§3.1 pg. 105

(1.) Prove $CT_a = T_{C(a)}C$

By def$^{\underline{n}}$ $T_a(x) = a + x$, $C(x) = Rx$ for some $R \in O(3)$ $(R^T R = I)$

Consider,

$$\begin{aligned}(CT_a)(x) &= C(a+x) \\ &= R(a+x) \\ &= Ra + Rx \\ &= C(a) + C(x) \\ &= T_{C(a)}(C(x)) \\ &= (T_{C(a)}C)(x) \quad \therefore \quad \boxed{CT_a = T_{C(a)}C}.\end{aligned}$$

(2.) Given isometries $F = T_a A$ and $G = T_b B$ find translational and orthogonal parts of FG & GF

$$\begin{aligned}(FG)(x) &= F(T_b B(x)) \\ &= T_a A(b + Bx) \\ &= T_a(A(b) + A(B(x))) \\ &= a + A(b) + AB(x) \\ &= \underline{T_c C \quad \text{for} \quad c = a + A(b), \quad C = AB}.\end{aligned}$$

$$\begin{aligned}(GF)(x) &= b + B(a) + BA(x) \\ &= \underline{T_d D \quad \text{for} \quad d = b + B(a), \quad D = BA}.\end{aligned}$$

(3.) If $F = T_a C$ then find F^{-1},

$$\begin{aligned}F(x) = T_a C(x) = y \quad \Rightarrow \quad a + C(x) &= y \\ C(x) &= y - a \\ x &= C^{-1}(y-a) = C^{-1}(-a) + C^{-1}(y)\end{aligned}$$

$$\underline{F^{-1} = T_{-C^{-1}(a)}C^{-1}}.$$

you can write $C^{-1} = C^T$ to obtain text's answer.

§3.1#7) Let $\mathcal{E}(3)$ = set of all isometries forms a group w.r.t composition

Observe, $F = Id$ is in $\mathcal{E}(3)$ and serves as the identity for function composition, moreover, fnct composition is associative. Closure under inverses and composition remains to check. We proved $F \in \mathcal{E}(3) \Rightarrow F^{-1} \in \mathcal{E}(3)$ in #3 note that $C \in O(3) \Rightarrow C^{-1} = C^T$ also in $O(3)$ as $C^T C^{T^T} = CC^T = I$ follows from $C^TC = I \Rightarrow C^{T^T}C^T = I^T \Rightarrow CC^T = I$. Oh, we also have $F, G \in \mathcal{E}(3) \Rightarrow FG \in \mathcal{E}(3)$ by our work on (2.); $FG = a + A(b) + AB = T_c C$ and clearly T_c is translation by $c = a + A(b) \in \mathbb{R}^3$ and $AB \in O(3)$ as $A, B \in O(3) \Rightarrow A^TA = I$, $B^TB = I$ thus, by socks-shoes,

$$(AB)^T AB = B^T A^T AB = B^T I B = B^T B = I.$$

Thus $FG = T_c C$ for orthogonal $C \Rightarrow FG \in \mathcal{E}(3)$. Thus $\mathcal{E}(3)$ forms a group.

§3.2#3 Given

$$\left.\begin{aligned} e_1 &= \tfrac{1}{3}(2,2,1) \\ e_2 &= \tfrac{1}{3}(-2,1,2) \\ e_3 &= \tfrac{1}{3}(1,-2,2) \end{aligned}\right\} \text{ at } p=(0,1,0) \quad \text{and} \quad \left.\begin{aligned} f_1 &= \tfrac{1}{\sqrt{2}}(1,0,1) \\ f_2 &= (0,1,0) \\ f_3 &= \tfrac{1}{\sqrt{2}}(1,0,-1) \end{aligned}\right\} \text{ at } q=(3,-1,1)$$

Find an isometry to transport $e_i \xrightarrow{F} f_i$

Define C by linearly extending $C(e_j) = f_j$ for $j = 1,2,3$.

$$F = T_a C \Rightarrow F(p) = T_a(C(p)) = a + C(p) = q$$

So we insist $a = q - C(p)$. It remains to find the explicit form of C and hence a.

(we follow the spirit of pg. 109 ≈)

§3.2 #3 continued

$$Ce_j^T = f_j^T \qquad A = \left[\begin{array}{c} e_1 \\ \hline e_2 \\ \hline e_3 \end{array}\right]$$

$$C\left[\text{Col}_j(A^T)\right] = \text{Col}_j(B^T) \qquad B = \left[\begin{array}{c} f_1 \\ \hline f_2 \\ \hline f_3 \end{array}\right]$$

By concatenation prop. these 3 column eq^{n}s yield

$$\Rightarrow CA^T = B^T$$

$$\Rightarrow C = B^T A \quad \text{as } A^T A = I.$$

Hence, calculate,

$$C = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} = \overbrace{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -2/3 & 1/3 & 2/3 \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix}}^{C}$$

$$a = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -2/3 & 1/3 & 2/3 \\ \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/3 \\ \frac{4}{3\sqrt{2}} \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ -4/3 \\ 1 - \frac{4}{3\sqrt{2}} \end{bmatrix}}}_{a}$$

In summary, $F = T_a C$ where C and a are as given above. (this matches O'neil)

Remark: the method we discovered here (with O'neil's help!) is nice to remember for later, any time we want to fit frames we just have multiply the attitude & the transpose of the new attitude as discussed above.

