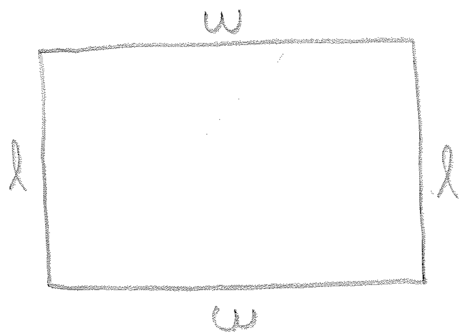
OPTIMIZATION:

We will now apply the techniques of calculus to some possibly interesting problems. Whenever we have some system that we want the biggest or smallest of something we can apply the min/max Th^{m}'s of the preceding notes. Before we can apply calculus to the problem we need to choose suitable dependent & independent variables to describe the quantities of interest,

$\boxed{\text{E1}}$. Given 400 ft of fencing what dimensions should you give a rectangular pen as to maximize the area?



$$2l + 2w = 400$$

$$A = lw$$

Notice that $l = 200 - w$

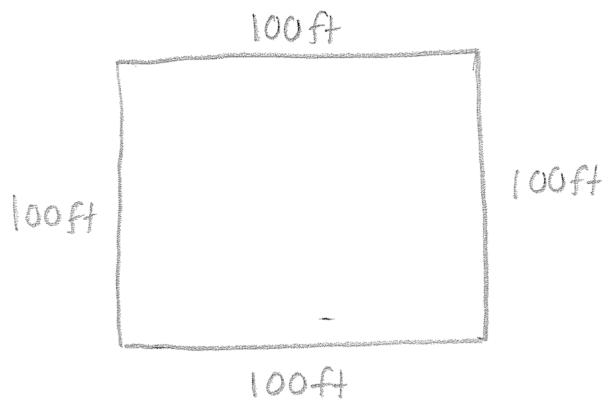
$$\Rightarrow \boxed{A = (200-w)w = 200w - w^2}$$

Maximize A as a function of w then,

$$\frac{dA}{dw} = 200 - 2w \Rightarrow \frac{dA}{dw} = 0 \text{ when } w = 100$$

Then note $\frac{d^2A}{dw^2} = -2 < 0 \Rightarrow A(100)$ is a maximum.

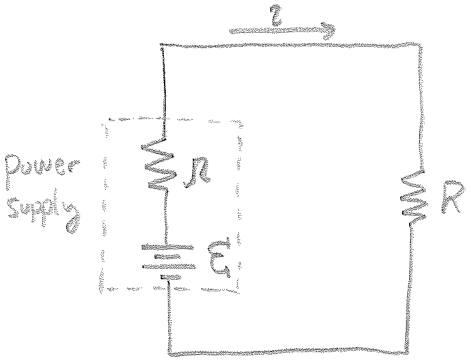
Thus by 2nd derivative test $w = 100$ gives max. area. Thus



$$l = 200 - 100 = 100$$

These dimensions maximize the area of the pen.

E2 Given a power supply with fixed voltage $\mathcal{E}$ and internal resistance r what load R will maximize the power delivered to the load?



$$\mathcal{E} = ir + iR \quad : \text{Kirchoff's Voltage Law}$$

$$P = i^2 R \quad : \text{Power delivered to } R \text{ by current } i$$

Solve to find $i = \dfrac{\mathcal{E}}{r+R}$ then substitute into $P = i^2R = \dfrac{\mathcal{E}^2 R}{(r+R)^2}$

<u>Now do calculus on P as a function of R,</u>

$$\frac{dP}{dR} = \mathcal{E}^2 \frac{d}{dR}\left(\frac{R}{(r+R)^2}\right)$$

$$= \mathcal{E}^2\left[\frac{(r+R)^2 - 2(r+R)R}{(r+R)^4}\right]$$

$$= \frac{\mathcal{E}^2}{(r+R)^4}\left[r^2 + \cancel{2rR} + R^2 - \cancel{2rR} - 2R^2\right]$$

$$= \frac{\mathcal{E}^2}{(r+R)^4}\left[r^2 - R^2\right]$$

$$= \frac{\mathcal{E}^2}{(r+R)^2}\left[(r-R)(r+R)\right] \quad : \ r, R > 0 \text{ for physical reasons so } R = r \text{ is only interesting critical point.}$$

$$\underset{0}{|}\ + + + + + + + + + + \underset{r}{|} - - - - - - - - - - \rightarrow \ \frac{dP}{dR}$$

Thus by 1st derivative test the max power is delivered to R when $R = r$. This is a simple case of the

"Max Power Transfer Thm"

Notice it implies that the Max Efficiency of a simple power supply is 50%. Electrical engineers can build more efficient power supplies called "switching" power supplies, in those supplies the output is automatically Taylored as to be best for the load. It's a tricky buisness.

