1.) If $V, W \in \mathfrak{X}(\mathbb{R}^3)$ which are point-wise LI then show that $E_1 = \frac{1}{\|V\|}V$, $E_2 = \frac{1}{\|\tilde{W}\|}\tilde{W}$, $E_3 = E_1 \times E_3$ is a frame field where $\tilde{W} = W - \text{Proj}_V(W) = \text{Orth}_V(W)$.

Observe $E_1 \cdot E_1 = 1$ by construction. Moreover $E_2 \cdot E_2 = 1$ by construction, consider, $V, W \neq 0$ as $\{V, W\}$ is LI at each p,

$$E_1 \cdot E_2 = \frac{1}{\|V\|}\frac{1}{\|\tilde{W}\|} V \cdot \left[W - \left(\frac{W \cdot V}{V \cdot V}\right)V\right] =$$

$$= \frac{1}{\|V\|}\frac{1}{\|\tilde{W}\|}\left(V \cdot W - \frac{(W \cdot V)\cancel{(V \cdot V)}}{\cancel{V \cdot V}}\right)$$

$$= 0.$$

Finally $\|E_3\| = \|E_1\|\|E_2\|\sin 90° = 1$ and $E_1 \cdot E_3 = E_2 \cdot E_3 = 0$ by construction of $E_3 = E_1 \times E_2$. Hence $\{E_1, E_2, E_3\}$ forms a frame field on $\mathbb{R}^3$.

Remark: this was point-wise Gram-Schmit paired with cross-product technique.

2.) Express the vector fields below in terms of
(i.) cylindrical frame with r, θ, z - based coefficients
(ii) spherical frame with ρ, θ, ϕ - based coeff. formulas.

(a.) $U_1 = (U_1 \cdot E_1)E_1 + (U_1 \cdot E_2)E_2 + (\cancel{U_1 \cdot E_3})^{0}E_3 = \underline{\cos\theta E_1 - \sin\theta E_2}$.

$U_1 = (F_1 \cdot U_1)F_1 + (F_2 \cdot U_1)F_2 + (F_3 \cdot U_1)F_3 = \cos\phi\cos\theta F_1 + \sin\theta F_2 + - \sin\phi\cos\theta F_3$

Recalled the following to do the calculations above,

$E_1 = \cos\theta U_1 + \sin\theta U_2$
$E_2 = -\sin\theta U_1 + \cos\theta U_2$
$E_3 = U_3$

$F_1 = \cos\phi(\cos\theta U_1 + \sin\theta U_2) + \sin\phi U_3$
$F_2 = -\sin\theta U_1 + \cos\theta U_2$
$F_3 = -\sin\phi(\cos\theta U_1 + \sin\theta U_2) + \cos\phi U_3$

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2b) $\cos\theta\, U_1 + \sin\theta\, U_2 + U_3 = X$

$$X = (X \cdot E_1) E_1 + (X \cdot E_2) E_2 + (X \cdot E_3) E_3$$

$$= (\cos^2\theta + \sin^2\theta) E_1 + (-\sin\theta\cos\theta + \cos\theta\sin\theta) E_2 + E_3$$

$$= \underline{E_1 + E_3}.$$

$$X = (X \cdot F_1) F_1 + (X \cdot F_2) F_2 + (X \cdot F_3) F_3$$

$$= (\cos^2\theta\cos\phi + \sin^2\theta\cos\phi + \sin\phi) F_1 + (-\sin\theta\cos\theta + \sin\theta\cos\theta) F_2 +$$

$$+ (-\sin\phi\cos^2\theta - \sin\phi\sin^2\theta + \cos\phi) F_3$$

$$= \underline{(\cos\phi + \sin\phi) F_1 + (\cos\phi - \sin\phi) F_3}.$$

2c.) $Y = x\, U_1 + y\, U_2 + z\, U_3$

$$Y = (Y \cdot E_1) E_1 + (Y \cdot E_2) E_2 + (Y \cdot E_3) E_3$$

$$= (x\cos\theta + y\sin\theta) E_1 + (-x\sin\theta + y\cos\theta) E_2 + z E_3$$

$$= (r\cos^2\theta + r\sin^2\theta) E_1 + (-r\cos\theta\sin\theta + r\sin\theta\cos\theta) E_2 + z E_3$$

$$= \underline{r E_1 + z E_3}.$$ (neat simplification)