§ 3.3 pg. 115

(2.) If H_0 is an orientation-reversing isometry of $\mathbb{R}^3$ then show that <u>every</u> orientation-reversing isometry is uniquely represented as $H_0 F$ where F is orientation-preserving

Let G be an orientation reversing isometry. There exists orthogonal R with $\det(R) = -1$ and $G = T_a R$

Notice $H_0 = T_b S$ with $\det(S) = -1$ hence H_0^{-1} has

(too lazy to look up where we already did this, I'll just calculate it again here,

$$H_0(x) = T_b S(x) = b + S(x) = y \hookrightarrow x = S^{-1}(y-b)$$

Thus $H_0^{-1} = T_{-S^{-1}(b)} S^{-1} = T_{-S^T(b)} S^T$ notice

$\det(S^T) = \det(S) = -1$ hence H_0^{-1} is also orientation reversing.

Finally, observe, let $F = H_0^{-1} G$

$$G = H_0 H_0^{-1} G = H_0 F.$$

Clearly F so constructed is unique. Furthermore, let $c = -S^T(b)$

$$F = H_0^{-1} G = T_c S^T T_a R$$

$$\begin{aligned} F(x) &= T_c\left(S^T(a + R(x))\right) \\ &= c + S^T(a + R(x)) \\ &= c + S^T(a) + S^T R(x) \\ &= T_{\bar{c}} B \quad \text{where } \bar{c} = c + S^T(a),\ B = S^T R \end{aligned}$$

Notice $\det(B) = \det(S^T R) = \det(S^T)\det(R) = (-1)(-1) = 1.$

Thus F is an orientation-preserving isometry as claimed. //

§3.3 #4) A rotation C has $C^TC = I$ and $\det(C) = 1$. Here I abuse notation and use $C = [C]$ as convenient. Prove that for each rotation C, $\exists \theta \in \mathbb{R}$ and points e_1, e_2, e_3 with $e_i \cdot e_j = \delta_{ij}$ and:

H63

$$C(e_1) = \cos\theta\, e_1 + \sin\theta\, e_2$$
$$C(e_2) = -\sin\theta\, e_1 + \cos\theta\, e_2$$
$$C(e_3) = e_3$$

Observe $C^TC = I \Rightarrow \det(C^TC) = \det(C^T)\det(C) = \det(C)^2 = 1$. However, $\det(C) = \lambda_1\lambda_2\lambda_3$ where $\lambda_1, \lambda_2, \lambda_3$ are the (possibly complex) eigenvalues of C which are sol$^{\underline{e}}$s to $\det(C - \lambda I) = 0$. Since conjugate roots come in conjugate pairs $\Rightarrow \lambda_3 = 1$, $\lambda_1 = \lambda_2^*$ hence $|\lambda_1| = |\lambda_2| = 1$. There is thankfully a Th$^{\underline{m}}$ of linear algebra which say each e-value has at least one e-vector hence $\exists U_3 \in \mathbb{R}^3$ s.t. $C(U_3) = U_3$. Rescale $e_3 = \frac{1}{\|U_3\|} U_3$ to obtain $e_3 \cdot e_3 = 1$. If $\lambda_1, \lambda_2 \in \mathbb{R}$ then $\lambda_1 = \lambda_2 = 1$ hence $C = I$ so we can use $e_i = v_i$ for $i = 1, 2, 3$. Suppose $\lambda_1, \lambda_2 \in \mathbb{C}$ then as $|\lambda_1| = |\lambda_2| = 1 \hookrightarrow \lambda_1 = e^{i\theta}, \lambda_2 = e^{-i\theta}$ and $\exists u = a + ib \in \mathbb{C}^3$ s.t. $a, b \in \mathbb{R}^3$ and

$$C(a+ib) = e^{i\theta}(a+ib)$$
$$\Rightarrow Ca + iCb = (\cos\theta + i\sin\theta)(a+ib)$$
$$\Rightarrow Ca + iCb = \cos\theta\, a - \sin\theta\, b + i(\sin\theta\, a + \cos\theta\, b)$$

I prove a, b LI in my Math 321 notes, I'm sure you could derive some $\rightarrow\leftarrow$ by β $b = ka$ etc... Hence $\{a, b\}$ are LI and

$$C(a) = \cos\theta\, a - \sin\theta\, b$$
$$C(b) = \sin\theta\, a + \cos\theta\, b$$

Let $e_1 = \frac{1}{\|a\|} a$ and $e_2 = \frac{1}{\|b\|} b$ and it follows that e_1, e_2, e_3 so constructed has $\underbrace{e_i \cdot e_j = \delta_{ij}}$.

gap in my argument,
I've not shown $e_1 \cdot e_3 = 0$ etc...

