c)\cdot B(0) = 0 \quad \text{as } N(0)\cdot B(0) = T(0)\cdot B(0) = 0$$

Moreover, $\|\gamma(s) - c\| = r$ as you can easily verify.

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H36

(7, 8, 9) shipped.

(10) Let α be unit-speed with $\kappa > 0$, $\tau \neq 0$.

(a.) If α lies on sphere of center c and radius r, show that $\alpha - c = -\rho N - \rho' \sigma B$ where $\rho = 1/\kappa$, $\sigma = 1/\tau$. Thus, $r^2 = \rho^2 + (\rho'\sigma)^2$ (this follows from by orthonormality of T, N, B).

$$(\alpha - c)\cdot(\alpha - c) = r^2 \Rightarrow \alpha' \cdot (\alpha - c) = 0$$

$$\Rightarrow \underline{\alpha - c = c_1 N + c_2 B} \quad \text{as show the T-comp. is trivial.}$$

We need to show $c_1 = 1/\kappa$ and $c_2 = 1/\tau$.

By othonormality of T, N, B, $c_1 = (\alpha - c)\cdot N$
$c_2 = (\alpha - c)\cdot B$

Curvature and torsion are related to α'' and α''' so we best differentiate again to find c_1 & c_2.

$$\alpha' \cdot (\alpha - c) = 0 \Rightarrow \alpha'' \cdot (\alpha - c) + \alpha' \cdot \alpha' = 0$$

$$\Rightarrow \kappa N \cdot (\alpha - c) + 1 = 0$$

$$\Rightarrow \kappa c_1 = -1 \quad \therefore \quad \underline{c_1 = -1/\kappa}.$$

Differentiate again,

$$\alpha''' \cdot (\alpha - c) + \alpha'' \cdot \alpha' + 2\alpha'' \cdot \alpha = 0$$

$$\alpha''' \cdot (\alpha - c) + 3\kappa N \cdot T \overset{0}{=} 0 \quad \star$$

Well, this may give c_2 easily, but, w/o help from §2.4 I'm a bit lost at $\star$. Instead, try, differentiating

$$\alpha' = \frac{d}{ds}\left(\frac{-1}{\kappa} N + c_2 B\right) = \frac{1}{\kappa^2}\frac{d\kappa}{ds} N - \frac{1}{\kappa}\frac{dN}{ds} + \frac{dc_2}{ds} B + c_2 \frac{dB}{ds}$$

$$\Rightarrow T = \frac{-1}{\kappa^2}\frac{d\kappa}{ds} N - \frac{1}{\kappa}(-\kappa T + \tau B) + \frac{dc_2}{ds} B - c_2 \tau N$$

$$\Rightarrow 0 = \left(\frac{1}{\kappa^2}\frac{d\kappa}{ds} - c_2\tau\right) N + \left(\frac{-\tau}{\kappa} + \frac{dc_2}{ds}\right) B$$

(10) <u>continued</u> (pg. 68-69)

$$C_2\tau = \frac{1}{\kappa^2}\frac{d\kappa}{ds} \Rightarrow C_2 = \frac{-1}{\tau}\underbrace{\left(\frac{-1}{\kappa^2}\frac{d\kappa}{ds}\right)}_{\rho' = \frac{d}{ds}\left(\frac{1}{\kappa}\right).} = -\rho'\sigma B$$

Thus, as claimed,

$$\boxed{\alpha - C = -\rho N - \rho'\sigma B \quad \text{with } \rho = \frac{1}{\kappa},\ \sigma = \frac{1}{\tau}}$$

(6.) Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$ show that α lies on sphere of radius r.

Consider curve α with $\rho^2 + (\rho'\sigma)^2 = r^2$ and $\rho' \neq 0$.
As hinted, consider the "center curve", show the following is constant

$$\gamma = \alpha + \rho N + \rho'\sigma B$$

$$\begin{aligned}
\gamma' &= \alpha' + \rho' N + \rho N' + \rho''\sigma B + \rho'\sigma' B + \rho'\sigma B' \\
&= T + \rho' N + \rho(-\kappa T + \tau B) + \rho''\sigma B + \rho'\sigma' B + \rho'\sigma(-\tau N) \\
&= T\underbrace{(1 - \rho\kappa)}_{0} + N\underbrace{(\rho' - \rho'\sigma\tau)}_{*} + B\underbrace{(\rho\tau + \rho''\sigma + \rho'\sigma')}_{**}
\end{aligned}$$

Notice $\rho^2 + (\rho'\sigma)^2 = r^2 \Rightarrow 2\rho\rho' + 2\rho'\sigma(\rho''\sigma + \rho'\sigma') = 0$

As $\rho' \neq 0$ we divide by ρ' $\Rightarrow \rho + \sigma(\rho''\sigma + \rho'\sigma') = 0$

$$\Rightarrow \underline{\rho\tau + \rho''\sigma + \rho'\sigma' = 0} \quad **$$

It remains to show $*$ is zero. But,

$$\rho' - \rho'\sigma\tau = \rho'(1 - \sigma\tau) = \rho'\left(1 - \frac{1}{\tau}\tau\right) = 0.$$

Hence $\gamma' = 0 \Rightarrow \gamma(s) = C \quad \forall s$

$$\Rightarrow \underline{\alpha(s) = C - \rho N - \rho'\sigma B}$$

Thus $\|\alpha(s) - C\|^2 = \|-\rho N - \rho'\sigma B\|^2 = \rho^2 + (\rho'\sigma)^2 = r^2.$

<u>Remark</u>: ouch.