$$Y = (Y \cdot F_1) F_1 + (Y \cdot F_2) F_2 + (Y \cdot F_3) F_3$$

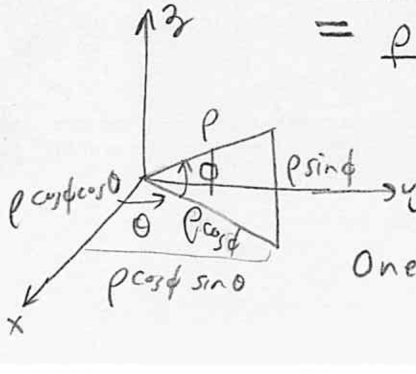
$$= (x\cos\phi\cos\theta + y\cos\phi\sin\theta + z\sin\phi) F_1 +$$

$$+ (-x\sin\theta + y\cos\theta) F_2 + (-x\sin\phi\cos\theta - y\sin\phi\sin\theta + z\cos\phi) F_3$$

$$= (\rho\cos^2\phi\cos^2\theta + \rho\sin^2\theta\cos^2\phi + \rho\sin^2\phi) F_1 + (-\rho\cos\theta\cos\phi\sin\theta + \rho\sin\theta\cos\phi\cos\theta) F_2 +$$

$$+ (-\rho\cos\phi\cos^2\theta\sin\phi - \rho\sin\phi\cos\phi\sin^2\theta + \rho\sin\phi\cos\phi) F_3$$

$$= \underline{\rho F_1}.$$



Oneil's Nonstandard Sphericals

(3.) Find frame field E_1, E_2, E_3 such that
$$E_1 = \cos(x)\, U_1 + \sin(x)\cos(z)\, U_2 + \sin(x)\sin(z)\, U_3$$

Notice $E_1 \cdot E_1 = \cos^2 x + \sin^2 x \cos^2 z + \sin^2 x \sin^2 z = 1$, good.
The answer here would not seem to be unique. That said, the following should suffice,

$$\boxed{E_2 = \sin(x)\, U_1 - \cos(x)\cos(z)\, U_2 - \cos(x)\sin(z)\, U_3}$$

Clearly $E_2 \cdot E_2 = 1$. Also, $E_1 \cdot E_2 = \sin(x)\cos(x)\left[1 - \cos^2 z - \sin^2 z\right] = 0$.
Then, we obtain E_3 via $E_3 = E_1 \times E_2$,

$$E_1 \times E_2 = \begin{vmatrix} U_1 & U_2 & U_3 \\ \cos x & \sin x \cos z & \sin x \sin z \\ \sin x & -\cos x \cos z & -\cos x \sin z \end{vmatrix}$$

$$= (0)\, U_1 - (-\cos^2 x \sin z - \sin^2 x \sin z)\, U_2 + (-\cos^2 x \cos z - \sin^2 x \cos z)\, U_3$$

$$= \boxed{\sin z\, U_2 - \cos z\, U_3 = E_3}$$

§2.7 #1 Let
$$E_1 = \frac{1}{\sqrt{2}}\left(\sin f\, U_1 + U_2 - \cos f\, U_3\right)$$
$$E_2 = \frac{1}{\sqrt{2}}\left(\sin f\, U_1 - U_2 - \cos f\, U_3\right)$$
$$E_3 = \cos f\, U_1 + \sin f\, U_3$$
show these form a frame field and find its connection forms

You can easily verify $E_i \cdot E_j = \delta_{ij} \;\; \forall i, j \in \{1, 2, 3\}$. Observe the attitude matrix A and dA yield $\omega = (dA)A^T$,

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix} \sin f & 1 & -\cos f \\ \sin f & -1 & -\cos f \\ \sqrt{2}\cos f & 0 & \sqrt{2}\sin f \end{bmatrix} \hookrightarrow dA = \frac{1}{\sqrt{2}}\begin{bmatrix} \cos f\, df & 0 & \sin f\, df \\ \cos f\, df & 0 & \sin f\, df \\ -\sqrt{2}\sin f\, df & 0 & \sqrt{2}\cos f\, df \end{bmatrix}$$

$$\omega = (dA)A^T = dA\,\frac{1}{\sqrt{2}}\begin{bmatrix} \sin f & \sin f & \sqrt{2}\cos f \\ 1 & -1 & 0 \\ -\cos f & -\cos f & \sqrt{2}\sin f \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

$$\Rightarrow \omega_{12} = \cancel{\sin f \cos f\, df} - \cancel{\cos f \sin f\, df} = 0, \quad \omega_{13} = (\cos^2 f\, df + \sin^2 f\, df)/\sqrt{2} = \frac{df}{\sqrt{2}}$$

$$\omega_{23} = \frac{-\sin^2 f\, df - \cos^2 f\, df}{\sqrt{2}} = -\frac{df}{\sqrt{2}} \quad \therefore \boxed{\omega = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & df \\ 0 & 0 & -df \\ -df & df & 0 \end{bmatrix}}$$