H64

§3.3 #5 Let a be point of $\mathbb{R}^3$ such that $\|a\| = 1$.
Prove that $C(p) = a\times p + (p\cdot a)a$ defines an orthogonal transformation. Describe its general effect on $\mathbb{R}^3$

$$C(p)\cdot C(q) = [a\times p + (p\cdot a)a]\cdot[a\times q + (q\cdot a)a]$$

$$= (a\times p)\cdot(a\times q) + \underbrace{\cancel{(a\times p)\cdot(q\cdot a)a}}_{a\perp(a\times p)}\overset{0}{} + \underbrace{\cancel{(p\cdot a)a\times(a\times q)}}_{a\perp(a\times q)}\overset{0}{} + (p\cdot a)(q\cdot a)(a\cdot a)$$

$$= (a\times p)\cdot(a\times q) + (p\cdot a)(q\cdot a)$$

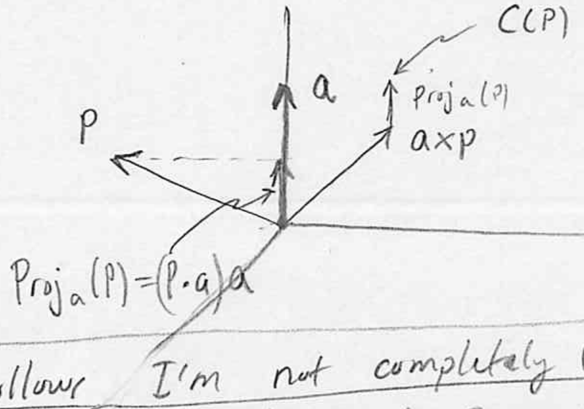
given $\|a\| = 1$ $\hookrightarrow a\cdot a = \|a\|^2 = 1.$

$$= \underbrace{(a\cdot a)(p\cdot q) - (p\cdot a)(q\cdot a)}_{} + (p\cdot a)(q\cdot a)$$

Lagrange's Identity. See Math 231 notes for proof. It involves the double ϵ_{ijk} identity.

$$= p\cdot q \quad \therefore \quad \underline{C \text{ orthogonal}}.$$

(Notice, C is clearly linear from $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$)



- Maybe rotating around the a axis
- $C(a) = a\times a + (a\cdot a)a = a$.
  Yep, it fixes the a-axis

What follows I'm not completely ☺ about, but it's the best I have now,

Let's see, what angle? To find $\angle(P, C(P))$ we calculate:

$$\underbrace{P\cdot C(P)}_{\|P\|^2\cos\theta_1} = P\cdot\left(a\times P + (P\cdot a)a\right) = \underbrace{(P\cdot a)}_{(\|P\|\cos\theta_2)^2}{}^2$$

Thus $\cos\theta_1 = \cos^2\theta_2$ where $\theta_1 = \angle(P, C(P))$ and $\theta_2 = \angle(P, a)$. There are two sol^{n}'s. ($P \neq a$ is interesting)

1.) $\cos\theta_1 = \cos^2\theta_2 = 1$ $(P = a)$, 2.) $\cos\theta_1 = \cos^2\theta_2 = 0$ $P \neq a \implies \boxed{\theta = \pi/2}$

§3.3 #6 Prove — a.) $O^+(3) = SO(3) \leq O(3)$.
— b.) $\mathcal{E}^+(3) = \{ F \in \mathcal{E}(3) \mid sgn(F) = 1 \} \leq \mathcal{E}(3)$.

I'll use the subgroup test Th$^{\underline{m}}$. Observe $I \in O^+(3)$ as $\det(I) = 1 \Rightarrow I \in O^+(3)$. Let $A, B \in O^+(3)$ then $\det(AB) = \det A \det B = (1)(1) = 1 \therefore AB \in O^+(3)$ also $A^{-1}A = I \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} = 1 \therefore A^{-1} \in O^+(3)$ thus $O^+(3) \leq O(3)$ (Note: $O^+(3) = \underline{SO(3)}$ in many texts) special orthogonal group.

Likewise $I \in \mathcal{E}^+(3)$ as $sgn(I) = \det(I) = 1$. If $F, G \in \mathcal{E}^+(3)$ then $F = T_a + A$, $G = T_b + B$ and F^{-1} has A^{-1} as its orthogonal part whereas FG has AB as its orthogonal part. Observe $\det(A^{-1})$ and $\det(AB)$ are both 1 hence FG, F^{-1} have $sgn(FG) = sgn(F^{-1}) = 1$ thus $FG, F^{-1} \in \mathcal{E}^+(3) \therefore \mathcal{E}^+(3) \leq \mathcal{E}(3)$.

