pg. 77 §2.4

(1.) Let $\alpha(t) = (2t, t^2, t^3/3)$

$\alpha'(t) = (2, 2t, t^2)$, $\alpha''(t) = (0, 2, 2t)$, $\alpha'''(t) = (0,0,2)$

$\|\alpha'(t)\| = \sqrt{t^4+4t^2+4} = \sqrt{(t^2+2)^2} = \underline{t^2+2}$

$\therefore$ $\boxed{T(t) = \dfrac{1}{t^2+2}(2, 2t, t^2)}$

I'm ~~x~~ following Example 4.4 to better appreciate the approach of §2.4.

$$\begin{aligned}\alpha' \times \alpha'' &= (2, 2t, t^2) \times (0, 2, 2t) \\ &= (4t^2 - 2t^2, -4t, 4) \\ &= (2t^2, -4t, 4)\end{aligned}$$

$\|\alpha' \times \alpha''\| = \sqrt{4t^4+16t^2+16} = \sqrt{4(t^4+4t^2+4)} = 2(t^2+2)$

Hence, $\boxed{B(t) = \dfrac{1}{t^2+2}(t^2, -2t, 2)}$. Next,

$$\begin{aligned}(\alpha' \times \alpha'') \cdot \alpha''' &= 2(\alpha' \times \alpha'')_z \\ &= 2(2(2) - 2t(0)) \\ &= 8\end{aligned}$$

Thus,

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} = \frac{2(t^2+2)}{(t^2+2)^3} = \boxed{\frac{2}{(t^2+2)^2} = \kappa(t)}$$

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2} = \frac{8}{[2(t^2+2)]^2} = \boxed{\frac{2}{(t^2+2)^2} = \tau(t)}$$

And finally,

$$\begin{aligned}N = B \times T &= \frac{1}{(t^2+2)^2}(t^2, -2t, 2) \times (2, 2t, t^2) \\ &= \frac{1}{(t^2+2)^2}(-2t^3-4t^2, 4-t^4, 2t^3+4t) \\ &= \frac{1}{(t^2+2)^2} \cdot (t^2+2) \cdot (-2t, 2-t^2, 2t) \\ &= \boxed{\frac{1}{t^2+2}(-2t, 2-t^2, 2t) = N(t)}\end{aligned}$$

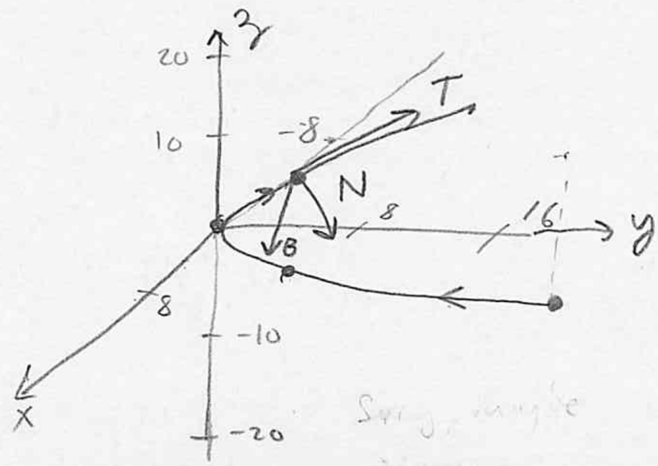
(1b) $x = 2t$, $y = t^2$, $z = t^3/3$ for $-4 \le t \le 4$ sketch and illustrate T, N, B at $t = 2$. Let's see,

$$t^2 = \left(\frac{x}{2}\right)^2 = y \Rightarrow y = \frac{1}{4}x^2$$

$$t^3 = \left(\frac{x}{2}\right)^3 = 3z \Rightarrow z = \frac{1}{24}x^3$$

it's on intersection of these surfaces.

t	x	y	z
-4	-8	16	$-64/3$
2	-4	4	$-8/3$
0	0	0	0
2	4	4	$8/3$
4	8	16	$64/3$



(1c) $\lim_{t \to \pm\infty} (T(t)) = \lim_{t \to \pm\infty} \left(\frac{2}{t^2+2}, \frac{2t}{t^2+2}, \frac{t^2}{t^2+2}\right) = (0,0,1).$

$\lim_{t \to \pm\infty} (N(t)) = \lim_{t \to \pm\infty} \left(\frac{-2t}{t^2+2}, \frac{2-t^2}{2+t^2}, \frac{2t}{2+t^2}\right) = (0,-1,0).$

$\lim_{t \to \pm\infty} (B(t)) = \lim_{t \to \pm\infty} \left(\frac{t^2}{t^2+2}, \frac{-2t}{t^2+2}, \frac{2}{t^2+2}\right) = (1,0,0).$