$\boxed{\S 2.7\#2}$ Find the connection forms of U_1, U_2, U_3 (HS1)

Observe $A = I \Rightarrow dA = 0 \Rightarrow \omega = dA\,A^T = 0.$

$\S 2.7\#3$ For any function f, show A below is the attitude matrix of a frame field, calculate the connection forms

$$A = \begin{bmatrix} \cos^2 f & \cos f \sin f & \sin f \\ \sin f \cos f & \sin^2 f & -\cos f \\ -\sin f & \cos f & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} \cos^2 f & \sin f \cos f & -\sin f \\ \cos f \sin f & \sin^2 f & \cos f \\ \sin f & -\cos f & 0 \end{bmatrix}$$

You can multiply to verify $AA^T = I$ (we need this to confirm orthonormality for the frame which A forms attitude)

$$dA = \left[\begin{array}{c|c|c} -2\cos f \sin f\, df & (\cos^2 f - \sin^2 f)\,df & \cos f\, df \\ (\cos^2 f - \sin^2 f)\,df & 2 \sin f \cos f\, df & \sin f\, df \\ -\cos f\, df & -\sin f\, df & 0 \end{array}\right]$$

To calculate connection forms find (1,2), (1,3), (2,3) components for $\omega = dA\,A^T$,

$$\omega_{12} = (\cos^2 f, \cos f \sin f, \sin f) \cdot \big((\cos^2 f - \sin^2 f)\,df, 2\sin f \cos f\, df, \sin f\, df\big)$$

$$\omega_{12} = \left(\cos^4 f - \sin^2 f \cos^2 f + 2\sin^2 f \cos^2 f + \sin^2 f\right) df$$

$$= \left(\cos^2 f \cos^2 f + \cos^2 f \sin^2 f + \sin^2 f\right) df \Rightarrow \boxed{\omega_{12} = df}$$

$$\omega_{13} = \left(-\cos^2 f \cos f - \sin^2 f \cos f\right) df \Rightarrow \boxed{\omega_{13} = -\cos f\, df}$$

$$\omega_{23} = \left(-\sin f\,(\cos^2 f - \sin^2 f) + 2\cos^2 f \sin f\right) df$$

$$= \left(\sin^3 f + \sin f \cos^2 f\right) df$$

$$= \sin f\, df \quad \therefore \quad \boxed{\omega_{23} = \sin f\, df}$$

$$\boxed{\omega = \begin{bmatrix} 0 & 1 & -\cos f \\ -1 & 0 & \sin f \\ \cos f & -\sin f & 0 \end{bmatrix} df}$$

this disagrees with text, I wonder if you all will agree with me or Oneil... I've likely made a mistake in here...

§2.7 #4), I leave to you

§2.7 #5 Let $W = \sum_i f_i E_i$ and prove $\nabla_V W = \sum_j \left[V[f_j] + \sum_i f_i \omega_{ij}(V) \right] E_j$ (typo in text)

$$\nabla_V W = \nabla_V \left(\sum_{i=1}^3 f_i E_i \right)$$

$$= \sum_{i=1}^3 \nabla_V (f_i E_i)$$

part (3.) of Cor 5.4

$$= \sum_{i=1}^3 \left(V[f_i] E_i + f_i \nabla_V E_i \right)$$

Th$^{\underline{m}}$ 7.2

$$= \sum_{i=1}^3 \left(V[f_i] E_i + f_i \sum_{j=1}^3 \omega_{ij}(V) E_j \right)$$

$$= \sum_{i=1}^3 V[f_i] E_i + \sum_{i=1}^3 \sum_{j=1}^3 f_i \omega_{ij}(V) E_j$$

$$= \sum_{i=1}^3 V[f_i] E_i + \sum_{j=1}^3 \sum_{i=1}^3 f_j \omega_{ji}(V) E_i$$

$$= \sum_{i=1}^3 \left(V[f_i] + \sum_{j=1}^3 f_j \omega_{ji}(V) \right) E_i$$

$$= \sum_{j=1}^3 \left(\underbrace{V[f_j]} + \sum_{i=1}^3 f_i \omega_{ij}(V) \right) E_j$$