§3.4 #1a (pg. 120) If $F = TC$ is an isometry of $\mathbb{R}^3$, β unit speed, then prove β cylindrical helix $\Rightarrow F(\beta)$ is cylindrical helix

β cylindrical helix $\Rightarrow \exists u \in \mathbb{R}^3$, $\theta \in \mathbb{R}$ (fixed, independent of t) such that $T(t) \cdot u = \cos\theta$ $\forall t$. Let $\gamma = F(\beta)$ and observe that $F_*(\beta') = \gamma'$ and we've shown $F_* = C$ thus $F_* u = Cu$ is the natural candidate for $\tilde{u}$ for γ. Let $\tilde{u} = Cu$, $\tilde{T} = \gamma'$,

$$\tilde{T}(t) \cdot \tilde{u} = (CT) \cdot (Cu) = T \cdot u = \cos\theta$$

Thus $\gamma = F(\beta)$ is a cylindrical helix (with same characteristic θ).

§ 3.5 # 2 Let $Y = (t, 1-t^2, 1+t^2)$ be a vector field on the helix $\alpha(t) = (\cos t, \sin t, 2t)$ and

H66

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Calculate $\bar{\alpha} = C(\alpha)$ and $\overline{Y} = C_*(Y)$ and check that

$C_*(Y') = \overline{Y}'$

$C_*(\alpha'') = \bar{\alpha}''$

$Y' \cdot \alpha'' = \overline{Y}' \cdot \bar{\alpha}''$

$$\bar{\alpha}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \cos t \\ \sin t \\ 2t \end{bmatrix} = \left(-\cos t, \frac{1}{\sqrt{2}}\sin t - t\sqrt{2}, \frac{1}{\sqrt{2}}\sin t + t\sqrt{2}\right)$$

$$\bar{\alpha}'(t) = \left(\sin t, \frac{1}{\sqrt{2}}\cos t - \sqrt{2}, \frac{1}{\sqrt{2}}\cos t + \sqrt{2}\right)$$

$$\bar{\alpha}''(t) = \left(\cos t, \frac{-1}{\sqrt{2}}\sin t, \frac{-1}{\sqrt{2}}\sin t\right)$$

$$\overline{Y} = C_* Y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} t \\ 1-t^2 \\ 1+t^2 \end{bmatrix} = (-t, -t^2\sqrt{2}, \sqrt{2}) \quad \big\}\, d/dt$$

$$C_* Y' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ -2t \\ 2t \end{bmatrix} = (-1, -2\sqrt{2}t, 0) \neq Y'(t)$$

Likewise, we can check $C_* \alpha'' = \bar{\alpha}''$. This will be true as $C \frac{d^2}{dt^2}(\alpha) = \frac{d^2}{dt^2}[C\alpha]$ which is a simple consequence of $\frac{dC}{dt} = 0$. On the other hand

$$\begin{aligned} \overline{Y}' \cdot \bar{\alpha}'' &= (CY)' \cdot (C\alpha)'' \\ &= CY' \cdot C\alpha'' \\ &= Y' \cdot \alpha'' \end{aligned}$$

def$^{\underline{n}}$ of orthog. C.

I could verify these for the given C & α but I tire of this problem. We go on,

§3.4 #4) If $F: \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism such that F_* preserves dot-products, show F is an isometry.

<u>Reminder</u>: $F: \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry if it is a smooth function such that $d(F(P), F(q)) = d(P, q) \ \forall P, q \in \mathbb{R}^3$.

Let $\alpha(t) = P + t(q - P)$ for $0 \leq t \leq 1$. Note $\alpha(0) = P$, $\alpha(1) = q$. Furthermore, the arclength of α is simply $\| q - P \|$. But, it is also $d(P, q) = \int_0^1 \sqrt{\alpha'(t) \cdot \alpha'(t)}\, dt$. Consider the image of α under F; $\bar{\alpha} = F(\alpha)$ notice

$$\bar{\alpha}(0) = F(P) \quad \text{and} \quad \bar{\alpha}(1) = F(q)$$

Naturally, we calculate the arclength of $\bar{\alpha}$,

$$\begin{aligned}
\bar{s}^{\alpha}_{F(P) \to F(q)} &= \int_0^1 \sqrt{\bar{\alpha}'(t) \cdot \bar{\alpha}'(t)}\, dt \\
&= \int_0^1 \sqrt{F_*(\alpha'(t) \cdot F_*(\alpha'(t))}\, dt \\
&= \int_0^1 \sqrt{\alpha'(t) \cdot \alpha'(t)}\, dt \\
&= d(P, q).
\end{aligned}$$

We'd be done except we don't know $F(\alpha)$ is a line-segment. O'neil's hint is if F preserves length of line-segments then so does F^{-1}. Note, $F^{-1} \circ F = Id \Rightarrow (F^{-1})_* \circ F_* = Id$ we know $[F_*] \in O(3) \Rightarrow [F_*]^{-1} \in O(3) \Rightarrow (F^{-1})_*$ preserves dot products. Consider the curve $\gamma(t) = F(P) + t(F(q) - F(P))$, $0 \leq t \leq 1$ clearly $\| F(q) - F(P) \| = \int_0^1 \sqrt{\gamma'(t) \cdot \gamma'(t)}\, dt$ and we may study $\beta = F^{-1}(\gamma)$ from $\beta(0) = F^{-1}(F(P)) = P$ and $\beta(1) = q$. By the argument already given, $s^{\beta}_{P \to q} = \| F(q) - F(P) \|$. It follows $d(P, q) = d(F(P), F(q))$. Sneaky.

