§4.1 #9 Let $\mathbf{X}(u,v) = (u+v, u-v, uv)$ (H76)

To show $\mathbf{X}$ is a proper patch is to show $\mathbf{X}$ is

1.) 1-1
2.) regular
3.) $\mathbf{X}^{-1} : \mathbf{X}(D) \longrightarrow D$ continuous.

$\Big\}$ pg. 130-131 Oneil.

show $\mathbf{X}$ is proper patch and $\mathbf{X}(D) = M : z = \frac{1}{4}(x^2 - y^2)$

$$\left.\begin{array}{l} x = u+v \Rightarrow x^2 = u^2 + 2uv + v^2 \\ y = u - v \Rightarrow y^2 = u^2 - 2uv + v^2 \end{array}\right\} \therefore \frac{1}{4}(x^2-y^2) = uv = z.$$

Thus $\mathbf{X}$ is a patch on $z = \frac{1}{4}(x^2-y^2)$. To find the inverse $\mathbf{X}^{-1}$ we must solve for u,v the eqns below:

$$\begin{array}{lll} x = u+v & x+y = 2u & \rightarrow u = \frac{1}{2}(x+y) \\ y = u - v & x-y = 2v & \rightarrow v = \frac{1}{2}(x-y) \\ z = uv & & \end{array}$$

Hence, $\mathbf{X}^{-1}(x,y,z) = \left(\frac{1}{2}(x+y), \frac{1}{2}(x-y)\right)$.

Notice $uv = \frac{1}{4}(x+y)(x-y) = \frac{1}{4}(x^2-y^2) = z$ for $(x,y,z) \in M : z = \frac{1}{4}(x^2-y^2)$. Clearly $\mathbf{X}^{-1}$ is continuous and existence of $\mathbf{X}^{-1}$ implies 1-1 as

$$\begin{aligned} \mathbf{X}(u,v) = \mathbf{X}(\bar{u},\bar{v}) &\Rightarrow \mathbf{X}^{-1}(\mathbf{X}(u,v)) = \mathbf{X}^{-1}(\mathbf{X}(\bar{u},\bar{v})) \\ &\Rightarrow (u,v) = (\bar{u},\bar{v}) \end{aligned}$$

Finally, regularity requires $d\mathbf{X}_p : T_pD \longrightarrow T_pM$ be injective $\forall p \in D$. Notice,

$$[d\mathbf{X}_p] = \left[\frac{\partial \mathbf{X}}{\partial u} \Big| \frac{\partial \mathbf{X}}{\partial v} \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ v & u \end{bmatrix}$$

Clearly rank $[d\mathbf{X}_p] = 2 \;\; \forall (u,v) \in \mathbb{R}^2$ hence $\mathbf{X}$ is everywhere regular. Alternatively, you could show $\mathbf{X}_u \times \mathbf{X}_v \neq 0 \;\; \forall (u,v)$.

§4.2 #1 Find a parametrization of surface obtained by revolving

(a.) $C: \; y = \cosh x$ around x-axis (CATENOID)

(b.) $C: \; (x-2)^2 + y^2 = 1$ around y-axis (TORUS)

(c.) $C: \; z = x^2$ around z-axis (PARABOLOID)

H77

On pg. 135 we're advised to revolve $f(x,y) = c$ in xy-plane around x-axis we can use $g(x,y,z) = f(x, \sqrt{y^2+z^2}) = c$.

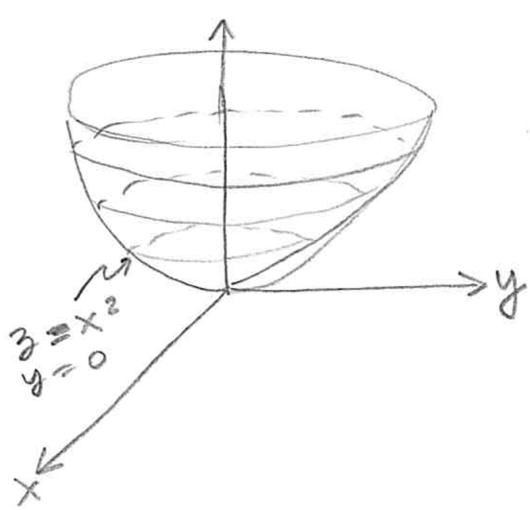
Then on pg. 143 we're given hint for curve parametrized by $u \mapsto (g(u), h(u), 0)$ for $h(u) > 0$ we can rotate by $\Phi(u,v) = (g(u), h(u)\cos(v), h(u)\sin(v))$. We'll follow this guide.