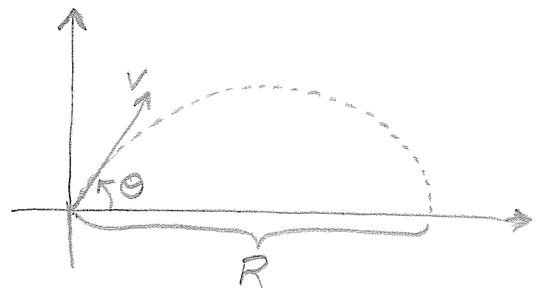
E3 The range of a projectile fired at angle Θ with velocity V can be shown to be $R = \frac{V\sin(2\Theta)}{g}$ assuming no friction and $g = 9.8\ m/s^2$. What angle Θ maximizes the range. What is R_{max} ?

We notice that V and g are constants, R is the dependent (like Y) variable and Θ is the independent variable (like x).

$$\frac{dR}{d\Theta} = \frac{d}{d\Theta}\left(\frac{V}{g}\sin(2\Theta)\right)$$

$$= \frac{V}{g}\frac{d}{d\Theta}\left(\sin(2\Theta)\right)$$

$$= \frac{2V}{g}\cos(2\Theta)$$



For physical reasons we restrict Θ to $0 \le \Theta \le \pi/2$.

$$\cos(2\Theta) = 0 \quad \text{for} \quad 2\Theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\Rightarrow \quad \Theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots$$

only $\Theta = \pi/4$ is a physically interesting critical value.

Calculate $\frac{d^2R}{d\Theta^2} = \frac{-4V}{g}\sin(2\Theta) \Rightarrow \frac{d^2R}{d\Theta^2}(\Theta = \frac{\pi}{4}) = \frac{-4V}{g}\sin(\pi/4)$

Thus we note $\frac{d^2R}{d\Theta^2}(\Theta = \frac{\pi}{4}) = \frac{-4V}{\sqrt{2}\,g} < 0$ hence by 2^{nd} derivative test we have that $R(\pi/4)$ is the max. range. $\boxed{\Theta = \pi/4}$

$$R_{max} = \frac{V\sin(2\frac{\pi}{4})}{g} = \boxed{\frac{V}{g} = R_{max}}$$

E4 The height of a projectile launched straight up with velocity V_0 is given by:

$$Y = V_0 t - \frac{1}{2} g t^2$$

What is the maximum height and at what time is it reached?

Notice that V_0 and g are just numbers so,

$$\frac{dY}{dt} = \frac{d}{dt}\left(V_0 t - \frac{1}{2} g t^2\right) = V_0 - gt \quad \text{(the velocity)}$$

$$\frac{d^2Y}{dt^2} = \frac{d}{dt}\left(V_0 - gt\right) = -g \quad \text{(the acceleration)}$$

Now $\frac{dY}{dt} = 0 = V_0 - gt \implies \boxed{t_{max} = V_0/g \text{ is critical time.}}$

We know that $Y(t_{max}) = Y_{max} = V_0\left(\frac{V_0}{g}\right) - \frac{1}{2} g \left(\frac{V_0}{g}\right)^2 = \boxed{\frac{V_0^2}{2g} = Y_{max}}$

is indeed the max height because $Y'' = -g < 0$. (2^{nd} der. test.)

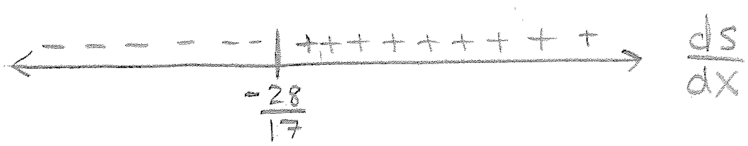
E5 Find the point on the line $Y = 4X + 7$ that is closest to the origin $(0,0)$.

We want to minimize the distance $S = \sqrt{X^2 + Y^2}$ where $Y = 4X + 7$ so consider

$$\frac{d}{dX}\left(\sqrt{X^2 + (4X+7)^2}\right) = \frac{2X + 8(4X+7)}{2\sqrt{X^2 + (4X+7)^2}} = \frac{dS}{dX}$$

So then $\frac{dS}{dX} = 0 \iff 2X + 8(4X+7) = 0 \iff 34X = -56$

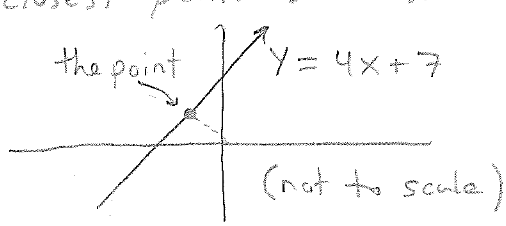
Thus $X = -28/17$ is the only critical point.