§ 3.4 #5 Let F be an isometry of $\mathbb{R}^3$. For each V let $\overline{V}$ be the vector field $F_*(V(P)) = \overline{V}(F(P))$ for all p. Prove that isometries preserve covariant derivatives. Show:

$$\overline{\nabla_V W} = \nabla_{\overline{V}} \overline{W}$$

The answer in O'neil is nice,

$$F_*(\nabla_V W) = F_*\big(W(P+tV)'(0)\big) \quad : \text{ def}^{\text{n}} \text{ of } \nabla_V W.$$

$$= \overline{W}(F(P)+tC(V))'(0) \quad : \text{ Cor. 4.1 on pg. 117}$$

$$= \nabla_{F_*(V)} \overline{W} \quad :$$

Exercise 5a from pg. 84

$W_\alpha \to \overline{W}(\gamma) = \overline{W}_\gamma$

$$(\overline{W}_\gamma)'(t) = \nabla_{\underbrace{\gamma'(t)}_{F_*(V)}} \overline{W}$$

$\gamma(t) = F(P) + t\,C(V)$

But, $F_*(\nabla_V W) = \overline{\nabla_V W}$ and $F_*(V) = \overline{V}$ hence

$$\overline{\nabla_V W} = \nabla_{\overline{V}} \overline{W}. //$$

Remark: I'd like to prove this by an explicit coordinate calculation... but, at the present I've not found the right attach. It should be simple....

§3.5 #1 Given $\alpha = (\alpha_1, \alpha_2, \alpha_3): I \longrightarrow \mathbb{R}^3$, prove $\beta: I \longrightarrow \mathbb{R}^3$ is congruent to α iff $\beta(t) = p + \alpha_1(t)e_1 + \alpha_2(t)e_2 + \alpha_3(t)e_3$ where $e_i \cdot e_j = \delta_{ij}$

α and β are congruent iff $\beta = F(\alpha)$ for some isometry $F: \mathbb{R}^3 \to \mathbb{R}^3$

Assume α, β are congruent. It follows $\exists\, F = TC$ an isometry. Assume $T(x) = x + P$ and let $e_i = C(U_i)$ for $i = 1, 2, 3$. Observe,

$$\beta(t) = F(\alpha(t)) = TC\Big(\sum \alpha_i U_i\Big) = P + C\Big(\sum_i \alpha_i U_i\Big) =$$

$$= P + \sum_i \alpha_i C(U_i) = P + \alpha_1(t)e_1 + \alpha_2(t)e_2 + \alpha_3(t)e_3$$

Assume β, α are curves for which $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta) = I$ and there exist e_1, e_2, e_3 s.t. $e_i \cdot e_j = \delta_{ij}$ and,

$$\beta(t) = p + \alpha_1(t) e_1 + \alpha_2(t) e_2 + \alpha_3(t) e_3$$

For all $t \in I$. We wish to show α, β congruent. We need to <u>construct</u> an isometry F for which $\beta = F(\alpha)$. It suffices to find T & C to construct F as $F = TC$.

$$T(x) = x + p$$

$$C(U_i) = e_i \quad \text{for } i = 1, 2, 3$$

To see C defines an orthogonal transformation, note that

$$C(U_i) \cdot C(U_j) = e_i \cdot e_j = \delta_{ij}$$

Extending linearly the identity above yields

$$C(p) \cdot C(q) = p \cdot q \quad \forall p, q \in \mathbb{R}^3.$$

Of course, we could also just argue

$$[C] = [e_1 | e_2 | e_3]$$

$$\text{and } e_i \cdot e_j = \delta_{ij} \implies [C]^T [C] = I$$

Thus, $F(x) = p + [e_1 | e_2 | e_3] x$ gives isometry for which $\beta = F(\alpha)$ $\therefore$ α, β are congruent.

H70

§3.5 #2 Let E_1, E_2, E_3 be frame field on $\mathbb{R}^3$ with dual forms $\theta_1, \theta_2, \theta_3$ and connection forms ω_{ij}. Prove $\alpha, \beta : I \longrightarrow \mathbb{R}^3$ are congruent $\Longleftrightarrow$ $\theta_i(\alpha') = \theta_i(\beta')$ and $\omega_{ij}(\alpha') = \omega_{ij}(\beta')$ for all $i,j = 1,2,3$.

Assume α, β are congruent. Then $\beta = F(\alpha)$ for an isometry F. Note, $\beta' = F_*(\alpha')$

$$\theta_i(\alpha'(t)) = \alpha'(t) \cdot E_i$$
$$\theta_i(\beta'(t)) = \beta'(t) \cdot E_i$$

Th$^{\underline{m}}$ (5.7) Let $\alpha, \beta : I \longrightarrow \mathbb{R}^3$ and E_1, E_2, E_3 a frame field on α and F_1, F_2, F_3 a frame field on β. If $\alpha' \cdot E_i = \beta' \cdot F_i$ <u>and</u> $E_i' \cdot E_j = F_i' \cdot F_j$ <u>then</u> α, β are congruent.