$\exists$ typo in O'neil here.

cylindrical connection.

$$\omega = \begin{bmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

§2.7 #6 Let E_1, E_2, E_3 be the cylindrical frame field, $V \in \mathfrak{X}(\mathbb{R}^3)$ such that $V[r] = r$, $\underline{V[\theta] = 1}$. Calculate $\nabla_V \left(\underset{f_1}{\underline{r\cos\theta}} E_1 + \underset{f_3}{\underline{r\sin\theta}} E_3 \right)$

$d\theta(V) = 1$ $\quad (f_2 = 0)$

Use the result of (5.),

see pg. 92

$$\nabla_V W = V[r\cos\theta]E_1 + V[r\sin\theta]E_3 + f_1 \omega_{12}(V) E_2 + f_1 \cancel{\omega_{13}}^{0}(V) E_3 + 2$$
$$+ f_3 \cancel{\omega_{31}}^{0}(V) E_1 + f_3 \cancel{\omega_{32}}^{0}(V) E_2$$

$$= (\cos\theta\, V[r] - r\sin\theta\, V[\theta]) E_1 + (\sin\theta\, V[r] + r\cos\theta\, V(\theta)) E_3 + f_1\, d\theta[V]\, E_2$$

$$= (r\cos\theta - r\sin\theta) E_1 + (r\sin\theta + r\cos\theta) E_3 + r\cos\theta\, E_2$$

$$= \underline{r(\cos\theta - \sin\theta) E_1 + r(\sin\theta + \cos\theta) E_3 + r\cos\theta\, E_2}.$$

§2.7 #8 Let β be unit-speed with $\kappa > 0$.
& E_1, E_2, E_3 is frame field on $\mathbb{R}^3$ s.t. when
E_1, E_2, E_3 restricted to β gives T, N, B of β. Prove

$$\omega_{12}(T) = \kappa, \quad \omega_{13}(T) = 0, \quad \omega_{23}(T) = \tau$$

We found in #5 of §2.5 the restriction of $W \in \mathfrak{X}(\mathbb{R}^3)$ to $\alpha : I \subseteq \mathbb{R} \to \mathbb{R}^3$ given by $(W|_\alpha)(t) = W(\alpha(t)) = W_\alpha(t)$ had $\nabla_{\alpha'(t)} W = (W_\alpha)'(t)$. Note $\beta' = T$ for unit-speed and $\omega_{ij}(V) = (\nabla_V E_i) \cdot E_j$ hence, at $\beta(t)$,

$$\begin{aligned}
\omega_{12}(T) &= (\nabla_T E_1) \cdot E_2(\beta(t)) \\
&= (\nabla_{\beta'(t)} E_1) \cdot N \\
&= (E_1(\beta(t)))'(t) \cdot N \\
&= T' \cdot N \\
&= \kappa N \cdot N \\
&= \boxed{\kappa.}
\end{aligned}
\qquad
\left.\begin{aligned}
E_1|_\beta &= T \\
E_2|_\beta &= N \\
E_3|_\beta &= B
\end{aligned}\right.$$

$$\begin{aligned}
\omega_{13}(T) &= (\nabla_T E_1) \cdot E_3(\beta(t)) \\
&= T' \cdot B \\
&= \kappa N \cdot B \\
&= \boxed{0.}
\end{aligned}$$

details similar to

$$\begin{aligned}
\omega_{23}(T) &= (\nabla_T E_2) \cdot E_3(\beta(t)) \\
&= (E_2(\beta(t)))' \cdot B \\
&= N' \cdot B \\
&= (-\kappa T + \tau B) \cdot B \\
&= \boxed{\tau,}
\end{aligned}$$

<u>Remark</u>: I'm curious, how would we calculate $\omega_{ij}(B)$ or $\omega_{ij}(N)$. We only see T-components here.