---

(2) Let $\alpha(t) = (\cosh t, \sinh t, t)$ find κ, τ as fncts of arclength s. (measure s from $t=0$)

$$\alpha'(t) = (\sinh t, \cosh t, 1)$$
$$\alpha''(t) = (\cosh t, \sinh t, 0)$$
$$\alpha'''(t) = (\sinh t, \cosh t, 0)$$

Note, $\cosh^2 t - \sinh^2 t = 1 \Rightarrow \cosh^2 t = 1 + \sinh^2 t$. Thus,

$$\|\alpha'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$$

Thus $s(t) = \int_0^t \|\alpha'(u)\| du = \int_0^t \sqrt{2}\cosh(u) du = \underline{\sqrt{2}\sinh t = s}.$

(2 continued)

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}$$

Calculate $\alpha' \times \alpha'' = (\sinh t, \cosh t, 1) \times (\cosh t, \sinh t, 0)$

$= (\sinh t, \cosh t, \cosh^2 t - \sinh^2 t)$

$= (\sinh t, \cosh t, 1)$

Thus, $\|\alpha' \times \alpha''\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$

We found $\|\alpha'\| = \sqrt{2}\cosh t$ thus,

$$\kappa = \frac{\sqrt{2}\cosh t}{(\sqrt{2}\cosh t)^3} = \frac{1}{2\cosh^2 t}$$

$$= \frac{1}{2(1+\sinh^2 t)}$$

$$= \frac{1}{2 + (\sqrt{2}\sinh t)^2}$$

$$= \boxed{\frac{1}{2+s^2} = \kappa(s)}$$

Now, lets use the nice §2.4 formula for torsion,

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2} = \frac{(\sinh t, \cosh t, 1) \cdot (\sinh t, \cosh t, 0)}{2\cosh^2 t}$$

$$= \frac{\sinh^2 t + \cosh^2 t}{2\cosh^2 t}$$

$$= \frac{1 + 2\sinh^2 t}{2 + 2\sinh^2 t}$$

$s^2 = 2\sinh^2 t$

$$= \boxed{\frac{1+s^2}{2+s^2} = \tau(s)}$$

(4.) Show for a regular curve the curvature is given by

$$\kappa^2 v^4 = \|\alpha''\|^2 - \left(\frac{dv}{dt}\right)^2$$

This follows almost immediately from Lemma 4.2,

$$\alpha'' = \frac{dv}{dt}T + \kappa v^2 N$$

$$\alpha'' \cdot \alpha'' = \left(\frac{dv}{dt}T + \kappa v^2 N\right) \cdot \left(\frac{dv}{dt}T + \kappa v^2 N\right) = \left(\frac{dv}{dt}\right)^2 + \kappa^2 v^4$$

By orthonormality of N, T thus,

$$\|\alpha''\|^2 = \left(\frac{dv}{dt}\right)^2 + \kappa^2 v^4.$$

<u>Remark</u>: we see this in PHYSICS 231 as $a^2 = a_T^2 + a_c^2$ where a_T = tangential acceleration and a_c = centripetal acceleration. (our N is center-seeking)

(6a) If α is a cylindrical helix then prove its unit vector $\hat{u}$ is given by $\hat{u} = \left(\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\right)T + \left(\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)B$

Def^{n}/ α is cylindrical helix iff $\exists$ constant $\hat{u}$ s.t. $T(t) \cdot \hat{u} = \cos\theta_o$ for some θ_o fixed for all t. From Lemma 4.2

$$\alpha' = vT \Rightarrow T = \frac{\alpha'}{v} \Rightarrow \alpha' \cdot \hat{u} = v\cos\theta_o$$

$$\Rightarrow \alpha'' \cdot \hat{u} = \frac{dv}{dt}\cos\theta_o$$

fun, but I must go on.

(actually I solved this while attempting §2.4 # 9)

§2.4 p. 78 (THIS CONTAINS SOLUTION TO #6, HOWEVER #9 REMAINS A MYSTERY CURRENTLY)

(9.) If α is curve with $\kappa > 0$ and κ, τ constant then show α is a circular helix

Th^m 4.6 reveals that as τ/κ is constant α is a cylindrical helix. It remains to show that cylinder is a circular cylinder. We know by Th^m 4.6, $\exists\, u$ s.t $T(t) \cdot u = \cos\theta_o \ \forall t$.

Observe, $T' \cdot u = 0 \Rightarrow \dfrac{T'}{\|T'\|} \cdot u = 0 \Rightarrow N \cdot u = 0.$

But, $u = c_1 T + c_2 N + c_3 B$ and we already know $c_1 = \cos\theta_o$ and $c_2 = 0$ hence $c_3 = \sin\theta_o$ as $u \cdot u = 1$.

Hence, $u = \cos\theta_o T + \sin\theta_o B$ for all t.