$\Longleftarrow$ If $\theta_i(\beta') = \theta_i(\alpha')$ and $\omega_{ij}(\alpha') = \omega_{ij}(\beta')$ $\forall\, i,j$.
then $\alpha'(t) \cdot E_i = \beta'(t) \cdot E_i$

(H71)

§3.5 #3 Show that $\beta(t) = (t + \sqrt{3}\sin t,\ 2\cos t,\ t\sqrt{3} - \sin t)$ is a helix by finding its curvature and torsion. Furthermore, find a helix $\alpha(t) = (a\cos t,\ a\sin t,\ bt)$ and an isometry such that $F(\alpha) = \beta$

$$\beta'(t) = (1 + \sqrt{3}\cos t,\ -2\sin t,\ \sqrt{3} - \cos t)$$

$$\|\beta'(t)\|^2 = (1+\sqrt{3}\cos t)^2 + 4\sin^2 t + (\sqrt{3} - \cos t)^2$$

$$= 1 + \cancel{2\sqrt{3}\cos t} + 3\cos^2 t + 4\sin^2 t + 3 - \cancel{2\sqrt{3}\cos t} + \cos^2 t$$

$$= 8 \quad \therefore \quad v = \sqrt{8}$$

$$\underline{T(t) = \frac{1}{\sqrt{8}}(1 + \sqrt{3}\cos t,\ -2\sin t,\ \sqrt{3} - \cos t)}$$

$$T'(t) = \frac{1}{\sqrt{8}}(-\sqrt{3}\sin t,\ -2\cos t,\ \sin t)$$

$$\|T'(t)\| = \frac{1}{\sqrt{8}}\sqrt{3\sin^2 t + 4\cos^2 t + \sin^2 t} = \sqrt{\frac{4}{8}} = \frac{1}{\sqrt{2}}$$

$$\underline{N(t) = \frac{1}{2}(-\sqrt{3}\sin t,\ -2\cos t,\ \sin t)} \qquad \frac{1/\sqrt{8}}{1/\sqrt{2}} = \sqrt{\frac{2}{8}} = \frac{1}{2}$$

$$B(t) = T(t) \times N(t) = \frac{1}{2\sqrt{8}}\Big(-2\sin^2 t + 2\cos t(\sqrt{3} - \cos t),$$
$$(\sqrt{3} - \cos t)(-\sqrt{3}\sin t) - (1 + \sqrt{3}\cos t)(\sin t),$$
$$(1 + \sqrt{3}\cos t)(-2\cos t) + (2\sin t)(-\sqrt{3}\sin t)\Big)$$

$$B = \frac{1}{\sqrt{8}}(-2 + 2\sqrt{3}\cos t,\ -4\sin t,\ -2\cos t - 2\sqrt{3})$$

$$B = \frac{1}{\sqrt{8}}(\sqrt{3}\cos t - 1,\ -2\sin t,\ -\cos t - \sqrt{3}) \qquad (\|B\| = 1 \text{ good} ✓)$$

$$B' = \frac{1}{\sqrt{8}}(-\sqrt{3}\sin t,\ -2\cos t,\ \sin t)$$

The non-unit speed Frenet Serret yield, (take dot-product on Lemma 4.1 pg. 70)

$$\kappa v = T' \cdot N$$

$$\kappa v = T' \cdot \left(\frac{T'}{\|T'\|}\right) \Rightarrow \kappa = \frac{T' \cdot T'}{v\|T'\|} = \frac{\|T'\|}{v} = \frac{1/\sqrt{2}}{\sqrt{8}} = \frac{1}{\sqrt{16}} = \underset{\kappa}{\boxed{\frac{1}{4}}}$$

§3.5 #3 continued:

$$v\tau = -B' \cdot N$$

$$\tau = \frac{-1}{v}\left[\frac{1}{\sqrt{8}}(-\sqrt{3}\sin t, -2\cos t, \sin t) \cdot \frac{1}{2}(-\sqrt{3}\sin t, -2\cos t, \sin t)\right]$$

$$\tau = \frac{-1}{\sqrt{8}\sqrt{8}} \cdot \frac{1}{2}\big(\underbrace{3\sin^2 t + 4\cos^2 t + \sin^2 t}_{4}\big) = \frac{-4}{8(2)} = \frac{-1}{4}$$

$$\therefore \boxed{\tau = -1/4}$$

By #9 of section 2.4 it follows β is a circular helix. The helix α has positive torsion hence $F(\alpha) = \beta$ $\underbrace{\text{needs } sgn(F) = 1}_{\text{well, actually}}$

∃ two choices $b = \pm 2$.