§2.7 #8 continued: (H54)

deduce Frenet formulas from connection eq^{n}'s.

$$\nabla_V E_1 = \omega_{12}(V) E_2 + \omega_{13}(V) E_3$$
$$\nabla_V E_2 = -\omega_{12}(V) E_1 + \omega_{23}(V) E_3$$
$$\nabla_V E_3 = -\omega_{13}(V) E_1 - \omega_{23}(V) E_2$$

We know $E_1 = \beta' = T$, $E_2 = \frac{1}{\kappa}\beta'' = N$ and $E_3 = T \times N$ along β. The Frenet eq^{n} $T' = \kappa N$ is essentially a def^{n} so, I'm not sure what is meant by this problem... I'll try to play along, how can we find T', N', B' in connection eq^{n}'s?

$$\nabla_{\alpha'(t)} W = W_\alpha'(t) \quad \text{exercise \#5 of §2.5}$$
$$\nabla_T W = W' \text{ along } \beta \text{ for } W = E_1, E_2, E_3.$$

$$\nabla_T T = \omega_{12}(T) N + \omega_{13}(T) B = T' \longrightarrow \underline{T' = \kappa N}.$$
$$\nabla_T N = -\omega_{12}(T) T + \omega_{23}(T) B = N' \longrightarrow \underline{N' = -\kappa T + \tau B}.$$
$$\nabla_T B = -\omega_{13}(T) T - \omega_{23}(T) N = B' \longrightarrow \underline{B' = -\tau N}.$$

I think this is what was intended here, note to find $\omega_{12}, \omega_{13}, \omega_{23}$ I need to know how T, N, B change in T, N, B directions.... here's what I take away, the connection eq^{n}'s are <u>consistent</u> with Frenet Serret as they restrict to appropriate frame on a $\kappa > 0$ curve.

§ 2.8 pg. 98 (H55)

1.) Let θ_i be dual to frame field E_1, E_2, E_3 and suppose a 1-form $\phi = \sum_i f_i \theta_i$ prove $d\phi = \sum_j \left(df_j + \sum_i f_i \omega_{ij} \right) \wedge \theta_j$

I'll use def^{n} & properties of $\wedge$ and d as given in §1.6,

$$\begin{aligned} d\phi &= \sum_i d(f_i \theta_i) \\ &= \sum_i \left(df_i \wedge \theta_i + f_i \, d\theta_i \right) \\ &= \sum_{i=1}^{3} \left(df_i \wedge \theta_i + f_i \sum_{j=1}^{3} \omega_{ij} \wedge \theta_j \right) \\ &= \sum_{j=1}^{3} \left[df_j + \sum_{i=1}^{3} f_i \omega_{ij} \right] \wedge \theta_j \end{aligned}$$

CARTAN'S FIRST STRUCTURAL EQ^{n}.

2.) Check structural eq^{n}'s of the spherical frame field.

$\theta_1 = d\rho \qquad \omega_{12} = \cos\phi \, d\theta$

$\theta_2 = \rho\cos\phi \, d\theta \qquad \omega_{13} = d\phi$

$\theta_3 = \rho \, d\phi \qquad \omega_{23} = \sin\phi \, d\theta$

$d\theta = \omega \wedge \theta$

$d\omega = \omega \wedge \omega$

Need to check $d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$, $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$

$$\theta = \begin{bmatrix} d\rho \\ \rho\cos\phi \, d\theta \\ \rho \, d\phi \end{bmatrix} \qquad d\theta = \begin{bmatrix} 0 \\ \cos\phi \, d\rho\wedge d\theta - \rho\sin\phi \, d\phi\wedge d\theta \\ d\rho \wedge d\phi \end{bmatrix}$$

matrix notation f+w.

$$\omega \wedge \theta = \begin{bmatrix} 0 & \cos\phi \, d\theta & d\phi \\ -\cos\phi \, d\theta & 0 & \sin\phi \, d\theta \\ -d\phi & -\sin\phi \, d\theta & 0 \end{bmatrix} \wedge \begin{bmatrix} d\rho \\ \rho\cos\phi \, d\theta \\ \rho \, d\phi \end{bmatrix} \overset{\checkmark}{=} \underbrace{\begin{bmatrix} \rho\cos^2\phi \, d\theta\wedge d\theta + \rho \, d\phi\wedge d\phi \\ -\cos\phi \, d\theta\wedge d\rho + \rho\sin\phi \, d\theta\wedge d\phi \\ -d\phi\wedge d\rho + 0 + 0 \end{bmatrix}}_{\text{simplifies to } d\theta}$$