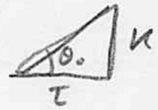
$$\Rightarrow \cos\theta_o \frac{dT}{dt} = -\sin\theta_o \frac{dB}{dt}$$

$$\Rightarrow \cos\theta_o \kappa v N = -\sin\theta_o(-\tau v N)$$

$$\Rightarrow \kappa v \cos\theta_o = \tau v \sin\theta_o$$

$$\Rightarrow \tan\theta_o = \kappa/\tau$$

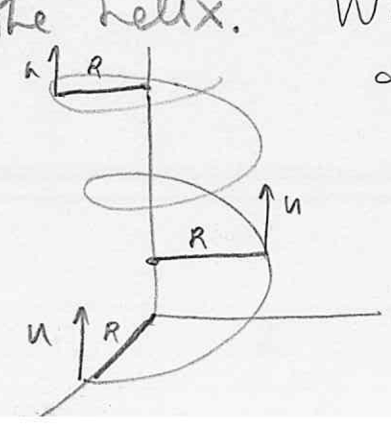
by Lemma 4.1 the Frenet-Serret non-unit speed version.

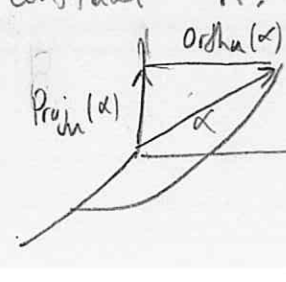


Funny, I just solved (6) by accident,

$$u = \cos\theta_o T + \sin\theta_o B = \frac{\tau}{\sqrt{\tau^2+\kappa^2}} T + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}} B.$$

We need a criteria to capture "circularity" of the helix. We need the distance from $\alpha(t)$ to axis $\parallel$ to u to be constant R.





$\text{Orth}_u(\alpha) = \alpha - (\alpha \cdot u)u$

$\|\text{Orth}_u(\alpha(t))\| = R \ \forall t$. Show this

(9) continued

$$\|\alpha - (\alpha \cdot u)u\|^2 = \alpha \cdot \alpha - 2(\alpha \cdot u)^2 + (\alpha \cdot u)^2$$

$$\Rightarrow \quad f = \alpha \cdot \alpha - (\alpha \cdot u)^2 .$$

$$\frac{df}{dt} = 2\alpha \cdot \frac{d\alpha}{dt} - 2(\alpha \cdot u)\frac{d}{dt}(\alpha \cdot u)$$

$$= 2\alpha \cdot \frac{d\alpha}{dt} - 2(\alpha \cdot u)\left(u \cdot \frac{d\alpha}{dt}\right)$$

$$= 2\left[\alpha \cdot (VT) - \alpha \cdot (uV\cos\theta_0)\right]$$

$$= 2V\alpha \cdot (T - u\cos\theta_0)$$

$$\left| \begin{array}{l} \alpha' = VT \\ T \cdot u = \cos\theta_0 \\ u \cdot \alpha' = V\cos\theta_0 \end{array} \right.$$

$\vdots$

I HAVE $\approx$ 10 MORE PAGES, BUT, IT'S GARBAGE UNTIL I SEE THE PATH CLEARLY.

Remark: H44a → H44? reserved for §2.4 problems. (9, 10)

§2.4 #17 Total curvature of unit-speed curve $\alpha : I \to \mathbb{R}^3$ is $\int_I \kappa(s)\,ds$. For regular, possibly non-unit-speed, $\int_I \kappa(t) v(t)\,dt$.

(a.) $\alpha(t) = (3t - t^3,\ 3t^2,\ 3t + t^3)$

$\alpha'(t) = (3 - 3t^2,\ 6t,\ 3 + 3t^2) = 3(1 - t^2,\ 2t,\ 1 + t^2)$

$\|\alpha'(t)\| = 3\sqrt{(1-t^2)^2 + 4t^2 + (1+t^2)^2} = 3\sqrt{2t^4 + 4t^2 + 2} = \sqrt{18(1 + 2t^2 + t^4)}$

$\alpha''(t) = 3(-2t,\ 2,\ 2t) = 6(-t,\ 1,\ t)$ $\qquad \underline{v = \sqrt{18}\,(t^2+1)}$.

following pg. 73, $\kappa = \dfrac{1}{3(1+t^2)^2}$. Thus,

$$\int_I \frac{\sqrt{18}\,(t^2+1)\,dt}{3\,(1+t^2)^2} = \int_I \frac{\sqrt{2}\,dt}{1+t^2} = \int_{-\infty}^{\infty} \frac{\sqrt{2}\,dt}{1+t^2} = \sqrt{2}\tan^{-1}(t)\Big|_{-\infty}^{\infty}$$

$$= \sqrt{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \boxed{\pi\sqrt{2}}$$

If $I = [0, \infty)$ then I'd obtain $\pi/\sqrt{2}$ as the ~~text~~ text found.