Furthermore,

$$\kappa = \frac{a}{a^2+b^2} \quad \text{and} \quad \tau = \frac{b}{a^2+b^2}$$

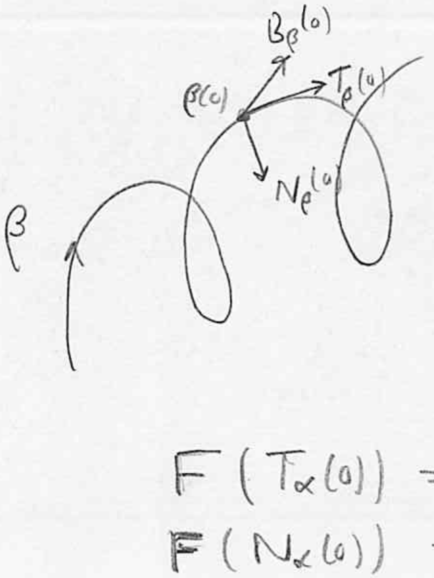
pg. 125 $\longrightarrow$ $a = \frac{\kappa}{\kappa^2+\tau^2}$ and $b = \frac{\tau}{\kappa^2+\tau^2} = \frac{-1/4}{1/16 + 1/16} = \frac{-16}{2(4)} = -2$

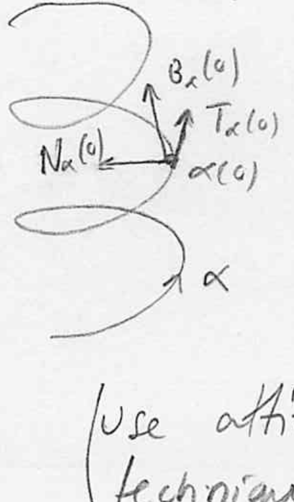
Hence $\alpha(t) = (2\cos t, 2\sin t, -2t)$.

$\alpha'(t) = (-2\sin t, 2\cos t, -2) \hookrightarrow T_\alpha(t) = \frac{1}{\sqrt{8}}(-2\sin t, 2\cos t, -2)$

$\alpha''(t) = (-2\cos t, -2\sin t, 0) \hookrightarrow N_\alpha(t) = (-\cos t, -\sin t, 0)$

$B_\alpha(t) = \frac{1}{\sqrt{2}}(-\sin t, \cos t, -1)$





$F(T_\alpha(0)) = T_\beta(0)$

$F(N_\alpha(0)) = N_\beta(0)$

$F(B_\alpha(0)) = B_\beta(0)$

(Use attitude matrix technique, just like H60-H61 # 3 of §3.2)

§3.5 # 3 continued (this is $\approx$ correct modulo the $\sqrt{3}$ issue in attitude matrix, sorry. I can't find my mistake currently H73

$$F = TC \quad \text{where} \quad \underbrace{T(X) = X - C\alpha(0) + \beta(0)}$$

$$F(\alpha(0)) = T(C\alpha(0)) = \beta(0)$$

$$C = B^T A \qquad \text{where } B \text{ is attitude of } \beta\text{-frame}$$
$$A \text{ is attitude of } \alpha\text{-frame}$$

$$= \underbrace{\left[\begin{array}{c|c|c} \frac{1}{\sqrt{8}}(1+\sqrt{3}) & 0 & \frac{1}{\sqrt{8}}(\sqrt{3}-1) \\ \hline 0 & -1 & 0 \\ \hline \frac{1}{\sqrt{8}}(\sqrt{3}-1) & 0 & \frac{1}{\sqrt{8}}(-1-\sqrt{3}) \end{array}\right]^T}_{\left[\begin{array}{c} T_\beta(0) \\ \hline N_\beta(0) \\ \hline B_\beta(0) \end{array}\right]^T} \underbrace{\begin{bmatrix} 0 & 2/\sqrt{8} & -2/\sqrt{8} \\ -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{\begin{bmatrix} T_\alpha(0) \\ N_\alpha(0) \\ B_\alpha(0) \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{8}}(1+\sqrt{3}) & 0 & \frac{1}{\sqrt{8}}(\sqrt{3}-1) \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{8}}(\sqrt{3}-1) & 0 & \frac{1}{\sqrt{8}}(-1-\sqrt{3}) \end{bmatrix} \begin{bmatrix} 0 & 2/\sqrt{8} & -2/\sqrt{8} \\ -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} 0 & \frac{2}{8}(\cancel{1}+\sqrt{3}) + \frac{1}{4}(\sqrt{3}-\cancel{1}) & \frac{-2}{8}(1+\cancel{\sqrt{3}}) + \frac{1}{4}(\cancel{\sqrt{3}}-1) \\ \hline 1 & 0 & 0 \\ \hline 0 & \frac{2}{8}(\cancel{\sqrt{3}}-1) + \frac{1}{4}(-1-\cancel{\sqrt{3}}) & \frac{-2}{8}(\cancel{\sqrt{3}}-1) + \frac{1}{4}(\cancel{1}+\sqrt{3}) \end{array}\right]$$

$$= \begin{bmatrix} 0 & \sqrt{3}/2 & -1/2 \\ 1 & 0 & 0 \\ 0 & -1/2 & 1/2 \end{bmatrix} \qquad C\alpha(0) = C\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

For $F = TC$ so constructed as above we can show $F(\alpha) = \beta$. Note, $\beta(0) - C\alpha(0) = (0,2,0) - (0,2,0) = (0,0,0)$.