$$\omega\wedge\omega = \begin{bmatrix} 0 & \cos\phi \, d\theta & d\phi \\ -\cos\phi \, d\theta & 0 & \sin\phi \, d\theta \\ -d\phi & -\sin\phi \, d\theta & 0 \end{bmatrix} \begin{bmatrix} 0 & \cos\phi \, d\theta & d\phi \\ -\cos\phi \, d\theta & 0 & \sin\phi \, d\theta \\ -d\phi & -\sin\phi \, d\theta & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\sin\phi \, d\phi\wedge d\theta & 0 \\ -\sin\phi \, d\theta\wedge d\phi & 0 & -\cos\phi \, d\theta\wedge d\phi \\ 0 & -\cos\phi \, d\phi\wedge d\theta & 0 \end{bmatrix}}$$

$$d\omega = \begin{bmatrix} 0 & -\sin\phi & 0 \\ \sin\phi & 0 & \cos\phi \\ 0 & -\cos\phi & 0 \end{bmatrix} d\phi\wedge d\theta \quad \overset{\checkmark}{\longleftrightarrow} \quad \begin{bmatrix} 0 & -\sin\phi & 0 \\ \sin\phi & 0 & \cos\phi \\ 0 & -\cos\phi & 0 \end{bmatrix} d\phi\wedge d\theta$$

§2.8 #3) Let $E_1, E_2, E_3 = \hat{r}, \hat{\theta}, \hat{z}$

(a.) use $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ to derive $\theta_1 = dr$, $\theta_2 = r\,d\theta$, $\theta_3 = dz$.

(b.) deduce $E_1[r] = 1$, $E_2[\theta] = 1/r$, $E_3(z) = 1$ and others are trivial (6)

(c.) Let $f = f(r,\theta,z)$ show $E_1[f] = \frac{\partial f}{\partial r}$, $E_2[f] = \frac{1}{r}\frac{\partial f}{\partial \theta}$, $E_3[f] = \frac{\partial f}{\partial z}$.

(a.)

$$E_1 = \cos\theta\, U_1 + \sin\theta\, U_2 \longrightarrow \theta_1 = \cos\theta\, dx + \sin\theta\, dy \overset{*}{=} dr$$

$$E_2 = -\sin\theta\, U_1 + \cos\theta\, U_2 \longrightarrow \theta_2 = -\sin\theta\, dx + \cos\theta\, dy \overset{**}{=} r\,d\theta$$

$$E_3 = U_3 \longrightarrow \theta_3 = dz$$

To see * and ** note $r^2 = x^2+y^2 \Rightarrow 2r\,dr = 2x\,dx + 2y\,dy$

$$\Rightarrow \underline{dr = \cos\theta\, dx + \sin\theta\, dy}.$$

Likewise, for **, $\tan\theta = y/x$

$$\hookrightarrow \sec^2\theta\, d\theta = \frac{x\,dy - y\,dx}{x^2}$$

$$\hookrightarrow r\,d\theta = \frac{\cancel{\cos^2\theta}}{r^2 \cancel{\cos^2\theta}}\left(r^2\cos\theta\, dy - r^2\sin\theta\, dx\right)$$

$$\Rightarrow \underline{r\,d\theta = \cos\theta\, dy - \sin\theta\, dx}.$$

(b.)

$$E_1[r] = dr(E_1) = (\cos\theta\, dx + \sin\theta\, dy)(\cos\theta\, U_1 + \sin\theta\, U_2) = \cos^2\theta + \sin^2\theta = 1.$$

$$E_2[\theta] = d\theta(E_2) = \left(\frac{-\sin\theta\, dx + \cos\theta\, dy}{r}\right)(-\sin\theta\, U_1 + \cos\theta\, U_2) = \frac{1}{r}(\sin^2\theta + \cos^2\theta) = \frac{1}{r}.$$

$$E_3[z] = dz(E_3) = dz(U_3) = 1.$$

these were correct, but there is better way, $\underline{\underline{\theta_i(E_j) = \delta_{ij}}}$

$$E_1[r] = dr[E_1] = \theta_1[E_1] = \delta_{11} = 1.$$

$$E_2[\theta] = d\theta[E_2] = \frac{1}{r}(r\,d\theta)[E_2] = \frac{1}{r}\theta_2[E_2] = \frac{1}{r}.$$