I'll leave b, c, d to you... the point of this problem was simply to make you aware of the concept of total curvature.

1b) Consider $V=(1,-1,2)$ at $(1,3,-1)$ compute $\nabla_V W$ from def$^{\underline{n}}$ where $W = x U_1 + x^2 U_2 - z^2 U_3$

$$W(p+tV) = W(\underbrace{1+t}_{x}, \underbrace{3-t}_{y}, \underbrace{-1+2t}_{z}) = (1+t)^2 U_1 + (1+t)^2 U_2 - (2t-1)^2 U_3$$

$$\frac{d}{dt}\Big[W(p+tV)\Big]\Big|_{t=0} = 1\cdot U_1 + 2(1+0)U_2 - 4(2(0)-1)U_3$$

$$\therefore \boxed{(\nabla_V W)(p) = U_1 + 2U_2 + 4U_3 = \langle 1,2,4\rangle \text{ at } p}$$

2.) $V = -yU_1 + xU_3$, $W = \cos x\, U_1 + \sin x\, U_2$

(a.)
$$\begin{aligned} \nabla_V W &= V(w_1)U_1 + V(w_2)U_2 + V(w_3)U_3 \\ &= (-yU_1 + xU_3)[\cos(x)]U_1 + (-yU_1 + xU_3)[\sin(x)]U_2 \\ &= \underline{y\sin(x)\,U_1 - y\cos(x)\,U_2} \end{aligned}$$

(b.)
$$\nabla_V V = (-yU_1 + xU_3)(-y)U_1 + (-yU_1 + xU_3)(x)U_2 = \boxed{-yU_2}$$

(c.)
$$\begin{aligned} \nabla_V(z^2 W) &= \nabla_V\left(z^2\cos(x)U_1 + z^2\sin(x)U_2\right) \\ &= V[z^2\cos(x)]U_1 + V[z^2\sin(x)]U_2 \\ &= \underline{\left(2xz\cos(x) + z^2 y\sin(x)\right)U_1 + \left(2zx\sin(x) + z^2 x\cos(x)\right)U_2} \end{aligned}$$

(e.)
$$\begin{aligned} \nabla_V(\nabla_V W) &= \nabla_V\left[y\sin x\, U_1 - y\cos(x)\, U_2\right] \\ &= V[y\sin(x)]U_1 - V[y\cos(x)]U_2 \\ &= \underline{-y^2\cos(x)\,U_1 + y^2\sin(x)\,U_2} \end{aligned}$$

§2.5 pg. 84

(3.) If $W \in \mathfrak{X}(\mathbb{R}^3)$ with $\|W\| = c$, prove for any $V \in \mathfrak{X}(\mathbb{R}^3)$ the $\nabla_V W$ is everywhere orthogonal to W

$$\begin{aligned}
(\nabla_V W) \cdot W &= \left(\sum_{i=1}^{3} V[w_i]\, U_i\right) \cdot \left(\sum_{j=1}^{3} w_j U_j\right) \\
&= \sum_{i=1}^{3} w_i\, V[w_i] \\
&= \frac{1}{2} \sum_{i=1}^{3} V[w_i^2] \\
&= \frac{1}{2} V\left(\sum_{i=1}^{3} w_i^2\right) \\
&= \frac{1}{2} V\big[\underbrace{\|W\|^2}_{\text{constant.}}\big] \\
&= 0.
\end{aligned}$$

as $\underbrace{V(w_i^2) = 2w_i\, V(w_i)}$

apply $w_i = f$, $w_i = g$ in Th$^{\underline{m}}$ 3.3 (3) on pg. 13 if you doubt me.

(4.) Let $X = \sum_{i=1}^{3} x_i U_i$ show $\nabla_V X = V \quad \forall V \in \mathfrak{X}(\mathbb{R}^3)$

$$\nabla_V X = \sum_{i=1}^{3} V(x_i)\, U_i = \sum_{i=1}^{3} v_i U_i = V.$$

$$\begin{aligned}
V[x_j] &= \left(\sum_{i=1}^{3} v_i U_i\right)[x_j] \\
&= \sum_{i=1}^{3} v_i \underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}} \\
&= v_j
\end{aligned}$$

(5.)

$$\begin{aligned}
\nabla_{\alpha'(t)} W &= \sum_{i=1}^{3} \alpha'(t)[w_i]\, U_i \qquad \alpha'(t) = \sum_{i=1}^{3} \frac{d\alpha_i}{dt} U_i \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{d\alpha_j}{dt} \frac{\partial w_i}{\partial x_j} U_i \\
&= \sum_{i=1}^{3} \frac{d}{dt}\Big((w_i \circ \alpha)(t)\Big) U_i \\
&= (W \circ \alpha)'(t).
\end{aligned}$$

I leave 5b to you...