Ha Ha!

$$F(x,y,z) = \begin{bmatrix} 0 & \sqrt{3}/2 & -1/2 \\ 1 & 0 & 0 \\ 0 & -1/2 & \boxed{1/2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

should be $\sqrt{3}/2$

I'm not sure where the mistake comes from

$$\underline{F(2\cos t, 2\sin t, -2t) = (\sqrt{3}\sin t + t,\ 2\cos t,\ -\sin t + \sqrt{3}\,t) = \beta(t)}$$

§3.5 # 6a Prove $\alpha, \beta : I \to \mathbb{R}^2$ are congruent if $\tilde{\kappa}_\alpha = \tilde{\kappa}_\beta$ and α, β have same speed

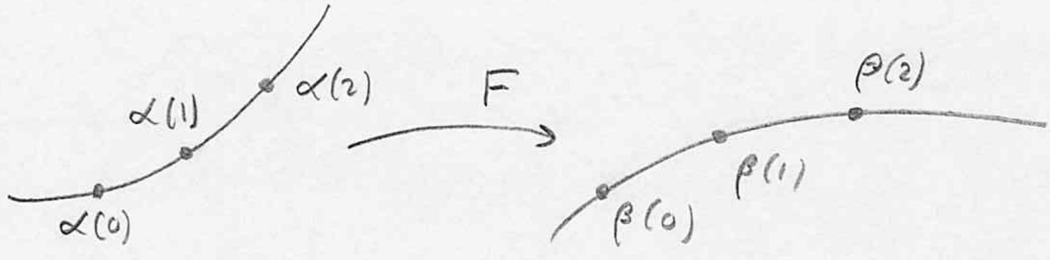
H74

Assume $\tilde{\kappa}_\alpha = \tilde{\kappa}_\beta$ and $\|\alpha'\| = \|\beta'\|$. Notice α, β are naturally extended to $\alpha, \beta : I \to \mathbb{R}^3$. Moreover, by assumption $\alpha(I), \beta(I) \subseteq \mathbb{R}^2 \times \{0\}$ supposing we extend via $\alpha \mapsto (\alpha, 0)$ and $\beta \mapsto (\beta, 0)$. Thus α, β are planar $\Rightarrow \tau_\alpha = \tau_\beta = 0$. Apply Cor. 5.6 to find α, β congruent. (note, by construction $\tilde{\kappa}_\alpha, \tilde{\kappa}_\beta > 0$)

§3.5 # 6b, pg. 128

$$\alpha(t) = (t\sqrt{2}, t^2, 0) \qquad \beta(t) = (-t, t, t^2)$$
$$\alpha'(t) = (\sqrt{2}, 2t, 0) \qquad \beta'(t) = (-1, 1, 2t)$$
$$\|\alpha'(t)\| = \sqrt{2 + 4t^2} = \sqrt{2}\sqrt{1+2t^2} \qquad \|\beta'(t)\| = \sqrt{2 + 4t^2}$$
$$T_\alpha(t) = \frac{1}{\sqrt{2+4t^2}}(\sqrt{2}, 2t, 0) \qquad T_\beta(t) = \frac{1}{\sqrt{2+4t^2}}(-1, 1, 2t)$$

Oh, so we could calculate $N_\alpha, N_\beta, B_\alpha, B_\beta$ and proceed as we did in #3. I'll try something different,



And, $C(T_\alpha(0)) = T_\beta(0)$ where $F = TC$. No need for T as $\alpha(0) = \beta(0) = (0,0,0)$. Notice, α is in $z = 0$ plane whereas β is in the $x+y=0$ plane. This suggests $F(0,0,1) = (1,-1,0)$.

§3.5 #66

H 75

$\alpha(1) = (\sqrt{2}, 1, 0)$ $\qquad$ $\beta(1) = (-1, 1, 1)$

$T_\alpha(0) = (1, 0, 0)$ $\qquad$ $T_\beta(0) = \frac{1}{\sqrt{2}}(-1, 1, 0)$

We have,

$$C\alpha(1) = \beta(1)$$
$$C\eta_\alpha = \eta_\beta \leftarrow \eta_\alpha = (0,0,1), \quad \eta_\beta = \frac{(1,-1,0)}{\sqrt{2}}$$
$$C\,T_\alpha(0) = T_\beta(0)$$

$$C\left[\alpha(1) \mid \eta_\alpha \mid T_\alpha(0)\right] = \left[\beta(1) \mid \eta_\beta \mid T_\beta(0)\right]$$

$$C \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} -1 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & \sqrt{2} & 0 \end{bmatrix}}_{N}$$

$$\left[\begin{array}{ccc|ccc} \sqrt{2} & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \sqrt{2} & 0 & 1 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -\sqrt{2} & 0 \end{matrix}}_{M^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$C = M^{-1}N = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & \sqrt{2} \end{bmatrix}$$

$$F(x,y,z) = \left(\tfrac{1}{\sqrt{2}}(z-x),\ \tfrac{1}{\sqrt{2}}(x-z),\ y+\sqrt{2}z\right)$$

$$F(t\sqrt{2}, t^2, 0) = (-t, t, t^2) = \beta(t)$$