(a.) $\boxed{\Phi(u,v) = (u, \cosh(u)\cos(v), \cosh(u)\sin(v))}$

(b.) Observe $x = 2 + \cos(u)$, $y = \sin(u)$ parametrizes the circle C. Then, by 143 idea, adjusted for y-axis rotation,

$$\boxed{\Upsilon(u,v) = \big((2+\cos(u))\cos(v), \sin(u), (2+\cos(u))\sin(v)\big)}$$

(c.) $Z(u,v) = (u\cos(v), u\sin(v), u^2)$



Seems to me $Z(0,v) = (0,0,0)$ so I'm not sure why the back of book says $(0,0,0)$ not covered. However, Z not 1-1 at $(0,0,0)$ clearly. That said, I do like $\Phi(u,v) = (u, v, u^2+v^2)$ better.

§4.2 #5abc

(H78)

Helicoid: $\Phi(u,v) = (u\cos v,\ u\sin v,\ bv)$, $b \neq 0$

(a.) $\Phi' = \begin{bmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \\ 0 & b \end{bmatrix}$ as $b \neq 0$ it is clear

$\operatorname{rank}(\Phi'(u,v)) = 2 \quad \forall (u,v) \Rightarrow d\Phi_{(u,v)}$ is injective
$\Rightarrow \Phi$ regular.

If $\Phi(u,v) = \Phi(\bar{u},\bar{v})$ then we find $bv = b\bar{v} \Rightarrow \underline{v = \bar{v}}$.
also $u\cos v = \bar{u}\cos\bar{v} \Rightarrow u\cos v = \bar{u}\cos v \Rightarrow \underline{u = \bar{u}}$.
thus Φ is 1-1.

(b.) $\alpha_{v_0}(u) = \Phi(u, v_0) = (u\cos v_0,\ u\sin v_0,\ bv_0)$
gives a ~~line~~ line $\perp$ to z-axis on $z = bv_0$ plane.

$\beta_{u_0}(v) = \Phi(u_0, v) = (u_0\cos v,\ u_0\sin(v),\ bv)$
gives a helix with radius u_0 and slope b which wraps around the z-axis.

(c.) implicit form: we eliminate u, v from eq^{ns}

$$\left.\begin{aligned} x &= u\cos v \\ y &= u\sin v \end{aligned}\right\} \tan(v) = \frac{y}{x}$$
$$z = bv \quad \} \quad v = z/b$$
$$\longrightarrow y = x\tan\left(\frac{z}{b}\right)$$

Thus $\boxed{g(x,y,z) = y - x\tan(z/b) = 0}$

H79

§4.3 #4) Let $\mathbf{X}$ be a patch in M.

(a.) Show $\mathbf{X}_*(U_1) = \mathbf{X}_u$ and $\mathbf{X}_*(U_2) = \mathbf{X}_v$

$$\mathbf{X}_*(U_1) = (U_1[x], U_1[y], U_1[z]) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \mathbf{X}_u$$

$x = x(u,v)$
$y = y(u,v)$
$z = z(u,v)$

U_1 is on $\mathbb{R}^2$ in this case... $U_1[f] = \frac{\partial f}{\partial u}$.

Likewise for $\mathbf{X}_v$. I'll show it in my prefered notation

$$d\mathbf{X}\left(\frac{\partial}{\partial v}\right) = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z} = \mathbf{X}_v$$

(b.) show $\mathbf{X}_u[f] = \frac{\partial}{\partial u}(f(\mathbf{X}))$ and $\mathbf{X}_v[f] = \frac{\partial}{\partial v}[f(\mathbf{X}))$

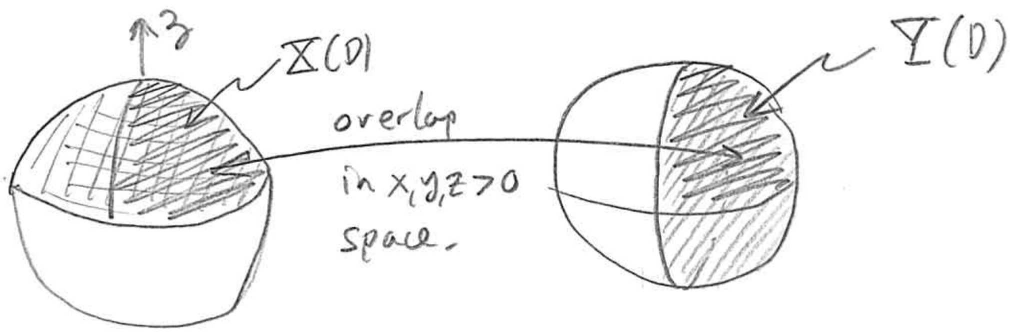
$$\mathbf{X}_u[f] = (x_u U_1 + y_u U_2 + z_u U_3)[f]$$
$$= \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial f}{\partial z}$$
$$= \frac{\partial}{\partial u}\left(f(\mathbf{X}(u,v))\right).$$

Likewise for $u \longrightarrow v$ we obtain $\mathbf{X}_v[f] = \frac{\partial}{\partial v}[f(\mathbf{X}))]$
this is just the chain-rule àla calculus III.