$$E_3[z] = dz[E_3] = \theta_3[E_3] = 0.$$

$$E_1[\theta] = d\theta[E_1] = \frac{1}{r}\theta_2[E_1] = 0$$

$$E_1[z] = dz[E_1] = \theta_3[E_1] = 0$$

$$E_2[r] = dr[E_2] = \theta_1[E_2] = 0$$

$$E_2[z] = dz[E_2] = \theta_3[E_2] = 0$$

$$E_3[r] = dr[E_3] = \theta_1[E_3] = 0$$

$$E_3[\theta] = d\theta[E_3] = \frac{1}{r}\theta_2[E_3] = 0$$

§2.8 #3 continued

(c.) $E_1[f] = \cos\theta\, U_1[f] + \sin\theta\, U_2[f]$

$$= \frac{\partial x}{\partial r}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial f}{\partial y}$$

$$= \underline{\frac{\partial f}{\partial r}}.$$

$x = r\cos\theta,\quad y = r\sin\theta$

$E_2[f] = -\sin\theta\, U_1[f] + \cos\theta\, U_2[f]$

$$= \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$= \underline{\frac{1}{r}\frac{\partial f}{\partial \theta}}.$$

I leave $E_3[f] = \dfrac{\partial f}{\partial z}$ to reader ☺

(4.) <u>FRAME FIELDS FOR $\mathbb{R}^2$</u>: Let E_1, E_2 form frame field on $\mathbb{R}^2$ then $\exists$ angle function ψ such that

$$E_1 = \cos\psi\, U_1 + \sin\psi\, U_2$$
$$E_2 = -\sin\psi\, U_1 + \cos\psi\, U_2$$

(a.) find connection form and dual forms in terms of x, y

(b.) what are struct. eq^{n}'s ?

Let $E_3 = U_3$ so we may apply the techniques of $\mathbb{R}^3$ w/o Δ.

$$\theta_1 = \cos\psi\, dx + \sin\psi\, dy$$
$$\theta_2 = -\sin\psi\, dx + \cos\psi\, dy$$
$$\theta_3 = dz$$

$$A = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\omega = dA\, A^T = d\psi \begin{bmatrix} -\sin\psi & \cos\psi & 0 \\ -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & d\psi & 0 \\ -d\psi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Structure Eq^{n}'s</u>:

$$d\theta = \omega\wedge\theta \longrightarrow \begin{bmatrix} 0 & d\psi \\ -d\psi & 0 \end{bmatrix}\begin{bmatrix} \cos\psi\, dx + \sin\psi\, dy \\ -\sin\psi\, dx + \cos\psi\, dy \end{bmatrix} = \underbrace{\begin{bmatrix} -\sin\psi\, d\psi\wedge dx + \cos\psi\, d\psi\wedge dy \\ -\cos\psi\, d\psi\wedge dx - \sin\psi\, d\psi\wedge dy \end{bmatrix}}_{\text{precisely } d\theta}$$

$$\omega\wedge\omega = \begin{bmatrix} 0 & d\psi \\ -d\psi & 0 \end{bmatrix}\wedge\begin{bmatrix} 0 & d\psi \\ -d\psi & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = d\omega$$

<u>Remark</u>: you can find the same working with 3x3 matrices...

Comment on ψ

we found for arbitrary ψ that

$$E_1 = \cos\psi\, U_1 + \sin\psi\, U_2$$
$$E_2 = -\sin\psi\, U_1 + \cos\psi\, U_2$$
$$E_3 = U_3$$

The connection form is

$$\omega = \begin{bmatrix} 0 & d\psi & 0 \\ -d\psi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

this is a bit interesting as $\psi = \tan^{-1}(y/x)$ gives polar coordinate θ (for half of $\mathbb{R}^3$ anyway) but, it seems we could just as well set $\psi = x^2+y^2$ and then $\omega = \begin{bmatrix} 0 & 2x\,dx + 2y\,dy & 0 \\ -2x\,dx - 2y\,dy & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

what would nonstandard ψ even mean? For example, $\psi = z$.

$$E_1, E_2, E_3 \xrightarrow{z=0} U_1, U_2, U_3$$
$$\xrightarrow{z=\pi/2} U_2, -U_1, U_3$$
$$\xrightarrow{z=\pi} -U_1, -U_2, U_3$$

etc...