Thus $S(-28/17 = X)$ is the min. distance to $Y = 4X + 7$ from the origin by 1^{st} derivative test. The Y-coordinate is

$$Y\left(\frac{-28}{17}\right) = 4\left(\frac{-28}{17}\right) + 7 = \frac{7}{17}$$

So the closest point is in fact $\boxed{\left(\frac{-28}{17}, \frac{7}{17}\right)}$



E6 Find the point on the parabola $y = x^2+4$ that is closest to the point $(1,4)$. How close is it?

We shond minimize the distance function,

$$S = \sqrt{(x-1)^2 + (y-4)^2}$$

Where $y = x^2+4$ so we can write S a a fnct of S,

$$S = \sqrt{(x-1)^2 + (x^2+4-4)^2} = \sqrt{2x^2-2x+1}$$

Thus,

$$\frac{dS}{dx} = \frac{1}{2\sqrt{2x^2-2x+1}}\left[4x-2\right]$$

Since the quantity in $\sqrt{}$ is only zero at $x=1$, $y=4$ we know it's never zero along the parabola ($(1,4)$ is not on parabola) Consequently the only critical value for x is when $4x-2=0 \Rightarrow \boxed{x = \frac{1}{2}}$

$$- - - - - - \;|\; + + + + + + + \rightarrow \frac{dS}{dx}$$

$$x = 1/2$$

Thus $S(1/2) = \sqrt{2(\frac{1}{2})^2 - 2(\frac{1}{2})+1} = \boxed{\frac{\sqrt{2}}{2}}$ is the closest the parabola comes to the point $(1,4)$. That is the distance from $(1,4)$ to $(\frac{1}{2}, (\frac{1}{2})^2+4) = \boxed{(\frac{1}{2}, \frac{17}{4})}$.

E7 Find two positive numbers whose product is 100 and whose sum is a minimum

Let m and n be the numbers. We have that

$$mn = 100$$

$$S = m+n = m + \frac{100}{m}$$

Now minimize S as a function of m,

$$\frac{dS}{dm} = 1 - \frac{100}{m^2} = 0 \Rightarrow m = \pm 10 \text{ are critical values of } m.$$

Notice that $m=0$ is also a critical value. However only $m=10$ matters because we're looking for <u>positive</u> #'s.

$$0 \;\; - - - - - \;|_{10}\; + + + + + + + + \rightarrow \frac{dS}{dm}$$

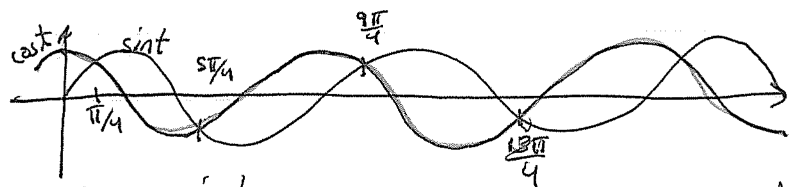
Thus S is minimized for $m=10$. Thus the numbers are $\boxed{m=10 \text{ and } n=10}$.

$ma = F_{spring} + F_{friction}(\text{velocity})$

$Y = A\sin(t)e^{-t}$ damp. harm. mot.

$Y' = (A\cos t)e^{-t} + (A\sin t)(-e^{-t})$
$= Ae^{-t}(\cos t - \sin t)$

$Y'(t) = 0 \Rightarrow \cos t - \sin t = 0 \Rightarrow \tan(t) = 1$
$t = \pi/4,\ 5\pi/4,\ 9\pi/4,\ 13\pi/4,\ \ldots$



$\tan(t) = \dfrac{\sin t}{\cos t} = 1$ when graphs match.

$Y'' = -Ae^{-t}(\cos t - \sin t) + Ae^{-t}(-\sin t - \cos t)$
$= Ae^{-t}(\cancel{\sin t} - \cos t - \cancel{\sin t} - \cos t)$
$= Ae^{-t}(-2\cos t)$
$= -Ae^{-t}\cos t$
$Y''(t) < 0$ for $t = \pi/4,\ 9\pi/4,\ \ldots$ (Maximums)
$Y''(t) > 0$ for $t = 5\pi/4,\ 13\pi/4, \ldots$ (Minimums)
So $Y(\pi/4)$ is the absolute maximum and $Y(5\pi/4)$ is abs. minimum
all subsequent points get smaller & smaller thanks to e^{-t}.