§4.3 #6) Consider patches on Σ below,

$$\mathbf{X}(u,v) = (u, v, \sqrt{1-u^2-v^2})$$
$$\mathbf{Y}(u,v) = (v, \sqrt{1-u^2-v^2}, u)$$

(a.) Sketch where $\mathbf{X}(D)$ and $\mathbf{Y}(D)$ cover on Σ and where they overlap



(b.) For what points of D is $Y^{-1} \circ X$ defined? Find formula for this function

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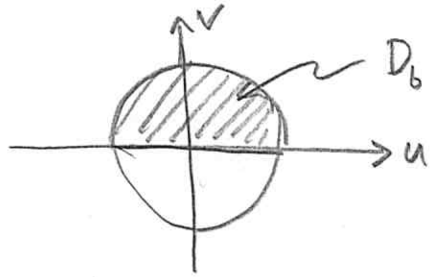
$$X(u,v) = (u, v, \sqrt{1-u^2-v^2})$$

$$Y(u,v) = (v, \sqrt{1-u^2-v^2}, u)$$

$$Y(D) = \{(x,y,z) \in \Sigma \mid y \geq 0\} = \text{dom}(Y^{-1})$$

We must eliminate $(u,v) \in D$ for which $(X(u,v)) \cdot U_2 < 0$. Thus, we need $v \geq 0$. Notice $D = \{(u,v) \mid u^2+v^2 \leq 1\}$.



$D_b = \text{dom}(Y^{-1} \circ X)$.

Oh, $Y^{-1}(x,y,z) = (z,x)$ for $(x,y,z) \in Y(D)$.

Hence, $\underline{Y^{-1}(X(u,v)) = Y^{-1}(u,v,\sqrt{1-u^2-v^2}) = (\sqrt{1-u^2-v^2}, u)}$.

Notice $Y^{-1} \circ X$ is clearly smooth on D_b.

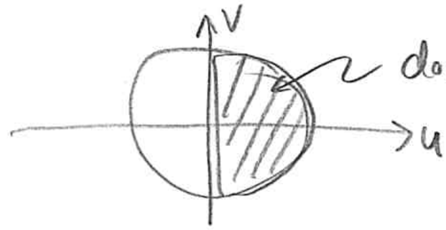
(c.) $X(D) = \{(x,y,z) \in \Sigma \mid z \geq 0\} = \text{dom}(X^{-1})$

Note $X^{-1}(x,y,z) = (x,y)$ hence

$$(X^{-1} \circ Y)(u,v) = X^{-1}(v, \sqrt{1-u^2-v^2}, u) =$$

$$\therefore \boxed{(X^{-1} \circ Y)(u,v) = (v, \sqrt{1-u^2-v^2})}$$

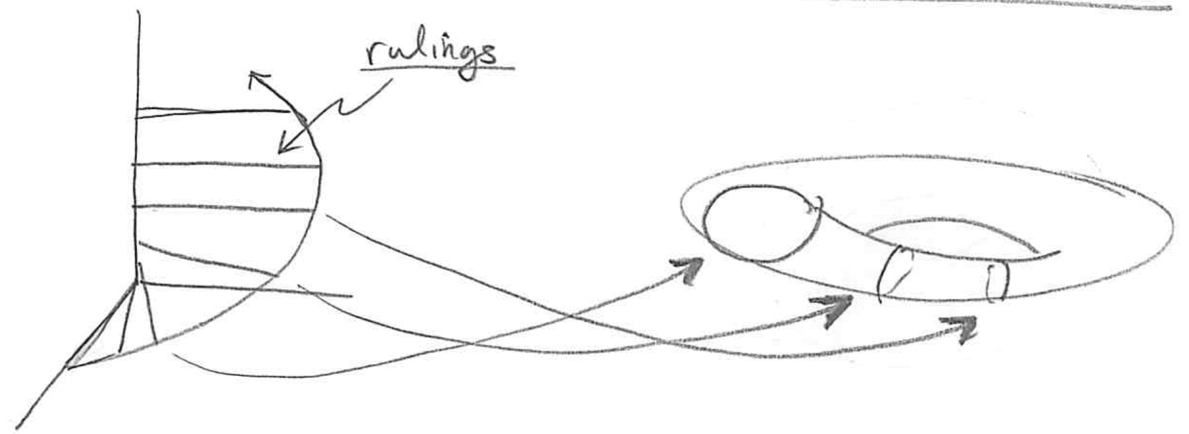
For $Y(u,v) \in \text{dom}(X^{-1})$ we need $(Y(u,v) \cdot U_3) \geq 0$ thus $u \geq 0$ so $\boxed{\text{dom}(X^{-1} \circ Y) = \{(u,v) \in D \mid u \geq 0\}}$



$\text{dom}(X^{-1} \circ Y)$

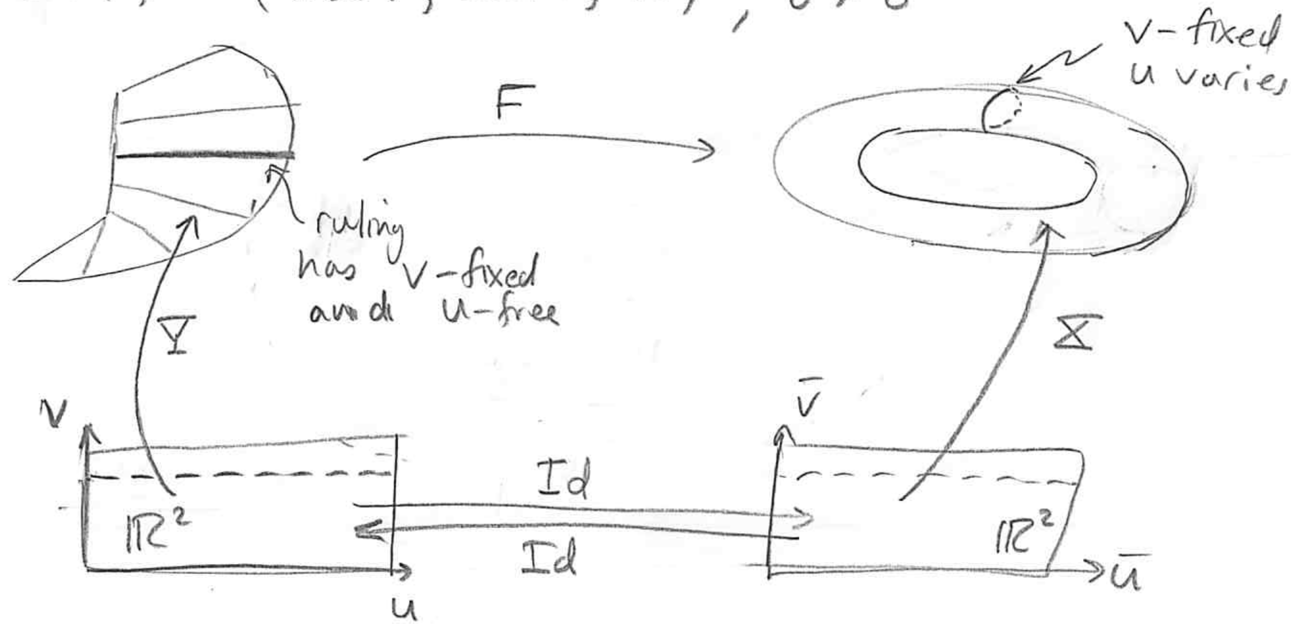
No, see p. 227, Ex 4.3

§4.5#4 | Map helicoid H (Ex. 2.5) onto torus T (Ex. 2.5) such that the rulings of H are carried to meridians of T



$$T: \ \mathbf{X}(\bar{u},\bar{v}) = \left((R + r\cos\bar{u})\cos\bar{v},\ (R + r\cos\bar{u})\sin\bar{v},\ r\sin(\bar{u}) \right),\ r, R \neq 0$$

$$H: \ \mathbf{Y}(u,v) = (u\cos v,\ u\sin v,\ bv),\ b \neq 0$$



$$F(x,y,z) = \mathbf{X} \circ \mathbf{Y}^{-1}(x,y,z)$$

$$= \mathbf{X}\left(\sqrt{x^2+y^2},\ \tfrac{1}{b} z\right), \quad \bar{u} = \sqrt{x^2+y^2},\ \bar{v} = z/b$$

$$= \boxed{\left((R + r\cos\sqrt{x^2+y^2})\cos(z/b),\ (R + r\cos\sqrt{x^2+y^2})\sin(z/b),\ r\sin\sqrt{x^2+y^2} \right)}$$

for $(x,y,z) \in H$

We can check my inverse for $\mathbf{Y}$

$$\mathbf{Y}(\mathbf{Y}^{-1}(x,y,z)) = \mathbf{Y}(\sqrt{x^2+y^2},\ z/b) = \big(\underbrace{\sqrt{x^2+y^2}\cos(\tfrac{z}{b}),\ \sqrt{x^2+y^2}\sin(\tfrac{z}{b})}_{\text{these are } x \,\&\, y \text{ why?}},\ z\big)$$

§4.7 #5 Consider plane, sphere, cylinder, torus:

(a.) which is connected? ALL

(b.) compact? Only sphere and torus. (the others are not bounded subsets of $\mathbb{R}^3$)

(c.) orientable? ALL.

(d.) simply connected? just the plane and sphere. There are loops not deformable to point on cylinder & torus.